DEPARTMENT OF DEFENSE

AD [illegible]

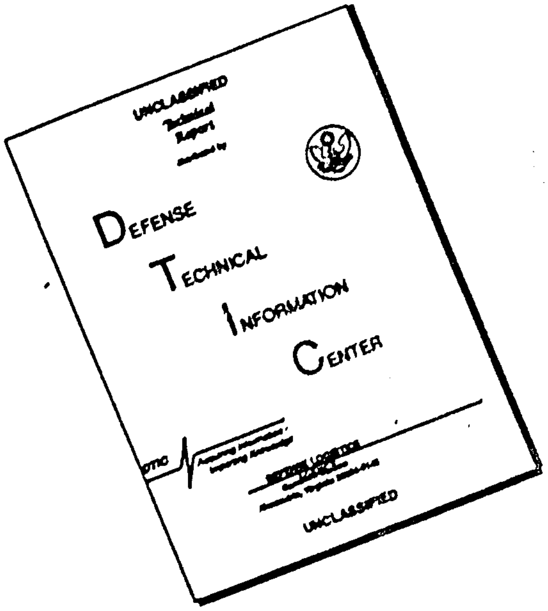
TRANSLATION

THEORY OF TRANSMITTING DISCRETE MESSAGES

NATIONAL TECHNICAL
INFORMATION SERVICE

[illegible]

DISCLAIMER NOTICE


UNCLASSIFIED
Technical
Report
DEFENSE
TECHNICAL
INFORMATION
CENTER
DTIC
UNCLASSIFIED

THIS DOCUMENT IS BEST QUALITY AVAILABLE. THE COPY FURNISHED TO DTIC CONTAINED A SIGNIFICANT NUMBER OF PAGES WHICH DO NOT REPRODUCE LEGIBLY.

Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY *(Corporate author)* DEPARTMENT OF DEFENSE	2a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED
	2b. GROUP

3. REPORT TITLE
THEORY OF TRANSMITTING DISCRETE MESSAGES

4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)*
TRANSLATION

5. AUTHOR(S) *(First name, middle initial, last name)*
FINK, LEV

6. REPORT DATE 710225	7a. TOTAL NO. OF PAGES 5,2	7b. NO. OF REFS 27

8a. CONTRACT OR GRANT NO	9a. ORIGINATOR'S REPORT NUMBER(S)
b. PROJECT NO.	
c.	9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)*
d.	

10. DISTRIBUTION STATEMENT
[illegible]

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY

13. ABSTRACT

DD FORM 1 NOV 65 1473 REPLACES DD FORM 1473, 1 JAN 64, WHICH IS OBSOLETE FOR ARMY USE.

LEV MATVEYEVICH FINK

THEORY OF TRANSMITTING DISCRETE MESSAGES

Editor: V. A. Smirnov
Art editor: V. T. Sidorenko
Technical editor: A. A. Belous, G. Z. Shalimova
Proofreaders: T. L. Knyazeva, L. A. Buyanova

APPROVED FOR PUBLIC RELEASE
DISTRIBUTION UNLIMITED

TABLE OF CONTENTS

Page

Foreword ii

Introduction i

Chapter I. Fundamental Concepts in the Theory of Transmitting Discrete Messages 4

1.1. Messages, Signals, Communication Channels. 4

1.2. Conversion of Message Into Signal. 7

1.3. Amount of Information in a Message 11

1.4. Entropy and Production of a Message Source 15

1.5. Interference and Distortion in a Channel 21

1.6. Decision System and Statistical Criteria 23

1.7. Amount of Transmitted Information 32

1.8. Carrying Capacity of a Channel 36

1.9. Principal Problems in Discrete Information Transmission Theory . 39

Notes 42

Bibliography 46

Chapter II. The Discrete Channel and Fundamentals of Coding Theory. . . 48

2.1. Discrete Channels and Their Classification 48

2.2. Carrying Capacity of a Discrete Noiseless Channel 51

2.3. Methods of Eliminating Message Redundancy 55

2.4. Discrete Noisy Channels. Interference-Resistant Codes 64

2.5. Constant Symmetrical Channel. Random Coding 69

2.6. Constant Symmetrical Channel. Regular Coding 77

2.7 Coding a Constant Symmetrical Erasure Channel 82

2.8. Coding in Asymmetrical Channels and Channels with Memory 84

2.9. Methods of Comparing Correcting Codes. 96

Notes 104

Bibliography 110

Chapter III. Constant-Parameter Channel with Additive Fluctuation Interference 114

3.1. Statement of the Problem 114

3.2. Representation of Signal and Interference Using Expansion into a Fourier Series 114

3.3. Decision Principle and Decision System 120

3.4. Probability of Error and Potential Resistance to Interference with a Binary Code 133

3.5. Error Probabilities and Potential Resistance to Interference with Code Base $m > 2$ 143

3.6. Decision System and Resistance to Interference with Normal Noise and Nonuniform Spectrum. 151

3.7. Carrying Capacity of Constant Parameter Channel with Additive Noise 158

Notes 167

Bibliography. 176

Chapter IV. Channel with Randomly Changing Signal Phase and Additive Fluctuation Interference 179

4.1. General Characteristics of Channel with Randomly Changing Initial Signal Phase. 179
4.2. Conjugate Signals, Envelope, Instantaneous Phase, and Instantaneous Frequency. Orthogonality in the Intensified Sense. . . 182
4.3. Decision System in the Case of Absolutely Incoherent Reception. . 186
4.4. Error Probabilities with Optimal Incoherent Reception 196
4.5. Nonoptimal Methods of Incoherent Reception 219
4.6. Channel with Slowly Fluctuating Signal Phase. 236
4.7. Carrying Capacity of a Channel with an Indeterminate Phase. . . . 250

Notes 255

Bibliography. 265

Chapter V. Channel with Slow General Fading (Single-Transmission Reception) 268

5.1. Nature of Fading Phenomena and Their Classification 268
5.2. Coherent and Incoherent Reception Under Conditions of General Fading at a Zero Rate 276
5.3. Memory in a Channel with Slow Fading. Several Problems in Coding 292
5.4. Effect of Rate of Fading on Probability of Error. 297
5.5. Carrying Capacity of Channel with Slow General Fading 307

Notes 310

Bibliography. 322

Chapter VI. Channel with Slow General Fading (Diversity Reception) . . . 325

6.1. Methods of Diversity Reception. 325
6.2. Coherent Diversity Reception. 328
6.3. Incoherent Optimal Diversity Reception 338
6.4. Nonoptimal Methods of Diversity Reception 346
Maximal Likelihood Selection System 346
Maximal Power Selection System. 348
General Comparison System 350
Linear Addition System 353

6.5. Discrete Addition 354

Notes 356

Bibliography 362

Chapter VII. Channels With Parameters Depending on Frequency and With Fast Fading 363

7.1. General Description of a Linear Communication Channel 363
Channel Models 364
Rate of Fading 371
Several Ideas About Optimal Decision Principle 373
7.2. Channel with Constant Frequency-Dependent Parameters 375
7.3. Fast General Fading 379
7.4. Slow Selective Fading 390
7.5. Multibeam Propagation 393
Methods of Single-Beam Discrimination 395
Methods of Using Several Beams 403
Fast Fading in a Multibeam Channel 407

Notes 410

Bibliography 415

Chapter VIII. Channel With Lumped and Impulse Interference 417

8.1. Definition and Basic Characteristics of Lumped and Impulse Interference 417
8.2. Optimal and Suboptimal Reception in the Case of Lumped Interference 421
8.3. Mutual Interference and Frequency Band Occupied by a Signal . . . 429
8.4. Mathematical Description of Impulse Interference 434
8.5. Possibilities in Principle of Suppressing Impulse Interference . . 438
8.6. Practical Methods of Protecting Against Impulse Interference . . . 444
8.7. Ways to Optimize a System with Joint Action of Lumped, Impulse, and Fluctuation Interference 446

Notes 451

Bibliography 455

Chapter IX. Multiplexing Communication Channels 457

9.1. Matching Channel Carrying Capacity with Source Productivity . . . 457
9.2. Classification of Multiplexing Methods 458
9.3. Reception Criteria and Decision Systems 462
9.4. Separable and Quasi-Separable Multiplexing Systems 465
9.5. Combination Multiplexed Systems 473

Orthogonal Systems. . . . . [illegible]
MPT Systems . . . . [illegible]
MRPT Systems. 484
Nonorthogonal MFT Systems 493
9.6. Interference-Resistant Coding in Channel Multiplexing . . . . . [illegible]
9.7. Discussion of Results . . . . [illegible]

Notes . . . . [illegible]

Bibliography . . . . [illegible]

Chapter 10. Reception of Redundancy-Coded Messages. . . . . [illegible]

10.1. Element-by-Element Reception and "Entirety" Reception . . . . . [illegible]
10.2. Entirety Reception with a Completely Known Signal and Fluctuation Noise . . . . [illegible]
10.3. Incoherent Entirety Reception . . . . [illegible]
10.4. Evaluating Noise Resistance in Entirety Reception . . . . [illegible]
10.5. Examples. . . . . [illegible]
10.6. Reception Based on the Most Reliable Symbols and Wagner Method . . . . [illegible]
10.7. Suboptimal Entirety Reception for Codes Permitting Majority Decoding. . . . . [illegible]
10.8. Conditions under which Use of Redundancy Codes is Advisable . . [illegible]

Notes . . . . [illegible]

Bibliography. . . . . [illegible]

Chapter 11. Feedback Systems . . . . [illegible]

11.1. Method of Statistical Sequential Analysis [illegible]
[illegible] . . . . [illegible]
11.2. Classification of Feedback Systems. . . . . [illegible]
11.3. Interrogation Systems in a Discrete Channel . . . . [illegible]
Principal Characteristics of the Simplest System . . . . [illegible]
Evaluation of Error Probability . . . . [illegible]
Blocking System . . . . [illegible]
Address Interrogation System . . . . [illegible]
Peculiarities of Duplex Interrogation Systems . . . . [illegible]
Selection of Code for an Interrogation System . . . . [illegible]
11.4. Interrogation Systems in a Continuous Channel . . . . [illegible]
11.5. Systems with Information Feedback . . . . [illegible]
System with Reverse Check and Repeat . . . . [illegible]
System with Transmission of Check Symbols over Reverse Channel . . . . [illegible]
Informational Feedback in a Continuous Channel . . . . [illegible]
11.6. Adaptive Methods of Encoding and Decoding . . . . [illegible]

  Notes . . . . [illegible]
  Bibliography . . . . [illegible]
Conclusion . . . . [illegible]
Appendix . . . . [illegible]
Principal Symbols Used . . . . [illegible]
Subject Index. . . . . [illegible]

This monograph is devoted to problems encountered in the theory of transmitting discrete messages. Designing optimal communication systems with given channel characteristics is considered and parameters of systems which differ to one degree or another from the optimal are described. The values of potential resistance to interference of various signal systems when transmitting in communication channels with constant and variable parameters are found. Expressions for the carrying capacity of such channels are presented. [illegible] attention is devoted to [illegible] [illegible] Recommendations are given for selecting optimal signal systems and optimal methods of reception in light of the properties of channels and the interference acting upon them.

The second edition is extensively supplemented and revised. Chapters devoted to multiplexing communication channels (Chapter IX) and to systems using feedback (Chapter XI) have been added. Chapters VII and VIII have been almost completely rewritten. Many changes and additions are to be found in other sections also. Much of the contents of the book constitutes original results obtained by the author over the last 20 years.

The book is intended for research engineers who are engaged in designing communication systems, for research students and teachers in higher educational institutions for the study of communications, and also for students in senior courses.

8 tables, 136 figures, 278 titles in bibliography.

Foreword

Over the past 20 years a scientific discipline which is referred to variously as a general theory of communication (A. A. Kharkevich), a mathematical theory of communication (K. Shannon), a statistical theory of communication (D. Middleton), a theory of communication in a broad sense (in distinction from a theory of communication in a narrow sense encompassing a quantitative determination of information and coding theory), and so on has appeared and grown rapidly at the junction of several technological and mathematical sciences. Many monographs and textbooks which differ from one another in the range of problems treated, the lines of thought pursued and the intended audience have been devoted to this theory. However, it would be difficult to find among them a work first, which gives sufficient treatment to problems in the theory of transmission and reception of discrete messages, these problems in all probability, being at the present time the most pressing for communication technology and, second, which is intended not for specialists in the field of mathematics but for research engineers engaged in developing communication systems. This situation motivated the author to gather and systematize extensive materials scattered throughout many magazines and to combine them with his own original work in a book which appeared for the first time in 1963.

This book is the second edition and it reflects extensive revision and addition. Many results which have appeared in publications or were obtained by the author following the printing of the first edition are included. The range of questions treated is somewhat extended and errors found by the author and readers have been corrected. Two new chapters devoted to multiplexing channels and building feedback systems, which received only superficial treatment in the first edition, have been added. A short table of Q-functions has been added in the form of an appendix inasmuch as no such table is to be found in available reference literature. Further, several sections which are no longer of any particular interest have been deleted. In view of the limited scope of the book it was not considered possible to give full treatment to numerous new results pertaining to diversity reception and to the reception of messages transmitted over-parallel channels. A special monograph prepared by I. S. Andronov with the author's help will be devoted to these problems.

I. S. Andronov gave the author a great deal of help in working on the second edition. Specifically, he wrote Remark 4 in Chapter V and part of the new material in Chapter VI. B. D. Kagan gave the author a great deal of help in arranging the format of the manuscript and preparing it for printing and also in writing Section 10.7.

Valuable comments and advice from many readers, especially L. F. Borodin, D. D. Klovskiy, V. I. Korzhik, B. R. Levin, Yu. S. Lezin, A. A. Sikarev, I. G. Khanovich, B. S. Tsybakov and also from many others were taken into consideration in revising the book. The author is indebted for many improvements [illegible] M. [illegible] and V. P. [illegible] who read the manuscript following revision and also to V. A. Smirnov who assisted in editing. I wish to take this opportunity to express sincere thanks to all my comrades who helped me.

Author

Introduction

Transmission of messages over electrical channels of communication is a process which continues to penetrate further into various spheres of human activity such as economy, culture and military, etc. V. I. Lenin, in the concluding part of his report on future tasks at the session of the All-Union Central Executive Committee on 29 April 1918, stated: "Socialism without a postal system, telegraph or machines is the emptiest of statements."[1] A start was made in the use of electrical processes for communication in the 1830s with the invention of the electric telegraph. Telegraphy makes it possible to transmit any text written in a particular alphabet and it constitutes an example of a system for transmitting discrete messages.

During the 130 years of its existence the technology of electrical communication has developed along various lines. A start was made in the transmission of continuous messages (telephony, transmission of half-tone images, etc.) along with discrete messages. In 1895 A. S. Popov demonstrated in practice the possibility of using electromagnetic waves propagated without the help of wire for the transmission of messages, thus laying the beginning of radio communication.

Until recently telegraphy by wire and radio was almost the only way to transmit discrete messages. The technology of telegraphy and radiotelegraphy developed continuously. Multichannel and multiple systems have been developed and new methods of keying and new receiving circuits which permit improvements in the quality of reception and making better use of communication lines have been introduced. During the past few years the technology of transmitting discrete messages has gone beyond the limits of transmitting text (telegraphy) and constitutes one of the most important links in the process of integrated automation in greatly varying fields (so-called data transmission systems or coupling systems). Wide use is also being made of systems for transmitting discrete messages for purposes of telecontrol. Finally, it can be pointed out, the most promising systems for transmission of continuous messages are based on conversion of them into discrete messages through so-called quantizing.

The theory of transmitting discrete messages is the most developed part in the general theory of communication. The main problem of this theory is finding methods for transmitting and receiving which provide for the required fidelity in the information received, for increasing the speed of transmission and for reducing the cost. These problems cannot be considered apart from one another. Indeed, each of them could be solved at the expense of the others. For example, it is easily possible to increase the fidelity of information received by decreasing the rate of transmission or by increasing the strength of the signal, etc. Therefore, only by taking all these indicated factors into consideration are we able to correctly formulate the problem of optimal design for a comminication system. The way in which this problem is posed depends on concrete conditions. In some cases the greatest possible economy (or the least possible expenditure of power) must be guaranteed while meeting demands for a given level of fidelity and rate of transmission. In other cases the rate of transmission and signal strength may be prescribed and providing for maximal fidelity may be required, etc.

---

[1] V. I. Lenin, Collected Works, [illegible], Vol. 27, p. 275.
[2] A definition of what is meant by "discrete messages" and also other concepts mentioned in the introduction will be given in Chapter 1.

Such problems constantly arise before engineers who are designing and operating various systems and information transmission lines and also developing suitable equipment. In order to solve them these engineers must have precise knowledge of the theory which permit them to find optimal (or close to optimal) conditions by comparing relatively simple calculations without resort to expensive experiments.

A general theory of communication came into existence relatively recently. It is closely associated, on one hand, with the cybernetics and, on the other hand, with the theory of probability, mathematical statistics, decision theory, theory of random processes, etc. In the main it has developed along two lines. The first line was begun with the works of V. A. Kotel'nikov in the USSR and D. Middleton and others in the USA. It amounts, in essence, to a theory of statistical detection and discrimination of signals or to a theory of potential resistance to interference. The second line, which is known as information theory, was begun by the works of C. Shannon (USA). It is based in large measure on the works of A. N. Kolmogorov and has found a rigorous foundation in the works of A. Ya. Khinchin and R. L. Dobrushin, A. Faynsteyn, and others. In these works, thanks to the introduction of the concept of "amount of information," it was found possible to think in a different way about the technical indicators of a channel of communication, such as carrying capacity and resistance to interference. During the past few years thought has been given to a synthesis of these two lines which supplement one another and are closely bound by the general nature of the problems to which they are applied.

As already noted, the case of transmission of discrete messages as the simplest case, has been developed in greatest detail in the general theory of communication. Nevertheless, this theory still fails to give an exhaustive answer to many problems which are advanced by modern communication practice. Thus, resistance to interference and carrying capacity of a line used for transmitting discrete messages have been studied only for the ideal case when the sole factor serving to distort a signal being received is additive interference. Furthermore, only one form of interference, which is expressed as a stationary random process with a normal distribution of probabilities of instantaneous values, has been thoroughly investigated.

In actual communication channels, along with additive interference, there are other factors which distort a signal, for example, fluctuations in phase and amplitude of a signal (fading), the existence of an echo, etc. Along with the thoroughly studied noise interference in radio channels, an essential role is played by mutual interference created by simultaneously operating radio stations, interference of an impulse nature, etc. All these hindering factors have been studied in the general theory of communication to a much less extent than normal noise interference. As a result, there is a serious danger of applying certain theoretical conclusions, which have enjoyed wide popularity lately, in situations fundamentally different from those for which the conclusions were drawn.

One example is the situation existing several years ago with respect to correcting codes. This theory was intended, until recently, to apply to a certain idealized "discrete channel" in which there exists a certain probability of incorrect reception of a transmitted symbol, regardless of how other transmitted symbols were received. Various correcting codes were developed in large numbers on the basis of this theory which, however, did not find practical application. Only lately has coding theory begun to allow for certain peculiarities of actual communication channels and this has made it possible to devise codes to increase the fidelity of a received message, not only theoretically but also in practice.

The purpose of this book is to set forth a modern theory of transmitting discrete messages encompassing as fully as possible the various conditions which prevail in actual communication channels. Where possible theoretical results are reduced to formulas for engineering calculations or graphs to aid in obtaining specific recommendations applicable in the design of systems and communication equipment.

The great variety and complexity of the problems involved in transmitting discrete messages prevents giving full treatment in one book to all the questions posed and giving complete solutions to problems which arise. Specifically, no mathematical exposition is given here for the "classical" theory of information. Those results of information theory which are required for solving the problems taken up are presented during the course of exposition, sometimes without strict conclusions but with reference to sources where they can be found. Further, no treatment is given to a number of technical problems such as specific equipment circuits or separate components, although, where possible, attention is devoted to the problems involved in technological feasibility of various methods for designing items of equipment, evaluating their complexity, and other technical problems attending implementation of various communication systems.

It is assumed that the reader has knowledge of the fundamentals of the theory of probability, including an elementary knowledge of the theory of stationary random processes (the concept of a correlation function and its connection with an energy spectrum). Several concepts in mathematical statistics will be explained as needed when they are introduced. The knowledge of other sections of mathematics needed is that usually reflected in courses taught at higher technical schools.

CHAPTER I

FUNDAMENTAL CONCEPTS IN THE THEORY OF TRANSMITTING DISCRETE MESSAGES

1.1. Messages, Signals, Communication Channels

Any message is a certain aggregate of information about the state of some material system which is transmitted by a man (or device) observing this system to another man (or device) that ordinarily has no way of getting this aggregate of information by direct observation. This material system, together with the observer, is the message source. In order that an item of information be transmitted usefully it is necessary to employ some physical process. A changing physical magnitude (for example, current in a wire, electromagnetic field, sound waves, etc.) reflecting the message is called a signal. The aggregate of means intended for the purpose of transmitting a signal is called a communication channel. Here by "means" is understood the physical medium in which a signal is propagated as well as the device itself. A signal is received by a recipient. By knowing the law joining message and signal the recipient is able to determine the information contained in the message. A signal is not known ahead of time by the recipient of the message and therefore it is a random process.

Along with a signal being transmitted there always are in a channel other random processes of various origin, called interference or noise. The presence of interference results in fundamental ambiguity in restoration of the message.

The communication channel together with the message source and the recipient, with given methods of conversion of a message into a signal and restoration of message based on the signal received, is called a communication system.

Sometimes a channel is used to transmit information from several sources to several recipients. Such a channel is called a multiplex channel and will be considered in Chapter IX.

A very general diagram of a communication system is shown in Figure 1.1. Here by transmitting device is meant all the equipment used to convert a message into a signal and by receiving device is meant all the equipment used to restore the message. Included in the channel may be such equipment as relay amplifiers.

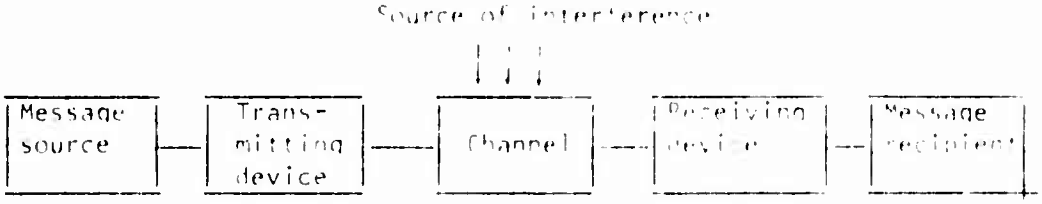


Figure 1.1. Diagram of a Communication System.

Figure 1.2. Pertaining to Definition of a Channel.

We note that the concept of "channel" is not strictly defined. For example, let a signal being transmitted from Point A to Point B (Figure 1.2) pass in sequence through several links a, b, c, ..., i, j, k, which may represent, for example, amplifiers, cable sections, medium through which electromagnetic or acoustical waves are propagated, etc. The aggregate of all these links can be called a channel. But it is also possible to consider some of the links, for example, from c to j, a channel, assigning links a and b to the transmitting device and link k to the receiving device. In the general theory of communication it is convenient to call a channel any part of a system of communication which, according to the conditions of the problem being solved, is impossible or undesirable to change. It is in this sense that we will understand the term "channel."

From a mathematical point of view to describe a channel means to indicate what signals can be delivered to its input and what the distribution will be of probabilities of a signal at its output with a given signal at its input. Finding such methods of converting a message into signals in a given channel and the reverse conversion of a signal received into a message in which, in a certain sense, the best possible transmission of messages is provided is the general problem in communication theory.

Any actual material system which is part of a message source can have a continuous series of states.¹ However, information transmitted about it never exhausts all peculiarities of a state and may in many cases form a discrete (i.e., finite or calculable) set. In these cases [illegible] that the message source is discrete.

---

[1] Here we depart from quantum laws which dictate that a series of the states of a system always be discrete.

In order to judge whether a certain message source is discrete or continuous it is essential, after selecting a finite interval of time of duration T, to consider the entire set of messages A_T which a given source could create during this time.[1] If this set is finite, the message source is discrete, otherwise it is continuous.

Understandable, with an increase in T, the number of various messages A_T increases and this can create a discrete source. This number increases exponentially for all sources [[illegible]]. Therefore, if interval of time T is not limited, set A_T is always infinite. However, for a discrete message source it will always be calculable. This means that all conceivable messages can be arranged in accordance with a certain law into a series and enumerated. For example, for a source creating messages in the form of text, written, say, in the Russian alphabet, it is possible to subdivide all possible messages into groups differing in the number of letters in a message, to arrange these groups in the order of increasing number of letters, within each group to arrange the messages in alphabetical order, and then to enumerate the sequence of messages obtained. It follows that such a message source is discrete. Any two messages from this source, if they are not identical, will differ by at least one letter.

A device transmitting the results of measurements of some continuous magnitude, say atmospheric pressure at a particular place, provides an example of a continuous source. If two messages from such a source are not identical, they may differ from one another by any amount however small. When this is so, no matter how little message A differs from B, it is always possible to have a certain message C which will differ from A less than from B. Such a set of items of information forms a continuum and cannot be enumerated.[2]

However, this continuous source will be discrete if two conditions are imposed upon it. First, it must give a message about the magnitude of atmospheric pressure at certain, previously prescribed instants of time. It must round the measured values off to a certain accuracy (say, to 0.01 mm Hg). It can easily be seen that such a modified source is discrete. At the same time, if the indicated instants of time are sufficiently frequent and the accuracy of approximated representation sufficiently great, from a practical point of view such a discrete source is in no way inferior to a continuous source. Nevertheless, resort is not always had to discretizing or quantizing a message. For example, a source transmitting the

---

[1] It should be stressed that here and later we speak of a set of messages which a source could create and not of actually created messages. From this set always one message is realized in practice. If a certain message lasts T_1 and following it a second message lasting T_2 is transmitted, then such a sequence of messages assigned to interval $T = T_1 + T_2$ is considered as one longer message.

[2] Perhaps it would be more correct to call such a source continual but the term "continuous" has found wide acceptance.

the magnitude of acoustic pressure in front of a microphone (in telephony or radio broadcasting) remains in most cases continuous.

In this work consideration is given only to messages created by discrete sources which for brevity will be called discrete messages.

Discrete as well as continuous sources can be subdivided into two types: sources with a controllable rate and sources with a fixed rate [3]. In sources of the first type messages are stored in recorded form and are issued on demand from the transmitting (coding) device. In sources of the second type messages are issued at certain instants of time which are determined by the source itself and do not depend on the functioning of the transmitting device.

The text of a telegram which is to be transmitted by telegraphic communication line, a phototelegram blank, or a perforated tape are examples of sources with a controllable rate. Many pickups in telemetric systems, electronic computers, a man speaking into a microphone, or a play transmitted by television provide examples of a source with a fixed rate.

Often an element of buffer memory is inserted between a source with a fixed rate and a transmitting device. If the capacity of the buffer memory is increased without limit, the conditions of message transmission approach those which prevail when the sources have a controllable rate.

1.2. Conversion of Message Into Signal

The form of a signal passing along a communication channel is determined by the physical peculiarities of the medium between the transmitter and the receiver. In electrical communication channels the signal amounts to a current in a conductor or the intensity of an electric field and in acoustical channels it is the sound pressure, etc. Ignoring the physical essence of a signal, we will consider a set of signals as set Z of a certain function z(t). Argument t is usually time and only this particular case will be considered here although in a more general theory t can have another meaning (for example, the coordinates of a point when recording a message on paper). Each signal of this set is defined in a limited sector of argument t.

For transmitting information with the help of signals it is possible to establish a certain mutual relationship between each of the possible messages in set A and certain signals selected from set Z. These selected signals form a subset Z_0.

Generally speaking, this relationship need not be mutually unique. However, in a reasonably designed communication system it must be unique in at least one direction. Specifically, to each signal in subset Z_0 must correspond one definite message in set A. If this condition is not met, then even in the absence of any factors at all which distort the signal, it is impossible with complete reliability to restore a message received in accordance with the signal received. The inverse relationship may be, and often is, ambiguous, this not precluding the possibility of valid restoration of the message.

Thus, the system for converting a message into a signal and back may be given in the form of a table, that is, a dictionary wherein certain signals of subset Z_0 are matched with all messages in set A. In the general case, if time T for message formation is not limited, such a dictionary would be infinite in size. But even with a limited time T, when set A is finite, in most cases it holdes such a large number of possible messages that compilation of such a dictionary and storage of it in the form of a recording on paper or in an electronic memory is impracticable. Only for the most primitive message sources, when the number of elements in set A is very small, is such a dictionary method of conversion suitable.

In other cases, instead of the direct dictionary method of conversion, use is made of a more convenient procedure which amounts to partitioning all possible messages in set A into a sequence of several "elements" or "elementary messages" or "letters," which form the finite set X having a rather small number of elements. Such partitioning is usually done by the message source itself and can be done in various ways. We will present several examples.

Example 1. Let a message measure a certain scalar magnitude with an accuracy of ϵ. Taking ϵ as unity, it is possible to depict every result of measurement by a whole number. This number can be written with digits in the decimal or any other system of numbers. Then any message which is the result of one measurement or a sequence of results of several measurements can be partitioned into digits in the selected system of numbers. Each digit represents an elementary message (or "letter") so that the set X (in the decimal system) can in this example contain 10 elements. In several cases it is advisable to include in X one additional elementary item of information (separating) indicating that a given result of measurement is terminated and an item about another result is beginning.

Example 2. Assume that source messages can be expressed in words and recorded in some language. Then a letter of the alphabet of the given language (after including in it separators between words and punctuation marks) can be taken as an elementary message. It would also be possible to take a word or sentence as an elementary message. All these methods of partitioning lead to a finite set X, however, partitioning by word or sentence is in practice inconvenient since in this case X will contain a very large number of elements.

Example 3. In the general case of any discrete source all possible messages, as indicated above, form a calculable set and can be enumerated. The law, in accordance with which enumeration is performed, can be selected in light of the peculiarities of a given specific source. The number of each message can be written in any number system and each digit of this number taken as an elementary message.

The last example shows that partitioning into elementary messages is in principle possible for any discrete message source. The set of elementary messages thus obtained will be called the source alphabet. The number of elements in the alphabet (size of alphabet) will be designated by the letter [illegible].

Thus each message a from set A can be represented in the form of a sequence of elements x of set X:

$$a = (x^{(1)} \ldots x^{(l)}) \qquad (1.1)$$

Here we will call l the length of the message a in alphabet X. The upper script in the designation of a message element indicates the ordinal number of the element and the subscript its place in the alphabet. Any element $x^{(i)}$ in (1.1) can have the values $x_1, x_2, \ldots,$ [illegible].

Now the task of conversion of a message into a signal is greatly simplified. Instead of compiling a "long" (in the general case even infinite) dictionary matching any possible message in set A with a signal, the compilation of only a very limited number of elementary messages in set X is sufficient. If in this process provision is made for mutually unique correspondence between signal and message (which, as already indicated, is not mandatory), it will be necessary to select [illegible] "fixed" signal samples and match each of them with one elementary message. Thus, for transmission of message a represented in the form of a sequence of elementary messages (1.1) signals [illegible] are sent in turn to the communication channel. In this case [illegible] is a signal corresponding to an element of message $x^{(i)}$.

However, with a large alphabet size it is usually [illegible] to use still another conversion (coding), which [illegible] size [illegible] a new alphabet represented by the set of symbols [illegible] is small. The rule of conversion of elementary messages of the alphabet X into symbols of alphabet [illegible] and back is called the code. The size of the code alphabet m is usually called the base of the code.

Here a certain amount of precision must be introduced. At first glance the operation of coding is a continuation of the operation of partitioning of messages into elements and, as we have seen, it should be possible to [illegible]

In books on information theory the term "coding" is used in different senses. In the broad sense coding is the term given to conversion of a message into a signal. In the narrow sense coding is defined as representation of discrete messages by a series of previously selected symbols. The reverse operation is called decoding. In this book the second definition is used unless otherwise stipulated.

immediately partition them to the size of an alphabet equal to m. In principle this is correct, however, there is a difference between partitioning and coding. In partitioning of a message into elements each of the elementary messages represents an undivisible segment which by itself carries a certain [illegible]. The method of partitioning is usually given by the message source itself and, as already noted, is accomplished in the [illegible].

[illegible] engineer who is engaged in transmission of messages over a [illegible] or in processing messages in electronic computers, [illegible] a message from a source already partitioned [illegible], in the form of a sequence of digits or letters.

The [illegible] engineer designing a communication [illegible] the message as well as [illegible] the channel of communication. Therefore, coding [illegible] message source [illegible].

[illegible]

[illegible] as a rule, [illegible] the receiver the [illegible]

[illegible] code symbol[1]

Thus, [illegible] the sequence of symbols [illegible] sent over the channel, [illegible] elements [illegible] beginning of transmission of the [illegible] ending of the first element, and [illegible] is the [illegible] of ending of the [illegible] signal element

---

[1] If the message source [illegible] continuous, but converted to a discrete source by [illegible] of the message in time and content, then partitioning of a message is accomplished in the very process of discretization.

In the general case several variants of the signal element, or even an infinite subset of set [illegible], which we will also designate as [illegible], may correspond to the code symbol.

The duration of each element of the signal, generally speaking, can vary for various code symbols, and also can have a random magnitude (with a particular distribution of probabilities for each). Examples of the latter case are provided by telegraphic transmission by key where the duration of an element of a signal ("dot" or "dash") fluctuates within certain limits depending on the qualifications of the operator and by start-stop transmission when the duration of the stop element corresponding to the last symbol of each code combination [illegible] wide limits. A communication system in which the [illegible] elements are strictly fixed and equal for all [illegible] multiple relationships) is called a synchronous system. [illegible] systems in wide use have many advantages and therefore [illegible] which for the most part will be considered here.

The process of conversion of a sequence of code symbols [illegible] sequence of signal elements is called, in communication theory, modulation. Sometimes in this sense the term "keying" is used with reference [illegible] telegraphy. Thus, conversion of a message emitted by a source in the form of a sequence of message elements into a signal [illegible] operations of coding and modulation.

At the receiving end of a communication channel the signal received must be identified with one of the possible messages transmitted. [illegible] most practical communication systems this operation is performed sequentially; first, signal elements are converted into a sequence of code symbols (demodulation) and then this sequence is converted [illegible] sequence of message elements (decoding) which are received by the recipient (Figure 1.3).

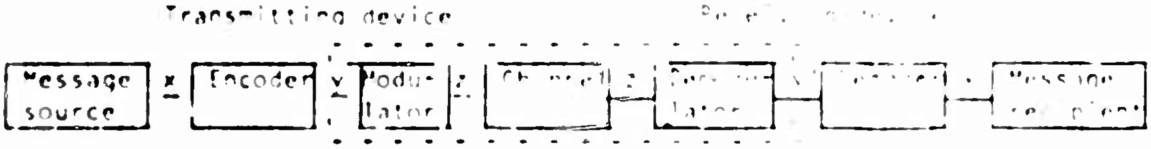


Figure 1.3. Diagram Showing Message Conversion in a Communication System

1.3. Amount of Information in a Message

In order to be able to compare various message [illegible] communication lines and channels, it is necessary to introduce a certain quantitative measure to permit evaluating the information [illegible] in a message and carried by a signal. Such a measure in the form of a quantity of information was introduced by C. Shannon [1]. It was based on the concept of selection. This made it possible for him to [illegible] general mathematical theory of communication.

---

[1] Prior to Shannon a logarithmic measure of information [illegible] and also Fisher (in developing asymptotic methods [illegible]

We will consider the basic ideas of this theory as applicable to a discrete source emitting a series of elementary messages. We will try to find a convenient measure of the quantity of information contained in a certain message. The principle idea in the theory of information is that this measure is defined not by the concrete content of each message but by the fact that the source selects a given elementary message from the finite set X. This idea is justified by the fact that on this basis it has been possible to obtain a series of significant and at the same time nontrivial results agreeing well with intuitive ideas about information transmission. The most important of these results will be further set forth.

And so, if a source performs selection of one elementary message x_k (k = 1,...,N) from alphabet set X, then the amount of information emitted by it does not depend on the specific content of this element but on how the selection is made. If the selected message element is determined in advance, it is natural to assume that the information contained in it is equal to zero. Therefore, we will consider that the selection of letter x_k takes place with a certain probability $p(x_k)$[1]. This probability can, generally speaking, depend on what sequence precedes the given letter. We will assume that the amount of information included in elementary message x_k is a continuous function of this probability $f[p(x_k)]$ and we will try to define the form of the function so that it will satisfy certain very simple intuitive ideas about information.

For this purpose we will make a simple conversion of a message which will amount to considering each pair of "letters" x_k, x_l formed successively by this source as one enlarged "letter." We will refer to this conversion as enlarging the alphabet. Set X^2 of enlarged "letters" forms an alphabet of size N^2 since after each of the N elements of alphabet X it is possible, generally speaking, to select any of the N elements. Letter $x_k x_l$ has the probability $p(x_k, x_l)$ of the simultaneous sequential selection of elements x_k and x_l.

It is naturally expected that the amount of information contained in the pair of letters selected, [illegible] the sum of the amounts of information [illegible] letters x_k and x_l of the original alphabet X. [illegible] equal to $f[p(x_l)]$ where [illegible] the probability of [illegible] letter x_l after all letters preceding it. For a determination of the information contained in letter x_l, it is necessary to consider the probability of selection of x_l after x_k [illegible] of all letters preceding x_k [illegible] will designate this conditional probability[1] by $p(x_k|x_i)$. Then the amount of information letter x_k will be expressed by function $\varphi[p(x_k|x_i)]$.

---

[1] This probability can depend on which message element preceded the given one.

On the other hand, the probability of selecting a pair of letters is, according to the rule of multiplication of probabilities

$$p(x_i, x_k) = p(x_i)p(x_k|x_i) \qquad (1.2)$$

The requirement of additivity of amount of information in enlarging an alphabet leads to the equation

$$\varphi[p(x_i)] + \varphi[p(x_k|x_i)] = \varphi[p(x_i, x_k)] = \varphi[p(x_i)p(x_k|x_i)]$$

Let $p(x_i) = p$ and $p(x_k|x_i) = q$. Then for any p and q(0 < p ≤ 1, 0 < q ≤ 1), the following equation must hold:

$$\varphi(p) + \varphi(q) = \varphi(pq) \qquad (1.3)$$

The cases p = 0 or q = 0 are excluded from consideration since as a consequence of the finite number of letters in the alphabet, these equations indicate that selection by the source of the letter pair x_i, x_k is an impossible event.

Equation (1.3) is a functional equation from which mode of function φ can be determined. We differentiate both parts of equation (1.3) with respect to p:

$$\varphi'(p) = q\varphi'(pq)$$

We will multiply both sides of the equation obtained by p and introduce the equality pq = r, then

$$p\varphi'(p) = r\varphi'(r) \qquad (1.4)$$

This equation must hold for any p(0 < p ≤ 1) and any r(0 < r ≤ p). The latter limitation (r ≤ p) is not significance since equation (1.4) is not significant since equation (1.4) is symmetrical with respect to p and r and, consequently, must be met for any pair of positive values of arguments not exceeding unity. But this is possible only if both sides of (1.4) represent a certain constant magnitude k, whence

---

[1] In essence all probabilities figuring in this consideration are conditional since they depend on letters preceding x_i. By introducing the designation $p(x_k|x_i)$ we only stress that in computing this probability, we must consider selection of the letter x_i itself as well as the letters preceding it.

$$p\varphi'(p) = k, \quad \varphi'(p) = \frac{k}{p}.$$

Integrating the equation obtained we find

$$\varphi(p) = k \ln p + C, \qquad (1.5)$$

where C is an arbitrary constant of integration.

Formula (1.5) defines the class of functions φ(p) expressing the amount of information in the selection of letter x_i having a probability of $p(x_i) = p$ and satisfying the condition of additivity. For a determination of the constant of integration C we use the condition expressed above in accordance with which a previously determined message element, i.e., having a probability p = 1, contains no information. It follows that φ(1) = 0, whence it immediately follows that C = 0.

As far as the proportionality factor k is concerned, it can be selected arbitrarily since it only defines the system of units in which the information is measured. However, inasmuch as ln p < 0, it is advisable to make k negative so that the amount of information will be positive. It is simplest to select k = -1. Then,

$$\varphi(p) = -\ln p = \ln \frac{1}{p} \qquad (1.6)$$

When this is done a unit of information is equal to that information contained in an elementary message having a probability of 1/e (e is the natural logarithm base) or, in other words, equal to the information contained in a message to the effect that an event has occurred, the probability of which is equal to 1/e. Such a unit of information is called a natural unit.

Most frequently we choose k = -1/ln2. Then, φ(p) = -lnp/ln2, or

$$\varphi(p) = -\log_2 p = \log_2 \frac{1}{p}. \qquad (1.6a)$$

With such a selection of k the unit of information is called a binary unit. It is equal to the information contained in a message that an event has occurred whose probability is equal to 1/2, i.e., which can happen or not happen with equal probability. Sometimes use is made of other units of information such as decimal units. We will use binary as well as natural units for amounts of information. In those cases when the selection of units does not matter, we will write

$$q(p) = -\log p \qquad (1.6b)$$

knowing that a logarithm can have any base as long as this base is retained throughout the solution of the problem.

Thanks to the property of additivity of information, expression (1.6) makes it possible to determine the amount of information not only in a letter of a message but also in any message, no matter how long. It is only necessary to take for p the probability of selection of this message from among all possible in light of previously selected messages.

1.4. Entropy and Production of a Message Source

In constructing a theory of communication it is not the amount of information contained in a specific message that has greatest importance but the average (mathematical expectation) of the amount of information contained in one elementary message of the source:

$$H(x) = \overline{-\log p(x_k)} \qquad (1.7)$$

Here, as everywhere in what follows, the horizontal line indicates mathematical expectation.

The magnitude H(x) characterizes the message source and is called the entropy of the source with respect to one element of a message.

In the simplest case of a source of independent messages in which the probability of selection of one message element or another does not depend on previously selected elements:

$$H(x) = -\sum_{k=1}^{l} p(x_k)\log p(x_k) \qquad (1.7a)$$

Here, l is the size of the source alphabet and $p(x_k)$ is the probability of selection of the k-th element (k-th letter).

Usually it is specified that entropy characterizes a given distribution of probabilities from the point of view of indeterminacy of outcome of test, i.e., indeterminacy of selection of one message or another. Indeed, it can immediately be seen that the entropy is equal to zero when and only when one of the probabilities $p(x_k)$ is equal to unity and all others are equal to zero. This indicates complete determinacy of selection. With a fixed size of alphabet l the entropy is a maximum when all $p(x_k)$ are the same. Then $p(x_k) = 1/l$ and

$$H_{max} = -\sum_{k=1}^{l} \frac{1}{l}\log\frac{1}{l} = \log l \qquad (1.8)$$

In this case the degree of indeterminacy of selection, based on intuition, is greater than in the case of probabilities which are not equal.

Finally, if we consider alphabets with equally probable elements but with different sizes, the entropy increases with an increase in the size [illegible]. This also agrees with the intuitive idea of degree of indeterminacy of selection.

After the source has made a selection of a specific message element, the indeterminacy which has existed is eliminated. In this respect the amount of information contained on an average in an element is measured by the indeterminacy which was eliminated as a result of the selection of this element, i.e., by the entropy of the source.

Another descriptive interpretation of the concept of entropy as a measure of "diversity" in messages created by a source is possible. It can easily be seen that the properties of entropy presented above agree completely with the intuitive idea of a measure of diversity. It is also natural to think that the amount of information contained in a message element is greater, the more varied are the possibilities in the selection of this element.

We will now define entropy for a more general class of message sources in which the probability of selection of an element depends on which elements were selected earlier. We will limit ourselves to sources in which probability relationships are expressed only for elements which are not far removed from one another. It is just such message sources which are most likely to be encountered in practical application.

For example, if a source emits information in the form of a text written in the Russian (or any other) language, the probability of occurrence of a certain letter depends strongly on several preceding letters but depends almost not at all on that part of the text which is far removed from it, say, by several tens of words. Indeed, if we find in some text the combination of the letters "raspredele..." there is a strong possibility that the following letters will be "niye." Further, if the text is mathematical, after the word "raspredeleniye" the word "veroyatnostey" will probably follow. However, the probability of occurrence of particular letters or words in a succeeding line depend's almost not at all on the letters written at the beginning of the preceding line. Somewhat more extended probabilities can be found in poetry (as a consequence of rhythm and rhyme) but even here, as a rule, they do not extend further than one stanza.

Another example is provided by a source which measures atmospheric pressure at a particular point with given precision at certain intervals of time. In this example the probability ties between results of observations are extended over long intervals of time on the order of several days or weeks and, consequently, include many elementary messages (if the measurements are performed with sufficient frequency, for example, every hour). However, even here, attention can be drawn to the rather long interval of time (several months or years) over which these ties for all practical purposes, do not extend.

Markov chains provide a mathematical representation of messages created by such sources.

A Markov chain of the n-th order is a sequence of dependent trials in which the conditional probability of a certain outcome x_l on the l-th trials when the outcomes of the n preceding trials are known, does not depend on previous outcomes. In other words, when $l > a_1 > a_2 \dots a_n > a_{n+1}$,

$$p(x_k^{(l)} | x^{(a_1)}, x^{(a_2)}, \dots, x^{(a_n)}) = p(x_k^{(l)} | x^{(a_1)}, x^{(a_2)}, \dots, x^{(a_{n+1})})$$

In a Markov source of the n-th order the distribution of probabilities $p(x_k)$ does not remain constant but depends on what the last n letters determine a certain state S_q of the source $(q = 1, 2, \dots, r)$ in which the probability of selection of the k-th letter of the alphabet is equal to $p_q(x_k)$.

The number of various possible sequences of n letters with a size of alphabet l is equal to l^n. It follows that the number r of various states of the Markov source is finite and does not exceed l^n. If for each state S_q probabilities $p_q(x_k)$ are given and it is known which state determines any sequence of n elements, then the probabilities P_q of each of the states $S_q (q = 1, \dots, r)$ can be computed. For several additional conditions, called ergodic conditions, which are met for all sources of practical interest, there exist unconditional probabilities $p(x_k)$ of selecting the k-th elementary message

$$p(x_k) = \sum_{q=1}^{r} P_q p_q(x_k) \tag{1.9}$$

The expression $H_q = -\sum_{k=1}^{l} p_q(x_k)\log_2 p_q(x_k)$, representing the mathematical expectation of the amount of information in a selected element for a source in the q-th state, may be called the entropy of this state. We obtain the entropy of a source (computed for one element) H_q in accordance with (1.7) by averaging over all possible states

$$H(x) = -\sum_{q=1}^{r}\sum_{k=1}^{l} P_q \, p_q(x_k)\log_2 p_q(x_k) \tag{1.10}$$

Expression (1.7a) is a particular case of (1.10) where r = 1, i.e., with the only state of the source. If we were not to consider the probabilities ties between message elements and base ourselves on unconditional probabilities $p(x_k)$ determined from (1.9), then for one element we would take for the entropy of the source

$$H \quad \sum_{i} \left(\sum p_{i} [illegible] \right) \left(\sum p [illegible] \right)$$

In information theory it is proved that always [illegible], i.e., the existence of probability ties decreases the entropy of a message [illegible]

A comparison of entropy H determined by expression (1.10) with the maximal entropy H_{max} = log [illegible] possible with a given alphabet is of interest for characterizing the alphabet of a message source. For this purpose we introduce the concept of alphabet redundancy in a given message source (or, as it is often stated, message redundancy)

$$r_{a} = 1 - \frac{H(x)}{H_{max}} = 1 - \frac{H(x)}{\log m}. \qquad (1.11)$$

From what has been said above it is clear that dissimilar probabilities of message elements and the existence of probability ties between close elements may cause redundancy.

We will present a simple example. Let the alphabet of a source consist of two elementary messages which we will designate A and B. Then the maximal entropy of such a source achieved with independent selection of A and B with equal probabilities and expressed in binary units is $H_{max} = \log_2 2 = 1$. If the elements are selected by the source independently, but with different probabilities p(A) = 0.3 and p(B) = 0.7, the entropy per element is $H_1 = -0.3 \log_2 0.3 - 0.7 \log_2 0.7 \approx 0.52 + 0.36 = 0.88$ bits.

In such a source the redundancy of the alphabet is equal to 0.12.

Now, let the probabilities of selection depend on one preceding element, namely: p(A|A) = p(B|B) = 0.8 and p(A|B) = p(B|A) = 0.2. Here p(A|B) indicates the probability of selection of element A on condition that the preceding one was element B, etc. It can easily be seen that the unconditional probabilities of both elements in such a source are the same and equal to 0.5. This source has two states determined by the last selected element and both states have probabilities equal to 0.5. Then, from (1.10) we obtain $H = -0.5 \cdot 0.8 \log_2 0.8 - 0.5 \cdot 0.2 \log_2 0.2 - 0.5 \cdot 0.8 \log_2 0.8 - 0.5 \cdot 0.2 \log_2 0.2 = 0.722$ bits. The alphabetic redundancy caused by the probability ties in this source is $r_x = 0.278$.

We will now consider the case when there are probability ties and the unconditional probabilities of elements are not the same. Let

$$p(A|A) = 0.3,\ p(B|A) = 0.7,\ p(A|B) = 0.1 \ \&\ p(B|B) = 0.9$$

It is easy to compute the unconditional probabilities[1] which prove to be p(A) = 0.125 and p(B) = 0.875. Such, obviously, are the probabilities of two possible states of the source. From (1.10) we obtain

---

[1] For this we may use the formula of complete probability p(A) = p(A)p(A|A) + + [1-p(A)]p(A|B) and solve the equation obtained with respect to p(A).

$$H = -0.125[0.3 \log_2 0.3 + 0.7 \log_2 0.7] - 0.875 (0.1 \log_2 0.1 +$$
$$+ 0.9 \log_2 0.9] \approx 0.52$$ binary units and the $r_x = 0.48$.

For many practical problems sources which emit messages in the form of a text written in some language are of interest. Specifically, for the Russian language, considering the number of letters in the alphabet[1] to be 32, we have $H_{max} = \log_2 32 = 5$ binary units.

If we consider the unequal probabilities of occurrence of letters in a text and the dependence of these probabilities on previous letters according to data provided by various authors, the entropy for one letter lies between the limits of 1 to 2.5 bits. Such a great spread in results is due to the difficulty of considering all probability ties spread over a large number of letters in sequence. Furthermore, the magnitude of the entropy depends in certain measure on the nature of the text. Based on these data the redundancy of the Russian alphabet lies between the limits of 0.5 and 0.8. Apparently the second number is closer to actuality.[2] Data close to these have been obtained for the alphabets of many other languages.

The entropy of a source defined above for an element in a message depends on how the message is divided into elements, i.e., on the selection of the alphabet. However, entropy possesses the important quality of additivity. Let a message with a size of alphabet Z_1 have an entropy per element of H_1 (considering all probability characteristics). We will enlarge the alphabet, considering each sequence of any n letters of the primary alphabet as one element of the new, secondary alphabet. Obviously the size of the secondary alphabet is $Z_2 = Z_1^n$. We will show that the entropy for one element of the secondary alphabet H_2 is equal to nH_1. From the definition of amount of information it follows that in a certain specific element of the secondary alphabet there is just as much information as there is n elements of the primary alphabet included in it. The amount of information in one specific element of the primary alphabet I is a random magnitude assuming various values for various elements. The amount of information in an element of the secondary alphabet I is the sum of n random magnitudes $\xi_1,\ldots,\xi_n$. The mathematical expectation of magnitude ξ, equal by definition to H_2, as is known [11], is equal to the sum of mathematical expectations of the terms of ξ_k ($k = 1,\ldots,n$) and since each of these is equal to H_1, then

---

[1] In the number of letters in an alphabet a gap between words must be included. In this case $Z = 32$ if the hard and soft signs are considered to be one letter.

[2] It must be remembered that these results pertain to sources which give the text the form of comprehensible Russian sentences united by a certain content. If Russian letters are used as symbols for certain events, knowledge about which is given by a source, the entropy for a letter must be computed based on probability characteristics of the source and can have any value, up to $H_{max} = 5$ bits.

$$H_2 = nH_1 \tag{1.13}$$

We will determine the redundancy of the secondary alphabet, r_{x2}. The maximal entropy for an alphabet os size $l_2 = l_1^n$ is equal to

$$H_{2max} = \log l_2 = n \log l_1,$$

whence, in light of (1.13),

$$r_{x2} = 1 - \frac{H_2}{H_{2max}} = 1 - \frac{nH_1}{n \log l_1} = 1 - \frac{H_1}{H_{1max}} = r_{x1}. \tag{1.14}$$

From expression (1.4) it follows that the redundancy does not change when the alphabet is enlarged.

We will point out that in enlargement of an alphabet the mutual probability ties among message elements become weaker. If the magnitude of n is so selected that it greatly exceeds the range of action of the probability ties in the primary alphabet, the probability ties between the enlarged elements can be ignored. Inasmuch as the redundancy does not change in the process of enlargement, it must be almost completely determined by the nonuniformity in the distribution of probabilities of the elements in the secondary alphabet. Thus, the operation of enlarging an alphabet can serve the purpose of "decorrelation" of the elements in a message, i.e., the purpose of eliminating mutual probability ties among them.

For sources with a fixed rate the productivity is an important characteristic, productivity being the average amount of information emitted in a unit of time. If on the average each elementary message occupies time T, then the productivity of a source is

$$H'(x) = \frac{H(x)}{T} \tag{1.15}$$

If messages are transmitted in a system of communication from a source with a controllable rate, the average time T consumed in the transmission of an elementary message is determined by the transmitting device. In this case the magnitude H'(x) determined by expression (1.15) should be called the productivity of the transmitting device. The difference between these two cases lies in the fact that the productivity of a source with a fixed rate may not be changed in designing a communication system while in the case of a source with a controllable rate the productivity of the transmitting device is selected by the designer in accordance with the various technological and economic demands made of the system.

It can easily be seen that the productivity of a source does not change in the operation of enlarging an alphabet.

1.5. Interference and Distortion in a Channel

In channels signals are transmitted in the form of certain processes of finite duration. The sequence of message elements x is converted into a sequence of code symbols y. To each code symbol y_k corresponds a certain element of a signal, i.e., a certain function z(t) defined in a finite length of time (or a particular set of such functions). If the signal at the output of a channel represents the same function z(t) as was delivered at the input, it would be possible with complete confidence to restore at the output the transmitted sequence of code symbols and then to decode it, i.e., to restore the transmitted message. The same would hold if in the channel there were only regular reversible distortions, i.e., if the signal at the output of the channel z'(t) represented function of the signal at the input $z'(t) = f[z(t)]$ and if there were an inverse function $f^{-1}[z'(t)] = z(t)$ which would permit restoring the transmitted signal exactly.

In actual channels there are irregular distortions along with such regular distortions, as a result of which the mutually unambiguous correspondence between signals at the output and the input of the channel is disrupted. The aggregate of all reasons causing indefiniteness in the signal received is usually called noise or interference. The term "interference" is also used in a narrower sense as the aggregate of voltages arriving at the input of a receiving device along with the signal. Such interference is added linearly with the signal and therefore is often called additive.

Along with additive interference in actual channels there is to be found nonadditive interference. It amounts to random distortions in a signal caused by the fact that the parameters describing the carrier of the signal fluctuate in the process of transmitting the message, i.e., the mode of the function $z'(t) = f[z(t)]$ changes irregularly over time.

On the basis of observations of various actual communication channels (specifically, various radio channels) the relation between the signal received[1] z(t) and that transmitted can be represented in a more general form as follows:

---

[1] We will call the received signal, for abbreviation, the total signal which is to distortion in the channel, and additive interference.

$$z(t) = \sum_{i=1}^{N} \mu_i s(t-\tau_i) + n(t) \qquad (1.16)$$

The meaning of this expression is as follows. Signal s(t) arrives along N different paths. In each of them it fades differently, this being characterized by transmission factors μ_i, and it is also delayed by different times τ_i. The sum of the signals which have arrived along different paths and the additive interferences which are expressed by the term n(t) is found at the input of the receiving device. The magnitudes μ_i and τ_i, generally speaking, change over time.

In many cases there is only one path of propagation for the signal and instead of (1.16) we may write

$$z(t) = \mu s(t-\tau) + n(t) \qquad (1.17)$$

As will be shown below, this expression can be approached approximately in the case of arrival of a signal along many paths if the spread of values of τ_i is very small in comparison with the duration of a signal element and with 1/Δf, where Δf is the effective width of the signal spectrum[1]. Those cases when expression (1.17) can be used we will call "single-beam propagation" of a signal and those cases when the full expression (1.16) must be used we will call "multi-beam propagation."

The magnitudes μ_i and τ_i in most actual channels change very slowly in comparison with the time of transmission of a signal element. If μ_i and τ_i change so slowly that during the time of transmission of a complete message ("period of communication") they can be considered constant, we will speak of a channel with constant parameters.

Additive interference n(t), as in the case of a signal, is a random process.

For exact determination of the effect of a signal plus interference on a receiving device, it is necessary to have a complete statistical description of the signal and interference. Only in a few particular cases is it possible to limit oneself to partial data about interference, i.e., with univariate distribution of probabilities and correlation functions (or spectral density). However, it is hardly reasonable to try to consider all the different peculiarities of various sources of interference. Therefore, we will limit ourselves to a discussion of several typical types of interference which amount to an idealization describing sufficiently well a greater part of actually observed additive interference.

---

[1] The concept of width of signal spectrum will be considered below.

Included in these types of interference is, first, fluctuation interference in which all univariate distributions of instantaneous values are normal. Such interference has been studied most completely and presents the greatest interest from a theoretical as well as practical point of view. Its theoretical significance lies in the fact that it has the greatest entropy with a given average strength and therefore the carrying capacity of the channel to the greatest degree [1]. The practical significance of normal fluctuation interference is associated with the fact that, in the first place, it is inevitably present in all actual channels in the form of thermal noise arising in an apparatus and, in the second place, it approximates sufficiently well the sum of any kind of interference arising from numerous sources which are always present in actual channels, especially in radio channels. In most cases normal fluctuation interference has a uniform spectrum in such a broad range of frequencies that it can be considered practically infinite. Such interference carries the name "normal additive white noise" and is completely described by spectral density.

Along with white noise and normal fluctuation interference which has a nonuniform spectrum we will consider impulse interference localized in brief duration (in comparison with the duration of a signal element) with a broad spectrum and also lumped interference which has a relatively narrow spectrum. Included in the latter is mainly mutual interference between signals when a medium is being used for transmission over several communication channels.

1.6. Decision System and Statistical Criteria

We will consider in general terms how restoration of a transmitted message is performed from a received signal in a channel with interference and irregular distortion. Inasmuch as in such a channel there is no unique functional tie between transmitted and received signals, the received signal can be used only to judge the probability that one signal or another was used from a set U of signals used in the given communication system and, consequently, the probability that one message or another was sent from set X of messages created by the source.

Here we are speaking of the conditional probability that the source selected, say, a message x_i if the signal received $z(t)$ arrived at the input of the receiving device. This conditional a posteriori or, using the terminology of P. Woodward [11], inverse probability can be determined using the Bayes formula [11]

$$p(x_i|z) = \frac{p(x_i)\,w(z|x_i)}{\sum_i p(x_i)\,w(z|x_i)}, \qquad (1.15)$$

where $p(x_i)$ is the a priori probability of selection of message x_i and $w(z|x_i)$ is the conditional probability density (generally speaking multivariate) of the signal received $z(t)$ when x_i is the transmitted message, this density being determined by the properties of the channel.

According to the point of view expressed by Woodward [1], nothing more can be demanded from the receiver other than data about the distribution of a posteriori probabilities of messages from the source, on the basis of which the recipient makes one decision or another. We will adhere to a more common idea, also including in the function of the receiving device the making of decision as to which message was transmitted. Incidentally, in some cases this decision, as we will see below, may not be final.

Thus, it is the task of the receiving device in a system for transmitting discrete information to issue a decision as to which elementary message of a possible set X is transmitted. The decision must be made on the basis of an analysis of the signal received z(t) in light of all knowledge available about the nature of the source, the coding system, and the properties of the communication channel. This means that no matter how the receiving device is built, the essence of its work lies in converting any arriving signal z(t) into a certain elementary message from set X. The possible ways that this conversion can be performed, with all the multitude of possible ways, reduce to breaking down set Z (generally speaking, infinite or incalculable) of all possible arriving signals into nonintersecting subsets, to each of which there is placed in correspondence one of the possible elementary messages of set X.

In most existing communication systems this correspondence is not set directly but with the help of set Y of code symbols. In these cases the set of arriving signals Z is broken down into nonintersecting subsets corresponding to each of the code symbols. Such a breakdown of a set of arriving signals into subsets will be called a decision system or first decision system. The process of identifying a signal element received with a certain code symbol based on the first decision system will be called demodulation of the received signal.

As a result of demodulation the sequence of signal elements is converted into a sequence of code symbols which must in turn be converted into a sequence of message "letters" emitted to the recipient. This conversion will be called decoding. It is done through a second decision system which amounts to breaking down the set of various sequences of code symbols into subsets, each of which is identified with a message letter.

Along with this method of receiving which is based on sequential use of two decision systems, sometimes use is made of another method with a single decision system in accordance with which an arriving sequence of signal elements is selected immediately and a sequence of message letters, i.e., the operations of demodulation and decoding are combined. In this process the set of incoming sequences of signal elements is broken down into subsets, each of which is identified with a message letter. Such a method of receiving with a single decision system has been named reception as a whole to distinguish it from the method using two decision systems which is called reception element by element. Below we will show that under certain conditions reception as a whole provides for higher noise stability than does reception element by element. However, element by element reception is much more widely used because of the simplicity of [illegible] problems involved. In this case we will be concerned [illegible] with reception element by element. [illegible] questions of reception as a whole [illegible] in Chapter [illegible].

---

[1] Generally speaking, it is not required that these subsets exhaust the entire set of arriving signals. Breakdowns are also possible in which some arriving signals are not identical with the code symbols and the question as to what was transmitted in the channel remains unanswered (so-called "erasure channels").

In any decision system the breakdown of the set of signals received into subsets corresponding to message elements can be done in a very large (or even infinite) number of ways. One of the most important problems in communication is the selection of an optimal system from among various possible ones. This is a typical problem in hypothesis testing by mathematical statistics. Here a hypothesis is meant an assumption that one message or another was transmitted. The decision system must select one of these hypotheses. In this process, obviously, the selected hypothesis will not always correspond to actuality.

Let set Z' of signals received be broken down into nonintersecting subsets Z'_i (i = 1, ..., n) and let subset Z'_i be compared with each message element x_i. Then there will be a set of conditional probabilities $p(z'_i|x_i)$ such that in the transmission of element x_i the signal received belongs to subset z'_i. If the signal received belongs to a subset z'_i, the receiving device "makes a decision" that element x_i was transmitted. We will say that in this case decision x'_i is made.

Probabilities $p(z'_i|x_i)$ depend on the way in which the message element x_i was converted into a signal, on the noise in the channel, and on the selected decision system. The probability that the transmitted element x_i is received correctly is equal to $p(z'_i|x_i)$ and the probability that it is received incorrectly is equal to

---

[1] For uniformity we speak here of a single decision system although all subsequent discussion can be applied to particular systems in reception by element.

$$1 = \sum p(x'_j | x_i)$$

If it were possible to break down set [illegible] into subsets [illegible] such that for each i the conditional probabilities were $p(x'_i|x_i) = 1$, it would be possible, despite the presence of interference and distortion in the channel, to receive all messages without error. In actual channels there usually are probabilities of error. [illegible], the complete probability of error [illegible] of a message element [illegible]

$$p = \sum p(x'_j | x_i) \ldots$$

Not every error in detecting [illegible] the message [illegible], of course, unless [illegible] however, [illegible] for various errors. [illegible] the message [illegible] the results of certain measurements in the form of [illegible] decimal figures, understandably, an error in detecting the first figure [illegible] hundreds of thousands) is much more harmful than an error in the last figure place indicating units. An error in any place changing a transmitted figure by several units is more harmful than an error resulting in a change of one unit, etc. For other sources completely different relationships are possible. Thus, it is possible to have a source also emitting five-place decimal figures, not reflecting the results of measurement but representing symbols (numbers) of certain messages. In this case an evaluation of the significance of errors may be altogether different and depend on the particular meaning of these messages.

In order to determine which of the possible decision systems is optimal, it is essential first of all to gain a clear impression of what is meant by optimality. In mathematical statistics use is made of a large number of various statistical criteria of optimality as applied to various problems. One of the most common is the so-called criterion of average risk suggested by Wald [15]. This criterion means that a certain "cost" $G(x_i, x'_j)$ which does not depend on the decision system is assigned to each pair of message x_i and decision x'_j. This cost is generally selected on an arbitrary basis but it must allow for concrete conditions prevailing in the communication system under consideration. It is higher the more undesirable is an error which amounts to making decision x'_j when actually x_i was transmitted.

The conditional mathematical expectation of cost is called the conditional risk, G_i, if it is known that element x_i was transmitted

$$G_i = \sum_{j=1}^{l} G(x_i, x'_j)\, p(x'_j | x_i) \qquad (1.20)$$

The unconditional mathematical expectation of cost G_{av}, which can be determined if the a priori probabilities of messages are known, is called the average risk

$$G_{av} = \sum_{i=1}^{l} p(x_i) G_i = \sum_{i=1}^{l} \sum_{j=1}^{l} p(x_i) G(x_i, x_j') p(z_j'|x_i) \quad (1.21)$$

According to Wald the optimal decision system is the one which provides for a minimum of average risk. This criterion pertains to the class of so-called Bayes criterion, i.e., of those for the application of which there must be knowledge of the a priori probabilities [illegible].

criterion is limited in that, in the first place, it is applicable only when the a priori probabilities are known and, in the second place, the magnitude of cost $G(x_i, x_j')$, [illegible] of the message, cannot but contain an element of subjectivity.

In most cases in designing systems for the transmission of discrete messages the properties of the source are known ahead of time, even though only approximately and, consequently, the a priori message probabilities are known approximately. Therefore, in most cases the Bayes criteria are altogether application to the selection of a decision circuit in such systems.

Solving the problem of cost is more complex. In the example above of a source emitting numerical results of measurements, it would seem logical to use as the cost $G(x_i, x_j') = |j - i|$, i.e., to consider that the cost is equal to the magnitude of absolute error in the received number relative to the transmitted number.[1] Then the criterion of minimal average risk amounts to the criterion of minimum absolute error. However, with no less reason it is possible to use $G(x_i, x_j') = (j - i)^2$, i.e., to consider the cost equal to the square of the error, and this leads to a criterion of minimal mean square error. Each of these approaches leads to the creation of its own decision systems and also each of them in a certain sense is optimal. Other possible methods of determining cost lead to various decision systems which are also optimal. The cost becomes even more indeterminate in those cases when a message is not associated with a quantitative measure. This greatly hinders determination of the optimal decision system in the general case.

In many cases, it can be considered, based on the nature of the use of a communication system, that any error in message reception entails the same degree of undesirability. From this point of view the cost should be considered the same, i.e., equal to 1 for all pairs (x_i, x_j') if $j \neq i$ and zero when $i = j$. With such an evaluation the average risk is

$$G_{av} = \sum_{i=1}^{l} \sum_{\substack{j=1 \\ j \neq i}}^{l} p(x_i) p(z_j'|x_i) = \sum_{i=1}^{l} p(x_i)[1 - p(z_i'|x_i)]. \quad (1.22)$$

---

[1] Here it is assumed that x_i corresponds to the number i.

But this expression represents nothing else than complete probability of incorrect reception of a message element in a decision system (1.19). Thus, the criterion of minimal risk with the same evaluation of all errors amounts to the criterion of minimal complete probability of error or, as it is usually called, the criterion of the ideal observer[1].

It can easily be determined how a decision system must be arranged for it to provide a minimum of complete probability of erroneous reception. Obviously the complete probability of error will be minimal if the decision system provides for a minimum of erroneous reception with each signal received.

Let signal $z'(t)$ arrive at a receiving device. For any letter x_i of the source alphabet it is possible to determine the a posteriori (inverse) probability (1.18). We will assume that the decision system assigns the signal to the subset z'_k. Then the probability that this signal will be received correctly amounts to nothing other than the a posteriori probability $p(x_k|z')$. If there is in the alphabet a certain other element x_n for which $p(x_n|z') > p(x_k|z')$, then it would be possible to increase the probability of correct reception of signal $z'(t)$ (and, consequently, to decrease the probability of error) by changing the decision system so that this signal is assigned to subset z'_n. From this it follows that the minimum probability of error in the reception of the given signal $z'(t)$ takes place in that case when it is interpreted as that message element x_k which has the greatest probability $p(x_k|z')$.

Inasmuch as this pertains to any of the signals received, the decision system based on the criterion of the ideal observer amounts to that breakdown of the set of signals received Z' wherein to subset z'_k belong all signals which differ in that for them the a posteriori probability of x_k is greater than or equal to the probability of any other message element

$$p(x_k|z'_k) \geq p(x_i|z'_k) \quad (i \neq k) \tag{1.23}$$

In many cases it is relatively easy to build such a decision system in a receiving device.

The probability of error with an optimal decision system based on the criterion of the ideal observer depends only on the properties of the channel which are determined by interference and characterizes the so-called potential resistance to interference of the communication system [2].

---

[1] This criterion is often called the Kotel'nikov criterion inasmuch as V. A. Kotel'nikov was the first to use it in 1946 in devising a theory of potential resistance to interference.

Instead of comparing the inverse probabilities $p(x_i|z')$, it is possible to compare the product $p(x_i)w(z'|x_i)$ which represents the numerators of the expression for a posteriori probability according to the Bayes formula (1.18). Indeed, the denominator in (1.18) with any given z' is a constant magnitude which does not depend on x_i. Therefore, inequality (1.23) which characterizes the decision system is equivalent to the inequality

$$p(x_k)w(z'|x_k) \geq p(x_i)w(z'|x_i) \quad (i \neq k)$$

or

$$\frac{w(z'|x_k)}{w(z'|x_i)} \geq \frac{p(x_i)}{p(x_k)} \quad (i \neq k) \tag{1.24}$$

The ratio in the left part of this inequality is called the likelihood ratio for x_k with respect to x_i. The decision system for the criterion of the ideal observer can be described thus: signal z' is assigned to subset z'_k if the likelihood ratio for x_k with respect to all x_i is greater than or equal to a magnitude which is the inverse ratio of the a priori probability. We will note that other Bayes criteria also (with a varying function of cost) can be reduced to a comparison of likelihood ratios analogous to inequality (1.24), with the difference that in the right part of the inequality instead of ratios of a priori probabilities there are other numbers dependent on i and k and determined by the cost function.

If the a priori probabilities of all x_i are the same, the right part of inequality (1.24) is equal to unity. In some cases it is assumed to be equal to unity even if the a priori probabilities are not the same or if they are unknown. The criterion thus obtained is called the criterion of maximal likelihood.

We will consider a very simple example to illustrate the criterion of the ideal observer. Let the alphabet of the source contain only two letters A and B and let the signal received $z'(t)$ be described by a single scalar parameter (i.e., current in a line) which will also be designated z'. Figure 1.4 depicts curves of $p(A)w(z'|A)$ and $p(B)w(z'|B)$ which represent conditional probability densities when transmitting letters A and B respectively multiplied by the a priori probabilities of these letters. When $z' > z'_0$ the likelihood ratio for B with respect to A is greater and when $z' < z'_0$ it is less than $p(A)/p(B)$. According to the criterion of the ideal observer the entire domain of z' is broken down into the two subsets Z'_A (in which are included all $z' < z'_0$) and Z'_B (in which are included all $z' > z'_0$). The point $z' = z'_0$ can be assigned to any of the subsets. The complete probability of erroneous reception of a letter is equal to the cross-hatched area in the figure. Indeed, this area is equal to

$$\int_{-\infty}^{z'_0} p(B)w(z'|B)dz' + \int_{z'_0}^{\infty} p(A)w(z'|A)dz'. \qquad (1.23)$$

The first integral represents the probability that letter B was transmitted and signal z' appeared in area Z'_A, i.e., the probability of erroneous reception of A instead of B and the second integral the probability of erroneous reception of B instead of A. If the area of values of z' is broken down into subsets Z'_A and Z'_B in another way, i.e., by selecting z'_1 instead of the "threshold" z'_0, then the probability of erroneous reception of A instead of B decreases and the probability of erroneous reception of B instead of A increases. However, the complete probability of erroneous reception increases by the magnitude of the area shown by the darkened triangle.

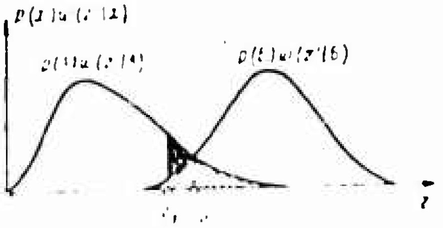


Figure 1.4. Graphic Definition of a Probability Mistake.

Application of the criterion of the ideal observer is very natural since the optimal decision system based on it provides for the least possible complete probability of erroneous reception of a message and, consequently, the greatest probability of error-free reception of the sequence of elements constituting the message. Below we will present several other conclusions in favor of this criterion. However, cases are possible when the result of applying the criterion of the ideal observer contradicts good sense.

For example, let the alphabet of the source contain two elementary messages A and B which are selected independently with probabilities p(A) = 0.999 and p(B) = 0.001.

We will consider two variation of the decision system. In the first variation the set of signals received Z' is broken down into subsets Z'_A and Z'_B so that $p(A|Z'_A) = p(B|Z'_B) = 0.999$ and $p(A|Z'_B) = p(B|Z'_A) = 0.001$. When this is so all messages are received with a probability of error of 0.001. In the second variation of the system the entire set of signals received Z' is taken as Z'_A since set Z'_B is empty. In this case all signals will be received as message A. Thus, A will be received in all cases correctly and B always incorrectly. Obviously, with such a "decision system"

$$p(Z'_A|A) = p(Z'_A|B) = 1 \quad \text{and} \quad p(Z'_B|A) = p(Z'_B|B) = 0$$

and the probability of error is equal to

$$p = p(A)p(Z'_B|A) + p(B)p(Z'_A|B) = 0.001$$

From the point of view of the criterion of the ideal observer the second pattern is closer to optimal inasmuch as it provides for a lesser complete probability of error than the first. But, on the other hand, it is quite clear that using the second variation of the decision system is meaningless since it does not give any idea about the message transmitted while the first variation, although with little certainty, makes it possible to judge which message was selected by the source.

Such a contradiction between the criterion of the ideal observer and good sense arose as a consequence of the fact that with such greatly varying a priori probabilities of A and B it was impossible to consider the cost of all errors the same. In fact, the information contained in B, according to (1.6), is much greater than the information contained in A. It follows that incorrect reception of A instead of B disturbs the information transmitted to a greater degree than reception of B instead of A.

It should not be thought that the paradox presented is peculiar to the criterion of the ideal observer. For many other statistical criteria it is possible to select more or less artificial situations in which they contradict good sense. Therefore, the selection of criteria should be made in light of the peculiarities of the decision system.

For ordinary communication systems the criterion of maximal likelihood is most convenient. According to this criterion the decision system assigns the signal received Z' to subset Z'_i if for all $j \neq i$

$$\frac{p(Z'|A_i)}{p(Z'|A_j)} > 1 \tag{1.26}$$

The advantage of this criterion is that it does not require knowledge of the a priori probabilities of messages. If the a priori probabilities of messages are known and are the same, the criterion of maximal likelihood coincides with the criterion of the ideal observer.

In all cases below where nothing to the contrary is stated, we will use the criterion of maximal likelihood.

In some cases the signal received depends not only on the communication transmitted and interference but also on one of several unknown parameters.

For example in expression (1.1[illegible]) the magnitudes of [illegible] or [illegible] may be unknown. If a parameter such as [illegible] represents a random variable with a known distribution of probabilities, it is possible to compute the conditional probability $p(z/x_i)$ which enters in the likelihood ratio, using the formula for complete probability

$$p(z/x_i) = \int p(z/x_i, \beta)\, w(\beta)\, d\beta \qquad (1.27)$$

where $w(\beta)$ is the probability density of parameter β and integration is performed over the entire area in which it is defined. Sometimes, however, nothing is known about parameter β. Then in order to construct a decision system resort is had to the generalized criterion of maximal likelihood in which the conditional probabilities $p(z/x_i)$ are computed with the most likely (with hypothesis x_i) value of parameter β, i.e., with that β which minimizes the magnitude of $p(z/x_i, \beta)$. In other words, the decision system assigns the signal received to subset z_i if for all $k \neq i$

$$\frac{\max_{\beta} p(z/x_i, \beta)}{\max_{\beta} p(z/x_k, \beta)} > 1 \qquad (1.28)$$

Examples of constructing a decision system with an unknown parameter will be presented in Chapter [illegible].

[illegible] Amount of Transmitted Information

Assume a message source in state [illegible] has selected a certain message x_i [illegible] x_i. The receiving device receives a [illegible] signal [illegible] on the basis of which the a posteriori probability $p(x_i/z)$ can be determined. We will first assume for simplicity that there is only a discrete set of signals received which we will designate z_1, z_2, ..., z_n.

If signal z_j is received, the a posteriori probability of the transmitted message is equal to $p(x_i/z_j)$. This probability would be equal to unity if there were no noise in the channel and the signal received were determined completely. The existence of noise leads to the fact that the probability $p(x_i/z_j)$ is less than unity. This can be explained as incomplete transmission of information [illegible] over the communication channel.

We will determine that amount of information which would have to be transmitted additionally after reception of signal z_j so that the transmitted message x_i would be known completely. Inasmuch as after reception of signal z_j the probability of transmission of x_i is $p(x_i/z_n)$, the required additional information can be obtained as [illegible] $p(x_1/z_n)$. But according to (1.6),

$$\tilde{\imath}[p_q(x_k|z'_n)] = -\log p_q(x_k|z'_n) = \log \frac{1}{p_q(x_k|z'_n)}. \qquad (1.29)$$

Thus, the amount of information transmitted over the communication channel in transmission of message x_k and reception of signal z'_n defined as the difference between the amount of information included in message x_k and the amount of information which remained untransmitted after reception of signal z'_n:

$$\begin{aligned} i_q(x_k, z'_n) &= \tilde{\imath}[p(x_k)] - \tilde{\imath}[p_q(x_k|z'_n)] \\ &= -\log p_q(x_k) + \log p_q(x_k|z'_n) = \log \frac{p_q(x_k|z'_n)}{p_q(x_k)}. \end{aligned} \qquad (1.30)$$

The average amount of information per elementary message transmitted in a noisy channel can be defined as the mathematical expectation of $i(x_k, z'_n)$, i.e., the result of averaging $i(x_k, z'_n)$ for all messages x_k, source states S, and signals received z':

$$I(x, z') = \sum_q \sum_k \sum_n P_q p_q(x_k, z'_n) \log \frac{p_q(x_k|z'_n)}{p_q(x_k)}, \qquad (1.31)$$

where P_q is as formerly the probability of source state S_q and $p_q(x_k, z'_n)$ is the joint probability of transmission of sign x_k and reception of signal z'_n.

Expression (1.31) can be considered as an amount of information about message x contained in signal received z', or, in a more general sense, as an amount of information contained in sequence z' with respect to sequence x.

This amount of information can be represented in another form:

$$\begin{aligned} I(x, z') &= \sum_q \sum_k \sum_n P_q p_q(x_k, z'_n) \log p_q(x_k|z'_n) \\ &- \sum_q \sum_k \sum_n P_q p_q(x_k, z'_n) \log p_q(x_k) \\ &= - \sum_q \sum_k P_q p_q(x_k) \log p_q(x_k) - \Big\{ \\ &- \sum_q \sum_k \sum_n P_q p_q(x_k, z'_n) \log p_q(x_k|z'_n) \Big\} = H(x) - H(x|z'), \end{aligned} \qquad (1.32)$$

where

$$\begin{aligned} H(x|z') &= - \sum_q \sum_n \sum_k P_q p_q(x_k, z'_n) \log p_q(x_k|z'_n) = \\ &= - \sum_q \sum_n \sum_k P_q p_q(z'_n) p_q(x_k|z'_n) \log p_q(x_k|z'_n) \end{aligned} \qquad (1.33)$$

is called the conditional entropy of message x in reception of signal z' (or, in the more general form, the conditional entropy of sequence x with the known sequence z'). It is also called the "unreliability" since it characterizes the loss of information during transmission. In expression (1.33)

$$p_q(z'_n) = \frac{p_q(x_k, z'_n)}{p_q(x_k|z'_n)}$$

is the probability of the signal received z'_n in state S_q.

It can easily be seen that in a channel without interference $H(x|z') = 0$ since $p_q(x_k|z'_n)$ can have values of 0 or 1, as a result of which all terms in (1.33) become zero. Therefore, as might be expected, in such a channel the amount of information transmitted is equal to the entropy of the source. It can be shown [3] that always $H(x|z') \geq 0$ and, consequently

$$I(x, z') \leq H(x), \tag{1.34}$$

in which the equality holds, for example, in the absence of interference in the channel. Specifically, if wet $z' \equiv x$, then

$$I(x, x) = H(x) \tag{1.35}$$

The amount of information transmitted can be expressed otherwise by using the identity

$$p_q(x_k, z'_n) = p_q(x_k)\, p_q(z'_n|x_k) = p_q(z'_n)\, p_q(x_k|z'_n)$$

After multiplying the numerator and the denominator of the logarithm in (1.31) by $p_q(z'_n)$, we find

$$I(x, z') = \sum_q \sum_n \sum_k P_q p_q(x_k, z'_n) \log \frac{p_q(x_k, z'_n)}{p_q(x_k)\, p_q(z'_n)}. \tag{1.36}$$

The expression obtained is symmetrical with respect to x and z', as a result of which we may conclude that

$$I(x, z') = I(z', x). \tag{1.37}$$

Therefore, from (1.34) it follows that

$$I(x, z') \leq H(z') \tag{1.38}$$

If we define the joint entropy of x and z' in the following way

$$H(x, z') = -\sum_q \sum_n \sum_k P_q p_q(x_k, z'_n) \log p_q(x_k, z'_n). \tag{1.39}$$

it can be shown

$$I(x, z') = I(z', x) = H(x) - H(x|z') = H(z') - H(z'|x) = H(x) + H(z') - H(x, z'). \qquad (1.40)$$

Until now we have considered that the signal received has only a discrete series of values. We will now consider a more practical case when $z'(t)$ takes a continuous series of values characterized by a probability density $w_q(z')$ and the conditional probability density $w_q(z'|x_k)$ with a known transmitted message. After setting approximately $p_q(z'_n) = w_q(z'_n)\Delta z'$, $p_q(x_k, z'_n) = p_q(x_k) w_q(z'|x_k)\Delta z'$, etc., and then performing the limiting transition $\Delta z \to 0$, we obtain from (1.36) the following integral expression

$$I(x, z') = \sum_q \sum_k \int_{Z'} P_q p_q(x_k) w_q(z'|x_k) \log \frac{w_q(z'|x_k) p_q(x_k)}{p_q(x_k) w_q(z')} dz' = \sum_q \sum_k \int_{Z'} P_q p_q(x_k|z') w_q(z') \log \frac{p_q(x_k|z')}{p_q(x_k)} dz', \qquad (1.41)$$

where integration is performed for the entire set Z'. Thus, we have obtained an expression for the amount of information contained in a continuous signal z' about a discrete message x.

Although we consider sources of discrete messages only, in some cases we will need an expression for the amount of information contained in one continuous process $z'(t)$ with respect to another continuous process $x(t)$. For this purpose we will assume that x is continuous as is z' and, performing the limiting transition in (1.36), we find

$$I(x, z') = \sum_q \iint_{X, Z'} P_q w_q(x, z') \log \frac{w_q(x, z')}{w_q(x) w_q(z')} dx\, dz', \qquad (1.42)$$

where $w_q(x)$ and $w_q(x, z')$ are the probability densities respectively of x and the joint process (x, z') with source state S_q.

Specifically, if the source has one single state, then

$$I(x, z') = \iint_{X, Z'} w(x, z') \log \frac{w(x, z')}{w(x) w(z')} dx\, dz'. \qquad (1.43)$$

If the average time taken to select one elementary message is equal to T, the amount of information transmitted in a communication channel in a unit of time, or the rate of transmission of information over the communication line is

$$I'(x, z') = \frac{1}{T} H(x) - \frac{1}{T} H(x|z') = H'(x) - H'(x|z'), \qquad (1.44)$$

where H'(x) = 1/T H(x) is the productivity of the source or the transmitting device and H'(x|z') = 1/T H(x|z') is the unreliability per unit of time.

1.8. Carrying Capacity of a Channel

Let there be given a certain communication channel, i.e., let there be defined a set of signals {z(t)} which can be delivered to the input of a channel, a set {z'(t)}[1] of signals at the output and a conditional distribution of probabilities ω(z'|z) of signals at the output with a known signal at the input.

If some a priori distribution of probabilities of input signals, for example with a density of w(z), is given, it is possible to determine the rate of transmission of information in a communication channel which, according to (1.43) and (1.44), is equal to

$$I'(z, z') = \frac{1}{T} \iint\limits_{z\,z'} w(z, z') \log \frac{w(z, z')}{w(z)\,w(z')}\, dz\, dz'. \qquad (1.45)$$

Here T is the average duration of a signal element which depends, generally speaking, on w(z), and the densities w(z') and w(z, z') included in this expression can be determined from the given densities:

$$w(z, z') = w(z)\,\omega(z'|z),$$
$$w(z') = \int\limits_{z} w(z, z')\, dz.$$

The value obtained for the rate of transmission of information depends on an arbitrarily selected distribution of probabilities of input signals. The maximal rate of transmission of information with all reliable distribution of probabilities of input signals (or, more exactly, least upper limit of rate of transmission of information) C is called the carrying capacity of the channel:

$$C = \sup_{w(z)} I'(z, z'). \qquad (1.46)$$

Sometimes in determination of a channel additional limitations are imposed on possible probability distributions of input signals. For example, it is possible to require that the average strength of the signal not exceed a given limit or that [illegible] elementary signals from a set [illegible] the upper limit is [illegible] imposed.

---

[1] Both these sets may be discrete as well as continuous. Below, in order to avoid repetition, all formulas are written for continuous sets of signals. In the case of discrete sets, it is necessary to replace distribution densities with probabilities and to replace integration with summation.

In all cases of practical [illegible] channel is a finite magnitude. [illegible] signals [illegible] the output [illegible] of the input signals [illegible] with [illegible]

The basic theorem [illegible] first formulated by [illegible] source may be coded [illegible] signals [illegible] at the output of the channel [illegible] can be made as close as desired [illegible] such [illegible]

Here [illegible] productivity of a transmitting [illegible] rate. [illegible] arbitrarily, then [illegible] must be formulated [illegible] rate can be encoded by [illegible] at the output of a channel [illegible] close as desired [illegible] messages [illegible] source for one message.

The [illegible] limitations not important [illegible] of the source and the channel [illegible] formation theory, for example [illegible] here a detailed exposition [illegible] steps which are essential [illegible] ourselves for simplicity [illegible]

We will consider [illegible] a source. Such various sequences [illegible] the [illegible] source alphabet and each of them has a particular [illegible] by statistical properties of the source. We will arrange them in order of decreasing probabilities and [illegible] and we will call those [illegible] the source with reference to an elementary message [illegible] brackets indicate the whole number part of the number included in the brackets. Let L be the summary probability of all [illegible] sequences of messages and [illegible] be any positive number. Based on the law of large numbers, with very broad assumptions about the source, it is possible to demonstrate the existence of a number such as n_1 such that when [illegible]

$$P \; [illegible]$$

We will further consider all signals [illegible] with a duration [illegible] allowable for a given channel. We will select from them [illegible] different signals, where [illegible] the carrying capacity [illegible] per [illegible], the time being [illegible] according to [illegible] these selected [illegible] signals. Let, at the output of the channel, signals be received which also have a duration [illegible], arrive at a decision system determining, based on the criterion of maximum likelihood, which of the selected signals was transmitted. [illegible] so, the decision system will err with a certain probability [illegible]. It can be shown, still with very broad assumptions about the channel, that [illegible] there is also a number [illegible], such that when [illegible]

select [illegible]

[illegible]

Now, with [illegible]

[illegible]

[illegible]

[illegible]

[illegible]

$$[illegible]$$

where [illegible]

We will select [illegible] signals [illegible] that the [illegible] exceed [illegible]. We will compare [illegible] the selected signals [illegible] the typical sequences of source messages. Since there are [illegible] sequences, there will be at least one unused signal among those selected. This signal will be sent to the channel every time the source emits a nontypical sequence, the probability of which is less than [illegible], accepting the fact that nontypical sequences will be received incorrectly. Then the total probability of incorrect reception of a sequence of messages will not exceed [illegible].

It can easily be seen that [illegible] messages are transmitted per second and that [illegible] can be as close as desired to [illegible]. This confirms the coding theorem.

Another coding system is possible in which nontypical sequences are transmitted correctly at the price of increased delay [1].

Approximately the same approach is used in the proof for a source with a fixed rate. It should be stressed that the closer I'(x) is to C and the less the allowable probability of error is, the greater must be the length of the block (sequences of messages) n. With an increase in n the delay between the instants of emission of messages by the source and reception of them by the recipient increases. It should be noted that the magnitude of this delay remains finite.

For most channels only one proof of the existence of the described method of coding is known (i.e., the possibility of selecting N signals distinguished by the decision system with as small a probability of error as is desired). Only for certain individual cases are there constructive proofs of the theorem showing how to select these signals.

1.4. Principal Problems in Discrete Information Transmission Theory

In its most general form the main problem in the theory of transmitting discrete information can be formulated as follows. A certain message source is given. Determine the best possible way to transmit this message to the recipient over a channel. However, the problem is not sufficiently defined [illegible] generally, it is necessary to stipulate [illegible] what constitutes the [illegible].

[illegible]

In such form the problem [illegible] was first [illegible] using the criterion [illegible] ideal observer for the case in which there is in the channel only additive normal white noise. Later this problem was solved by many authors using various statistical criteria and various channel properties.

Another no less important problem in the theory is the optimal selection of a set of signals and the method of converting a message into a signal. By optimal here is meant that selection in which the average risk (in the sense of statistical criterion used) takes a minimal value. In this case it is usually (but not always) assumed that the decision system is optimal.

For this wording of the problem to be meaningful, it is necessary to impose certain limiting conditions on the method of selecting the set of signals. As the reader probably knows (this will be discussed in detail in the following chapters), it is possible to bring about as small a magnitude of average risk as is desired with any statistical criterion by losing a great deal of signal strength or, under certain conditions, by using complex encoding and decoding devices, the use of which leads to a great delay between the instant of emission of a message by the source and the instant of decoding. Requirements of practice impose certain limitations on signal strength, the width of the spectrum, the degree of complexity of equipment, the magnitude of allowable delay for the communication, etc. If these limitations are formulated, the problem of optimal selection of a set of signals can in principle be solved.

A combination of the two problems indicated above can be viewed as the general problem in communication theory and amounts to the selection of an optimal communication system for a given message source, ensuring with certain limiting conditions a minimum of average risk. One possible approach to the solution of such a problem widely used in subsequent chapters amounts to the following. For given characteristics of a channel and for signals satisfying the limitations imposed but stated in very general form, an optimal decision system is determined and an average risk computed. The magnitude of this average risk depends on the set of signals used and on the method of transformation of the message into a signal. In many cases it is possible to determine such a set of signals (or class of such sets) and also such a method of conversion of a message into a signal that the indicated average risk is minimal and this amounts to a solution of the problem posed.

The problem of determining the optimal method of converting the message into a signal is greatly simplified when this conversion is broken down into two operations, i.e., into coding and modulation as was indicated in Section 1.2. This makes it possible to make an optimal selection of a set of signals based on the characteristics of the channel and the limiting conditions imposed on the signal without considering the peculiarities of the source. Then a search is made for an optimal method of coding which converts a message into a sequence of code symbols unambiguously linked to the selected signals and which allows for not only the characteristics of the channel but also the statistical properties of the source. It would be methodologically more convenient to first consider the second aspect of the problem, i.e., coding (Chapter II is devoted for the most part to this aspect).

It should be noted that, as a consequence of the multiplicity of limiting conditions dictated by practice and also as a consequence of the various characteristics of channels and various statistical criteria suitable for various concrete cases, there is no general solution to the problem pose.. For individual and more typical cases this task will be solved in subsequent chapters. However, the y far from exhaust all conditions which are encountered.

As already noted, in many (although not in all) cases of practical importance in the transmission of discrete messages, the criterion of the ideal observer serves as an adequate statistical criterion. In these, and also in several other cases, fidelity is the measure of quality of communication transmission. By fidelity we will understand the probability of complete coincidence between the message received and that transmitted. Such complete coincidence (in the case of a channel with interference) is in principle possible only in the transmission of discrete communications.

With this definition of fidelity we must indicate the length of the message for which a probability of complete coincidence between messages transmitted and received is given. A comparison of various communication systems with respect to degree of fidelity should always be made using the same amount of transmitted information.

In speaking of the fidelity of transmission of information it must be borne in mind that in principle as indicated in Section 1.8, it is possible to obtain as high fidelity as desired if the speed of transmission of the information is less than the carrying capacity of the channel. In this connection, for determining the potential capabilities of the communication system, it is important to know how to compute the carrying capacity of various communication channels. This is also a very important problem in communication theory. It can be solved relatively simply only for several mathematical models of a channel which usually only in rough outline describe the properties of actual channels.

It would probably not be amiss to stress once more that the carrying capacity of a channel determines only the potential capability of information transmission at a certain rate (less than the carrying capacity) with a certain given (perhaps very small but still differing from zero) probability of error. If the required probability increases or the required rate of transmission of information approaches the carrying capacity of the channel, then, as follows from the discussion presented in Section 1.8, it is necessary to increase the length of the sequence of messages from the source compared with the signal in coding. When this is done the complexity of the encoding and decoding devices increases very rapidly to a limit where they are technologically infeasible. In view of this it is customary in practice to be satisfied with a relatively low level of fidelity or to transmit messages at a rate much less than the carrying capacity of the channel. One problem in communication theory on which many have worked during the past few years is developing methods of coding such that lengthening the coding sequence of messages leads to a relatively slight increase in the complexity of the system.

All the problems listed will be discussed in some measure in subsequent chapters.

The properties of a channel, specifically the interference and distortion existing in it, are considered given. In engineering practice problems are also encountered in building or improving existing communication channels and reducing the relative level of interference (for example, by improving the design of the cable in electric wire communication, by improving directivity in receiving antennas in radio communication, and many other methods up to and including creation of artificial ionization in the upper layers of the atmosphere). Such problems go beyond the framework of general communication theory and will not be considered here.

Notes

1. (See Sections 1 and 2) In most works on communication theory a mutually unambiguous correspondence between message and transmitted signal is assumed. However, in actual communication systems this is far from always so. Very often a particular message may be converted into different signals. For example, two signals differing only in constant coefficient in many communication systems correspond to one message. With a shift of the signal in time by a certain magnitude, usually a signal is obtained which corresponds to that message. With narrow-band signals a change in phase of the high-frequency filling and even a change within narrow limits in its average frequency in many cases likewise does not change the message to which the signal corresponds.

Because of this the transmitted signal amounts to an element of not a discrete but a continuous set of possible signals.

Many authors completely ignore this fact and it sometimes leads to important divergences between theoretical results and practice. Other authors (for example, Feinstein) view the ambiguity in a signal as a result of the effects of "interference" occurring at the instant of signal transmission.

2. (See Section 1.4) The entropy of a source of information as determined by equations (1.7) and (1.10) is the mathematical expectation of an amount of information per element of a message. So that this magnitude have the actual meaning of the average value of the amount of information per element in a certain, sufficiently long sequence of message elements, it is essential that the source satisfy certain conditions of ergodicity [1]. Strictly speaking, actual sources of information are not ergodic but can be considered approximately so if they are considered over a stretch of not very long segments of time.

The concept of entropy of a message source is closely related to the thermodynamic entropy of a physical system which forms, together with the observer, a source of messages. The greater the thermodynamic entropy (depending, specifically, on the number of degrees of freedom) the greater the amount of information required to describe its state. Brillouin [15] made a detailed investigation of the relationship between informational and thermodynamic entropy.

3. (See Section 1.6) The Bayes criteria which are based on minimizing average risk (1.21) are suitable for designing a decision system only in those cases when the a priori probability of elementary messages $p(x_i)$ are known. In some cases in designing a communication system (specifically if it is intended for connection to several sources previously unknown) the a

priori probabilities cannot be determined. In such a case the minimax criterion is often used as a basis for designing the decision system.

The minimax criterion is a method for evaluating a decision system in which maximal values of conditional risk (1.20) are compared and the maximum is taken for messages x_i for each decision system.

$$G_{max} = \max_i \sum_{j=1}^{l} G(x_i, x'_j) p(x'_j / x_i) \qquad (1.22)$$

A decision pattern which provides for a minimal value of G_{max}, i.e., for which the maximal (for all messages) conditional risk is less, or at least not greater, than for any other decision system is considered optimal.

It can be shown (1,3) that the minimax criterion leads to the same decision system as does the Bayes criterion of minimal average risk with the condition that the a priori distribution of probabilities of messages is selected so as to be the least favorable. If in actuality the a priori distribution of probabilities is not the least favorable, then, for this distribution, it would be possible to design a decision system based on the Bayes criterion which would provide a lesser magnitude of average risk than a minimax decision system. But, on the other hand, it is always possible to find an a priori distribution of probabilities with which the average risk in the Bayes decision system (built for another a priori distribution) will be greater than in the minimax system.

We will note [9] that if for all cases of error the cost is considered the same, the minimax decision system coincides with a system designed in accordance with the criterion of maximal likelihood.

Inasmuch as a communication system is intended for the transmission of information, it seems reasonable to define an optimal decision system as the one which provides for the completest possible use of the information contained in the signal received with respect to the message transmitted. In this way it is possible to establish an information criterion of optimality. Unfortunately, there are great difficulties inherent in this. If the cost $G(x_i, x'_j)$, which figures in expression (1.21), of average risk is made inversely proportional to the amount of information contained in x'_j with respect to x_i, then it would seem that the criterion of average risk would lead to minimization of loss of information in the decision system. But a certain peculiarity arises in connection with the fact that the joint probability entering into the expression for the amount of information itself depends on selection of the decision system [9].

If in (1.21) we set the cost $G(x_i, x'_j)$ with $x'_j \neq x_i$) of the inversely proportional a priori probability $p(x_i)$ (on the basis that the informational content of the message increases with a decrease in the a priori probability), then it can easily be seen that the criterion of minimal average risk coincides with the criterion of maximal likelihood.

Several authors [3, 14] suggest that the informational criterion be that according to which the optimal decision system selects that one of hypotheses x_i in which the particular amount of information contained in an arriving signal z' with respect to message x_i, equal to $\log(w(z'|x_i)/w(z'))$ is maximized. This approach also leads to the criterion of maximal likelihood.

4. (See Section 1.7) Expression (1.43) for the amount of information contained in one continuous process z' with respect to another continuous process can be changed as follows:

$$\begin{aligned} I(x, z') &= \iint_{XZ'} w(x, z') \log \frac{w(z')\, w(x|z')}{w(x)\, w(z')}\, dx\, dz' \\ &= \iint_{XZ'} w(x, z') \log \frac{w(x|z')}{w(x)}\, dx\, dz' = -\int_X w(x) \log w(x)\, dx \;+ \\ &+ \iint_{XZ'} w(z')\, w(x|z') \log w(x|z')\, dx\, dz' = h(x) - h(x|z') \end{aligned} \tag{1.50}$$

C. Shannon calls the magnitude $h(x) = -\int_X w(x) \log w(x)\, dx$ the entropy of a continuous process and the magnitude $h(x|z') = -\iint_{XZ'} w(z') \times w(x|z') \log w(x|z')\, dx\, dz'$ the conditional entropy of a process x with a known process z'. However, such terminology is not especially fortunate since the indicated magnitudes do not possess those properties which are had by entropy and the conditional entropy of discrete sequences H(x) and H(x|z'). Therefore, following A. N. Kolmogorov [6], we will call h(x) and h(x|z') the differential entropy and the differential conditional entropy respectively.

As was shown in Section 1.7 the entropy of a discrete message H(x) can be defined as the amount of information included in x with respect to itself, H(x) = I(x,x). The differential entropy does not have this meaning. Indeed, from (1.43) or from (1.7) it can be shown by means of passage to a limit that, for a continuous process, $I(x,x) = \infty$.

This is altogether natural since for an exact description of a finite segment of a continuous process its value must be regarded at an infinite number of points. Even for an exact setting of the value of a continuous random variable describing the distribution density w(x), any finite magnitude assigned to the amount of information proves to be insufficient.

This can be explained as follows. We will break the entire domain of values of the variable x into segments of size Δx and we will produce this magnitude with a precision of $\Delta x/2$. Obviously, the amount of information needed for this can be defined as the entropy of a discrete magnitude, adopting the values of x_i with a probability of $w(x_i)\Delta x$:

$$H_\Delta(x) = -\sum_i w(x_i)\Delta x \log[w(x_i)\Delta x] = -\sum_i w(x_i)\Delta x \log w(x_i) -$$
$$-\sum_i w(x_i)\Delta x \log \Delta x = -\sum_i w(x_i)\Delta x \log w(x_i) + \log \frac{1}{\Delta x}. \tag{1.51}$$

If now Δx is allowed to approach zero, i.e., the precision with which the continuous random variable is set increases without limit, the first term in (1.51) approaches the differential entropy $h(x) = -\int_x w(x) \log w(x)\, dx$, and the second term increases without limit.

This result can only be explained in the following way. If there is a channel with complete absence of noise so that the signal received, z'(t), is identically equal to the transmitted continuous signal z(t) (or any value of z' is a regular inverse function of z), then I(z,z') = ∞ and it should be possible to transmit over such a channel an error-free message from any source, no matter what the productivity. If the dependence between x and z' is not regular but inverse, then I(z,z') is a finite positive magnitude. We will point out again that the term "continuous signal" is to be understood not in the sense of continuousness of the function z(t) but in the sense that the signals z(t) are elements of a continuous set.

The differential entropy h(x), in contrast to the entropy of a discrete source H(x), may assume negative values. Furthermore, the differential entropy may at will change its value and even its sign with a change in the unit of measurement of magnitude x since when this occurs the value of w(x) changes. As far as the difference in differential entropy (1.50) which is equal to the amount of information contained in one continuous process with respect to another, it does not depend on the unit of measurement but on the logarithm base (i.e., on the selected unit for amount of information). In actuality with continuous processes only the amount of information has a physical meaning, i.e., the difference of differential entropies and not the differential entropy itself.

We will point out that by simple transformation it is possible to express I(x,z') by analogy with (1.40) as follows:

$$I(x, z') = h(z') - h(z'|x) = h(x) + h(z') - h(x, z'), \qquad (1.52)$$

where

$$h(x, z') = \iint_{xz'} w(x, z') \lg w(x, z')\, dx\, dz'.$$

BIBLIOGRAPHY

1. Shannon, C., [illegible] [Mathematical Theory of Communication]. In the collection: Works on the Theory of Information and Cybernetics, Foreign Literature Press, 1963.

2. Kotel'nikov, V. A., [illegible] [Theory of Potential Resistance to Interference], Gosenergoizdat, 1956.

3. Fano, R., [illegible] [Information Transmission. A Statistical Theory of Communication], "Mir" Press, 1965.

4. Khinchin, A. Ya., [illegible] [Concept of Entropy in Probability Theory], Uspekhi matematicheskikh nauk, No. 3, 1953.

5. Khinchin, A. Ya., [illegible] [Basic Theorems of Information Theory], Uspekhi matematicheskikh nauk, No. 1, 1956.

6. Kolmogorov, A. N., [illegible] [Theory of Information Transmission], Press of the USSR Academy of Sciences, 1956.

7. Feynstein, A., [illegible] [Fundamentals of Information Theory], Foreign Literature Press, 1960.

8. Dobrushin, R. L., [illegible] [Mathematical Problems in the Shannon Theory of Optimal Coding of Information], In the collection: Problems in Information Transmission, Issue 10, 1961.

9. Middleton, D., [illegible] [Introduction to Statistical Communication Theory], "Sovetskoye radio" Press, Vol. I, 1961; Vol. II, 1962.

10. Romanovskiy, V. I., [illegible] [Discrete Markov Chains], Gostekhizdat, 1949.

11. Gnedenko, B. V., [illegible] [Course in Probability Theory], Gostekhizdat, 1954.

12. Vudvord, F., [illegible] [Designing Radar Receivers Based on Statistical Methods], In the collection: Theory of Transmission of Electrical Signals in the Presence of Interference, Foreign Literature Press, 1953.

13. Val'd, A., [illegible] [Sequential Analysis], Fizmatgiz, 1960.

14. [illegible], N. I., [illegible] [Information Fundamentals in Transmitting Messages], Sovetskoye Radio Press, 1966.

15. Brillouin, L., [illegible] [Science and Information Theory], Fizmatgiz, 1960.

16. Levin, B. R., [illegible] [Theoretical Fundamentals in Statistical Radio Technology], Sovetskoye Radio Press, Vol. I, 1966; Vol. II, 1968.

17. Kharkevich, A. A., [illegible] [Essays on General Communication Theory], Gostekhizdat, 1955.

18. Vol'fovits, Dzh., [illegible] [Coding Theorems in Information Theory], Mir Press, 1967.

19. Vozenkraft, D. Zh. and I. Dzhekobs, [illegible] [Theoretical Fundamentals in Communication Engineering], Mir Press, 1969.

CHAPTER II

THE DISCRETE CHANNEL AND FUNDAMENTALS OF CODING THEORY

2.1 Discrete Channels and Their Classification

In many problems in communication theory the structure of the modulator and demodulator is given. In these cases the channel is that part of a communication line which is encircled in Figure 1.3 by the broken line. Discrete code symbols v are delivered to the input of such a channel and from the output are taken symbols v' which, generally speaking, do not coincide with v (Figure 2.1).

Such a channel is called discrete. In studying transmission of messages over a discrete channel the main problem is to find methods of encoding and decoding which permit, in one sense or another, transmitting a message of a discrete source.

We will note that in almost all actual communication lines a discrete channel contains within itself a continuous channel at the input of which signals $z(t)$ are delivered and from the output of which signals $z'(t)$, distorted by interference, are taken. The properties of this continuous channel, along with the characteristics of the modulator and demodulator, unambiguously determine all parameters of the discrete channel. Therefore, sometimes a discrete channel is called a discrete representation of a continuous channel. However, in mathematical investigation of a discrete channel the continuous channel and the interference acting on it are neglected and attention is paid to a discrete channel, giving an alphabet of code symbols v_j ($j = 1, \ldots, m$) arriving at its input, an alphabet of code symbols v'_i ($i = 1, \ldots, m'$) taken from its output, the number v of code symbols transmitted per unit time, and the values of the probabilities of conversions $p(v'_i|v_j)$, the probability that symbol v'_i appears at the output when symbol v_j is delivered to the input. These probabilities depend on what symbols were transmitted and received previously. The alphabet of symbols at the input and output of a channel need not be the same, specifically, it is entirely possible that $m' \neq m$. The variable v is sometimes called the technical speed of transmission.

If the conversion probabilities $p(v'_i|v_j)$ for each pair i,j remain constant and do not depend on what symbols were transmitted and received previously, the discrete channel is called constant or uniform. Sometimes other names are used: a channel without memory or a channel with independent errors. If, however, the conversion probabilities depend on time or on conversions which have taken place previously, the channel is called nonuniform or a channel with memory.

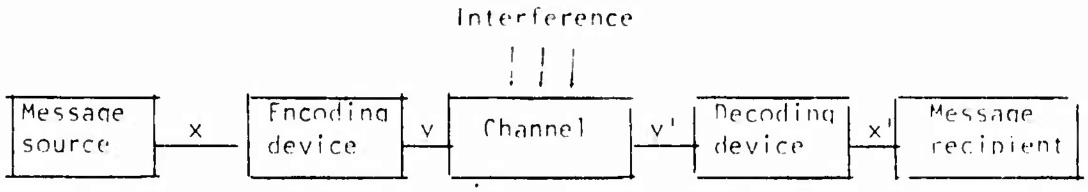


Figure 2.1. Communication System with a Discrete Channel.

In a channel with memory the probability ties, at least in the first approximation, are distributed only over a certain finite segment. This means that the conversion probabilities $p(y'_j|y_i)$ depend on what conversions took place in the transmission of the preceding k symbols and do not depend on earlier conversions. Such a channel is said to have a series of discrete states $S_i, \ldots S_r$ which are determined by preceding conversions, the relationships $r \leq (mm')^k$ holding. For each state S_q conditional conversion probabilities $p_q(y'_j|y_i)$ are defined. At the same time, only the last k symbols transmitted and received determine the channel states S_q.

The average unconditional conversion probabilities are determined by averaging the conditional probabilities over all the channel states

$$p(u'_j|u_i) = \sum_{q=1}^{r} P_q p_q(u'_j|u_i). \qquad (2.1)$$

where P_q is the probability of state S_q.

In actual channels in element-by-element reception the conversion probabilities $p(y'_j|y_i)$ are not given, but are determined, on the one hand, by interference and signal distortion in the channel and, on the other, by the transmission rate v of the code symbols and by the first decision system. By selecting, on the basis of one criterion or another, the optimal decision system, it is possible to change the conversion probability in the desired direction. In order to consider a channel discrete, it is necessary to select the first decision system and, taking into account the effective interference and distortion in the channel, to compute the conversion probabilities. Obviously, in those cases when the parameters of an actual channel are constant and the active interference in the channel is a stationary random process, its discrete representation is a constant channel. If these conditions are not met the discrete representation, as a rule, is a channel with memory.

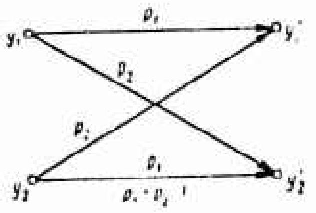


Figure 2.2 Probabilities of Conversion in a Symmetrical Binary Channel.

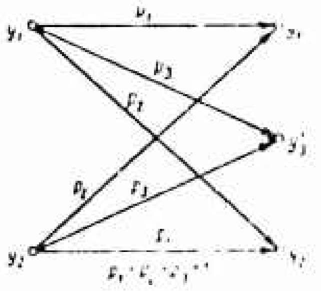


Figure 2.3. Probabilities of Conversion in a Symmetrical Erasure Channel.

If the alphabets at the input and output of a uniform channel are identical and for any pair i ≠ j the probabilities $p(y'_j/y_i) =$ [illegible] const, then this channel is called symmetrical. We will also call a variable channel symmetrical if in every state S_q for any pair i ≠ j is fulfilled the condition

$$p_q(y'_j/y_i) = p_q = \text{const}(i,\ j). \tag{2.2}$$

From (2.2) it obviously also follows that

$$p(y'_j/y_i) = \text{const}, \tag{2.3}$$

but it would not be true to assert the converse. Channels with memory in which (2.3) is fulfilled, but (2.2) is not, or not for all q, will be called average symmetrical channels. Figure 2.2 schematically shows the conversion probabilities in a symmetrical channel.

Among the channels in which the alphabets at input and output are not identical, the so-called erasure channel, in which m' = m + 1, is of interest. It gets its name from the fact that its output alphabet contains the additional symbol y'_{m+1} signifying "erasure" besides the symbols $y_1, \ldots, y_m$ common to both the input and the output alphabets. The appearance of y'_{m+1} means that the transmitted symbol has been distorted by interference and cannot be recognized. Therefore, a portion of the received code sequence has been erased.

As will be shown subsequently, introduction of this erasure symbol does not disturb the feasibility of correctly decoding the received code sequence, but, on the contrary, facilitates it when the coding method and decision systems have been rationally selected.

Let us observe that the code alphabet at the output is determined by the choice of the first decision system and is therefore considered to be given only because we are examining the discrete representation of the channel. The choice of the first decision system also to a considerable degree determines the characteristics of channel symmetry. Figure 2.3 shows the conversion probabilities in a symmetrical erasure channel.

2.2. Carrying Capacity of Discrete Noiseless Channel

If the alphabet of code symbols x at the input of a discrete channel and that of x' at the output are identical and

$$P(x'_j/x_i) = \begin{cases} 1 \text{ when } i = j \\ 0 \text{ when } i \neq j \end{cases}$$

i.e., the input and output symbols always are in agreement, then this channel is called a discrete noiseless channel. It is completely characterized by code base m and average symbol quantity v transmitted in the time unit.

In such a channel the amount of information I(x,x') contained in received symbols x' with respect to the transmitted symbols x is always equal to the entropy of the source H(x). This follows from the fact that

$$I(x, x') = H(x) - H(x/x') = H(x),$$

since H(x/x') = 0, inasmuch as with a known x', element x is determined unambiguously.

The carrying capacity C of a noiseless discrete channel which is equal, according to definition, to the maximum of I(x,x'), is achieved in that case when symbols with a maximal allowable rate of v arrive at the input of the channel from a source having a maximal entropy H(x). For this purpose it is necessary and sufficient that the source emit symbols of an alphabet with a size of m with equal probabilities and independently of one another. In this case, in accordance with (1.5) $H_{max}(x) = \log m$ and, consequently,

$$C = vH_{max}(x) = v \log m \qquad ([illegible])$$

The coding theorem for a noiseless discrete channel in the case of a source with a controllable rate can be formulated as follows [1].

If the message source has an entropy of H(x) (binary units per letter), it is possible to encode the source messages in such a way that they can be transmitted as accurately as desired over a noiseless discrete channel at an average rate of

$$\frac{v \log m}{H(x)} - \epsilon \text{ letters/sec}, \qquad ([illegible])$$

where ε is a positive magnitude which is as small as desired.

It is impossible to transmit them as accurately as desired at a rate greater than v log m/H(x).

The following assertation is an obvious consequence of this theorem which is applicable to a source with a fixed rate.

Messages from a source with a productivity of H(X) binary units per second can be encoded so that they can be transmitted as accurately as described over a noiseless discrete channel when the following condition is met:

$$H(X) < C\log m$$

This is impossible for

$$H(X) > C\log m$$

These theorems are particular cases of Shannon's coding theorem which was discussed in Chapter I. We will consider several very simple cases of their application.

1) Let the source select message elements independently of the preceding ones and with identical probabilities equalling $1/M$; then $H = H_{max} = \log_2 M$. If, moreover, $M = m$, then coding reduces simply to establishing by any method a mutually unique correspondence between each of the message elements x_i and a code symbol y_i.

It is apparent that in so doing v message elements can be transmitted over the channel each second. But $v = C/\log_2 m = C/\log_2 M = C/H$. Therefore, under these conditions the theorem is true even when $\varepsilon = 0$. In this case it is sometimes thought, but without any special reason, that the message is transmitted without coding.

2) Let $H = H_{max} = \log_2 M$ as before, but let $M = m^a$, where a is a whole number. Let us form all possible code symbol sequences ("code combinations") of length a. It is apparent that there are m^a of them. Let us establish a mutually unique correspondence between each message element x_i and a code combination $(y^{(1)},\ldots,y^{(a)})$. Therefore, every combination of a symbols transmitted over the channel corresponds to a single message element and hence the transmittal rate of the message elements is

$$w = \frac{v}{a} = \frac{C}{a\log_2 m} = \frac{C}{\log_2 M} = \frac{C}{H}$$

Thus in this case, too, the theorem is true even when $\varepsilon = 0$. A code in which all code combinations are of the same length a is called a uniform a-digit code.[1]

---

[1] The term "a-digit code" owes its origin to the fact that in the case examined may be imagined as numbering all the elements of the source alphabet with a-digit numbers divisible by m.

3) For the same source with $H = H_{max}$ [illegible] log [illegible], let [illegible] phabet [illegible] not be a whole power of the number m. Thus, for example, [illegible] source independently and with equiprobability [illegible] 0,1,...9 while the channel has two symbols designated [illegible] mits transmission of v symbols per second.

Here [illegible] = 10, m = 2, $H = H_{max}$ = log [illegible] 3,332 bits and [illegible]

According to the theorem the transmission rate [illegible] may be made as close as desired to [illegible] 3,332.

Let us try to achieve this by encoding each [illegible] digit code. This reduces essentially to representing each of the ten [illegible] 0,1,...,9 by a binary number 0000, 0001,...1001. It is obvious that [illegible] digit binary numbers are needed for this. Therefore, every message element (decimal digit) in this coding requires four code symbols, whereas the theorem asserts that coding can be done more "economically," approximating 3,332 code symbols per digit.

We will show that this is possible if the source alphabet [illegible] before encoding. We will consider every pair of digits emitted by [illegible] to be a two-digit decimal number, i.e., we will convert from [illegible]

ℓ_2 = 100 and will, as before, reduce coding to representation of these [illegible] in the binary system. Every number less than 100 can be represented [illegible] seven-digit binary number (because 2^7 = 128 [illegible] 100 and [illegible] six-digit binary numbers do not suffice to represent [illegible] numbers). In this coding every two message [illegible] symbols, i.e., an average of 3.5 code symbols per digit, [illegible] alphabet enlargement.

Let us continue enlarging the alphabet by regarding [illegible] emitted by the source as a three-digit decimal number. [illegible] in the binary system by a 10-digit number [illegible] sequently in this coding one digit requires 10/3 ≈ 3.33 [illegible] is already very close to the theoretical 3.332.

Further enlargement of the alphabet will make it possible [illegible] digit transmission rate over this binary channel even closer [illegible] v/log$_2$10, but, of course, not to go beyond it. [illegible] emitted by the source are combined into an x-digit decimal [illegible] be expressed by a y-digit binary number on condition that 10^x [illegible], or [illegible] x log$_2$10 [illegible], whence

$$[illegible]$$

---

[1] A. N. Kolmogorov first point out this [illegible]

In the given case the equal sign is even impossible since with whole-number values of x and y the equality x log₂10 = y means that log₂10 is a rational number, while in reality it is irrational.

In general, when $H = H_{max}$, but l is not a whole or rational power of m, there may be found for any number b a whole number a which makes these inequalities true:

$$b\frac{\log l}{\log m} < a < b\frac{\log l}{\log m} + 1. \qquad (2.10)$$

We will regard every b message letters as a letter of an enlarged alphabet containing l^b elements and will establish a mutually unique correspondence between the enlarged letters and the code combinations of a uniform a-digit code, as can always be done on the basis of (2.10) (since $l^b < m^a$), and some of the combinations will also remain unused. Then every combination of a symbols transmitted over the channel corresponds to b letters of the primary message alphabet. Here the transmission rate is

$$w = \frac{b}{a} = \frac{Cb}{a\log_2 m}. \qquad (2.11)$$

From (2.10) we have $b \log l < a \log m < (b + 1) \log l$, where $\gamma = \log m/\log l$.

This expression may be written in the form

$$a\log_2 m = (b + \delta)\log_2 l, \qquad (2.12)$$

where $0 < \delta < \gamma$.

From this

$$w = \frac{C}{\log_2 l}\,\frac{b}{b+\delta} = \frac{C}{H}\,\frac{b}{b+\delta} > \frac{C}{H}\,\frac{b}{b+\gamma} = \frac{C}{H}\left(1 - \frac{\gamma}{b+\gamma}\right). \qquad (2.13)$$

If number b is chosen from condition $b > \left(\frac{C}{H\varepsilon} - 1\right)\gamma$, then

$$w > \frac{C}{H} - \varepsilon, \qquad (2.14)$$

and this is what we wanted to prove.

The result derived shows that in a channel with carrying capacity C we can, by using a uniform code, transmit the messages of any source which has an alphabet of size l, and can do so at a rate as close as desired to $C/H_{max} = C/\log_2 l$ letters per second. We will call this the primary method of coding. This confirmation is, of course, less rigorous than requirement of the theorem according to which any source with entropy $H < H_{max}$ can have a transmission rate as close as desired to C/H.

2.3. Methods of Eliminating Message Redundancy

If the message source has no redundancy its productivity is $H' = H'_{\max}$ and, as was shown in Section 2.2, a simple procedure can encode the message for transmission over a noiseless channel with carrying capacity $C \geq H'_{\max}$. In the case of a source with redundancy $H' < H'_{\max}$. Therefore the problem often arises of transmitting its message over a channel with carrying capacity $C < H'_{\max}$ which is in principle possible provided that $C \geq H'$. The appropriate coding must transform the message element sequence with redundancy into a code symbol sequence without redundancy or with considerably less of it. Therefore this coding operation may be called redundancy elimination.

Let us first study the case where a source selects elementary messages which do not have the same probabilities, but selects them independently of each other. In this case the entire redundancy of the source is the result of the unequal probabilities of the elements. This redundancy may be eliminated completely or partially if during encoding the most probable elements are represented by short code symbol sequences and the less probable by long sequences. From this it is apparent that this efficient code must be nonuniform.

If message element x_k is represented by a code symbol sequence consisting of n_k symbols the average number of symbols n in the code sequence per element is

$$\bar{n} = \sum_{k=1}^{l} p(x_k) n_k \tag{2.15}$$

Maximum entropy $H_{\max}(y)$ of the code sequence per symbol equals log m, where m is the number of different code symbols. Consequently, the average code sequence entropy per combination (corresponding to a message element) is

$$\bar{n}H(y) \leq \bar{n}H_{\max}(y) = \sum_{k=1}^{l} p(x_k) n_k \log m \tag{2.16}$$

The chief requirement made of any code is that it be possible to unambiguously decode the code sequence. This leads to the stipulation that $\bar{n}H(y) = H(x)$, whence

$$\bar{n} = \frac{H(x)}{H(y)} \geq \frac{H(x)}{H_{\max}(y)} = \frac{1}{\log m} \sum_{k=1}^{l} p(x_k) \log \frac{1}{p(x_k)}. \tag{2.17}$$

The expression derived gives an estimate from below of the average code combination length. The task of economical coding is to select a code enabling us to move n as close as possible to that estimate. If in expression

(2.17) equality is achieved[1], this means that $H(y) = H_{\max}(y)$ and that the code sequences obtained will have no redundancy. Otherwise, residual redundancy will remain and will be

$$r_y = 1 - \frac{H(y)}{H_{\max}(y)} = 1 - \frac{H(x)}{n \log m}. \tag{2.18}$$

This redundancy, however, may always be made substantially less than message redundancy $r_x = 1-[H(x)]/\log'$.

Actually, by using primary coding (Section 2.2) with a uniform a-digit code we may derive n as close as desired to log '/log m, which, by substitution in (2.18), will give $r_x = r_y$. But by using a nonuniform code we may always shorten the average code combination length $\bar{n}$ by employing the shorter combinations for the more probable signs (provided that the message letters are not equiprobable); and we may thus derive $r_y < r_x$, i.e., eliminate at least some of the redundancy. Let us note that residual redundancy can be made as small as desired by enlarging the alphabet.

When designing an optimum nonuniform code which permits maximum limitation of redundancy we must take into consideration the requirement for unambiguous decoding. It is easy to see that this requirement will be met if not one combination of a given code coincides with the beginning of another longer combination. This code property is called "irreducibility." In decoding a code symbol sequence the property of irreducibility permits the unambiguous division of this sequence into code combinations and the comparison of the corresponding message element with each code combination, i.e., decoding[2].

For example, a binary code with base m = 2 containing code combinations[3] 00, 01, 100, 101, 110, and 111 is irreducible, whereas a code containing the combinations 00, 01, 10, 11, 000, 001, 010 is not because combination 01 coincides with the beginning of 010 and combination 00, with the beginning of combinations 000 and 001. The first of these codes permits unambiguous decoding. If, for example, the code sequence

---

[1] It is easy to prove that this equality is possible and always attainable if each of the probabilities $p(x_k)$ equals m^{-n_k}, where n_k is a whole number.

[2] The property of irreducibility is sufficient, but not necessary, for unambiguous decoding. It may, however, be shown that the stipulation of irreducibility does not limit the degree of redundancy elimination attainable (Remark 2 to Chapter II).

[3] It is convenient to designate code symbols by numerals. When the code base is m we will designate the symbols by the numerals 0,1,...,(m-1).

000110101011010100100110010001101 . . ,

is received, it may be divided into code combinations only in the following way:

00 01 101 01 01 101 01 00 100 110 01 00 01 101 . .

If the second code without the property of irreducibility were employed there are different ways in which the same sequence could be divided into code sequences, for example,

00 01 10 10 10 11 01 01 00 10 01 10 01 00 01 10 . .

or

000 11 010 10 11 010 10 010 01 10 010 001 10 . .,

or

000 11 01 010 11 01 01 001 000 10 01 000 11 01. . .

etc.

Various methods (algorithms) are known [1,3,4] for constructing irreducible nonuniform codes of base m which make possible the greatest degree of elimination of message redundancy for a given source. We will describe the most general-purpose method which was proposed by Huffman [4,38].

All the letters of the message alphabet ([illegible] in number) are written down in order of diminishing probability. If the number [illegible]-1 is not divisible by m-1, additional "letters" are added to the alphabet and are ascribed a probability of zero so that for the size [illegible]' of the alphabet thus obtained the stipulation of divisibility of [illegible]'-1 by m-1 is fulfilled. Then the following m elements of the derived alphabet are consolidated into an "enlarged" element and the probability is computed and noted in the appropriate place in the alphabet. The same procedure is followed with the last m elements of the derived alphabet (including the enlarged ones), and this is continued until there remains an "alphabet" consisting of m elements. The single-digit code combinations 0,1,...,(m-1) are ascribed to these m elements in any order. If these remaining m elements include any which belonged to the original alphabet (i.e., not derived by consolidation of other elements) they prove to be encoded one-digit code combinations. For those elements derived by consolidating m letters of the original alphabet into code combinations the second symbols are written out so that these signs prove to be encoded two-digit combinations. If among these two-digit combinations there are any which correspond to elements also derived by consolidation, then third symbols are ascribed to these combinations, and so on until all the elements of the original alphabet have been encoded.

As an example we will construct a nonuniform code of base m = 4 by using code symbols 0,1,2, and 3 for the source with an alphabet consisting of 16 elements (which we will designate with Russian letters) with the following a priori probabilities:

x_i	$p(x_i)$	x_i	$p(x_i)$	x_i	$p(x_i)$	x_i	$p(x_i)$
A	0,3	D	0,04	I	0,015	N	0,01
B	0,2	E	0,03	K	0,01	O	0,01
V	0,2	Zh	0,03	L	0,01	P	0,01
G	0,1	Z	0,02	M	0,01	R	0,005

The entropy of this source is

$$H(x) = -\sum_{i=1}^{16} p(x_i)\log_2 p(x_i) = 2{,}907$$

bits per sign and its redundancy is

$$r_s = 1 - \frac{H(x)}{\log_2 l} = 1 - \frac{2{,}907}{4} = 0{,}275.$$

In the given case l-1 = 15 is divisible by m-1 = 3, so there is no need to introduce supplementary letters into the alphabet.

Let us consolidate the last four letters. The derived consolidated element has a probability of 0,035 and must be inserted in the alphabet between D and E. Let us go through the same operation with the last four letters and write the derived consolidated element with a total probability of 0,045 in the appropriate place in the alphabet between G and D. We continue in the same way until there remain four letters (A, B, V, and a consolidated element), to which are ascribed the code symbols 0,1,2 and 3. We then compile code combinations for the letters which have entered into the consolidations (by groups of m letters). The whole process of constructing the code is clear from Table 2.1.

Finally we will have derived the following code table:

A	0	D	13	I	120	N	1000
B	2	E	101	K	121	S	1001
V	3	Zh	102	L	122	P	1002
G	11	Z	103	M	123	R	1003

It is easy to see that the derived code is irreducible and that the most probable signs have the shortest code combinations. The average number of symbols per letter is

$$n = \sum_{i=1}^{16} p(x_i)n_i = 1{,}49.$$

The redundancy of the derived code is, according to (2.18)

$$r_k = 1 - \frac{H(x)}{n\log_2 m} = 1 - \frac{2{,}907}{1{,}49\cdot 2} = 0{,}027,$$

i.e., almost 10 times less than the redundancy of the message source.

TABLE 2.1.

x_i	$p(x_i)$	First Consolidation	Second Consolidation	Third Consolidation	Fourth Consolidation	Construction of Code Combinations
A	0,3	0,3	0,3	0,3	0,3	0
					0,3	1
B	0,2	0,2	0,2	0,2	0,2	2
V	0,2	0,2	0,2	0,2	0,2	3
				0,115		10
G	0,1	0,1	0,1	0,1		11
			0,045	0,045		12
D	0,04	0,04	0,04	0,04		13
		0,035	0,035			100
E	0,03	0,03	0,03			101
Zh	0,03	0,03	0,03			102
Z	0,02	0,02	0,02			103
I	0,015	0,015				120
K	0,01	0,01				121
L	0,01	0,01				122
M	0,01	0,01				123
N	0,01					1000
O	0,01					1001
P	0,01					1002
R	0,005					1003

It may be shown [4] that the code derived by this algorithm is optimum in the sense that with given source and a given code base m it is impossible to construct a code with less redundancy. In the case where $(N-1)/(m-1)$ is a whole number this code is full; this means that any sequence of symbols may be obtained as a result of encoding some message. It is obvious that this condition is always fulfilled for binary codes ($m = 2$).

Let us observe that since the redundancy of such a nonuniform code is very small all symbols in any typical code sequence must be encountered with almost identical frequency and the probabilistic connections between the symbols must be very weak (i.e., the conditional probability of some symbol appearing when the preceding symbols are known differs very little from the total probability that this symbol will appear).

Let us go on to the more general case where the message source is Markovian and its redundancy is determined not only by the nonuniform probability distribution of the letters but also by the dependence of these probabilities on what letters preceded a given letter. We will scrutinize two possible methods of eliminating redundancy in coding the messages of such a source.

The first method is as follows. For every state s_i of the source there is a "particular" code constructed as described above with respect to the conditional probabilities of the letter which occur in this state. In other words, the code is constructed as an aggregate of the individual codes for every state S_i $(i = 1,\ldots,r)$ of the source. Since state S_i is uniquely determined by the preceding message elements it is known both in the transmitting and the receiving devices. We remind the reader that here we are discussing a noiseless channel in which every transmitted element is received without error. Therefore it is always known which of the particular codes is being used for the following element, and this ensures the uniqueness of encoding and decoding.

This coding method completely eliminates redundancy caused by probabilistic connections between the message elements or, as is said, effects complete decorrelation. At the same time, because the optimum nonuniform code is employed in every state, redundancy caused by nonuniform distribution of probabilities of the elements is entirely, or almost entirely, eliminated.

The use of this method in practice is generally made difficult by the need to use automatic change-over devices for encoding and decoding or to use a group of such devices for all states of the source. Sources are, however, frequently encountered with such probabilistic ties between the elements such that the method of decorrelation described above reduces to extrapolation of a sequence of elements, i.e., the prediction of an expected element from knowledge of the preceding elements. Here the "difference" between the predicted element and the one actually chosen is coded. When the concept "difference" is properly defined for a given source these differences have far less powerful probabilistic ties than do the original elements of the message.

We shall show this by a simple example. Let a communications system serve to transmit messages about the strength of a current put out by a remote-controlled electric power station. In order for these messages to be discrete they must correspond to the strength of the current at certain moments of time (e.g., every 0.1 sec), and the measurements must also be made with a certain accuracy (say, accurate to one ampere). The alphabet of the source consists of whole numbers from zero to the value of the maximum permissible current, which the station's protective apparatus prevents from being exceeded. These numbers are not, of course, equiprobable since the average values have the greatest probability. In addition, they are independent, since the probability that an abrupt current change will occur in a tenth of a second is very small. It is most probable that the current strength will remain (with accuracy to one ampere) the same that it was at the previous reading; current changes of ±1 ampere have considerable probability, changes of ±2 amperes have less probability, etc., and changes by tens or hundreds of amperes have very little probability at all.

Here we have in essence a Markovian source whose state is determined in a first approximation only by the last message (current strength at a given moment) and for every state is prescribed its own particular distribution of the probabilities of the other message elements (the other current values). Moreover, all particular probability distributions differ from each other only by the position of the most probable value (equalling the value at the last moment of reading) and can be found by shifting the graph of probabilities along the current axis, as Figure 2.4 shows. This figure gives the distributions of conditional probabilities $p(I_n/I_{n-1})$ of the measured current strength I_n if the result of the preceding measurement was I_{n-1}. Consequently, the differences $I_n - I_{n-1}$ do not depend (more exactly, loosely depend) on the preceding values of current strength[1]. We will regard the value $I^*_n = I_{n-1}$ as an extrapolated (predicted) value of current strength I_n. Then only the difference between true read-out value I_n and extrapolated value

$$I_n - I^*_n = I_n - I_{n-1}$$

need be transmitted over the channel.

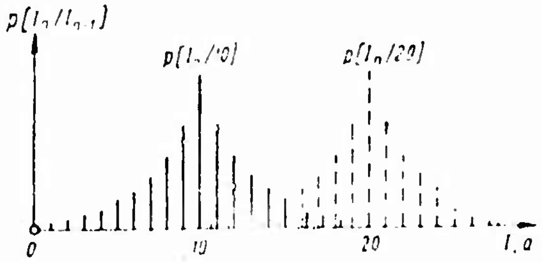


Figure 2.4. Example Distribution of Conditional Probabilities of Messages from a Markovian Source.

We shall use an optimum nonuniform code for these differences. Then, of course, the shortest code combinations will mean that the current strength has not changed, or has changed by ±1 ampere, while longer code combinations will correspond to more substantial changes in current strength. It is easy to see that this method of decorrelation by transmitting the differences between current strengths is one of the variants of the method of particular codes for every state of the source. In reality, every code combination does in fact carry a message about current strength at the given moment, but in order to decode the combination the state of the source must be known, i.e., (in the given example) the current strength at the preceding reading.

---

[1] In reality, of course, these distributions are not entirely identical and are particularly distorted when I_{n-1} is close to zero or to current maximum, but the probability of these extreme states is small and the distributions may be considered identical in shape without any particular error.

The particular codes in this example differ from each other only by the "initial reading point." Let us note that this method of eliminating redundancy in telemetric systems is of practical value because it enables us to reduce the requisite carrying capacity of a channel and to utilize with more efficiency an existing multichannel line of communication to transmit the results of measuring a number of physical magnitudes.

Another method of decorrelation is to utilize the alphabet-enlarging operation with which we are already familiar. We thus form a new source alphabet and if the number of letters of the initial alphabet which have gone into one "letter" of the enlarged alphabet substantially exceeds the range of action of the probabilistic connections, the connection between the elements of the enlarged alphabet may be disregarded.

As was shown in Chapter I (see (1.14)) total redundancy does not increase when an alphabet is enlarged; hence the decrease in redundancy caused by reciprocal connections must be accompanied by a corresponding increase in redundancy as the result of unequal probabilities of the appearance of various elements. Actually, the enlarged alphabet of a message source is always characterized by a more uneven distribution of element probabilities than is the original alphabet.

By use of an optimum nonuniform code to encode the enlarged alphabet we may practically completely eliminate the redundancy contained in the message. Therefore, the process of eliminating the redundancy of a Markovian message source reduces to two operations--decorrelation (by using the particular-code or the alphabet-enlarging method) and encoding with the optimum nonuniform code [5].

Many countries use special codes for official telegraphic correspondence between the different ministries and departments. These codes use short combinations for transmitting often recurring sentences and expressions [6] (typical sequences for a given message source), while rarely encountered sentences are transmitted in the ordinary way. This is a typical example of the elimination of redundancy by enlarging the alphabet (to whole sentences and parts thereof) and by efficient encoding. A considerable saving in telegraph expenses is also realized.

The described methods of eliminating redundancy permit effective use of the carrying capacity of a noiseless channel. They are also useful in cases where a large volume of information must be stored in various memory devices. Let us note that economy in channel carrying capacity or in memory device capacity results in the last analysis in actual monetary savings, in reduction in size and weight of instrumentation, etc.

The elimination of redundancy, however, also has a significant negative aspect. Code symbol sequences of an optimum nonuniform code, when they are stripped of redundancy, prove to be very "brittle" under the effect of noise in actual channels or in actual memory devices. This brittleness consists in distortion of one of the symbols in a sequence being enough to make impossible correct decoding of the letter containing the symbol as well as a number of following letters[1].

Let us clarify this with an example. Let us assume that the following telegram is transmitted over a communication channel: "WHEAT HARVEST FINISHED 8000 CWT GATHERED." If this telegram was encoded in a primary uniform code without the redundancy removed, then erroneous reception of one or even several isolated code symbols will lead only to erroneous decoding of one or several letters. Let the telegram be received in the following form: "WHEAT HARNEST FENESHED 8000 DWT GATHERED." It is obvious that the recipient of this telegram easily reads it and restores the sense according to context, the four mistakes notwithstanding. This is because of the redundancy of the English language. Since the entropy per letter is considerably less than the maximum entropy the overwhelming number of random sequences of letters form atypical (meaningless) concatenations. Those include what was received instead of the transmitted telegram and for this very reason the typical (meaningful) sequence which in all probability was transmitted over the communication channel may be determined. Let us make the a propos observation that if the number 6000 had been erroneously received instead of 8000 it would have been impossible to correct this error from the context. The reason for this is that when numbers are expressed in numerals redundancy is abruptly reduced since every numeral sequence has meaning. The number could have been written in letters ("eight thousand") to increase fidelity.

Let us suppose that the same telegram was encoded with the special departmental code which was mentioned above so that redundancy was reduced to a minimum. Encoded with a literal alphabet this telegram may, for example, have the form KTSHA 8000, which can be transmitted over a telegraphic communication channel considerably faster and more cheaply than with a primary code. Let us assume that one letter was distorted in transmission and the code sequence received was KTMHA 8000. If a code eliminates redundancy efficiently enough it has the property of completeness, i.e., any symbol sequence is typical or, in other words, has a meaningful context. In the given case the telegram received may, for example, mean: "Because of natural calamities the crop on 8000 hectares has been lost." It is no longer possible to correct the error from the context.

Therefore, every elimination of redundancy is associated with the risk of losing fidelity when messages are transmitted in a noisy channel. The problem of using code sequence redundancy to increase the reliability of a received message will be examined below.

---

[1] It should not, however, be thought that after one erroneously received symbol the whole remainder of the sequence will be wrongly decoded. As Gilbert and Moore [39] have shown, optimum nonuniform codes generally possess the property of autophasing, as a result of which the possibility of correct decoding is restored after several erroneously decoded signs.

2.4. Discrete Noisy Channels. Interference-Resistant Codes

In a discrete noisy channel a received symbol y' is not unambiguously determined by transmitted symbol y. There are certain conversion probabilities $p(y_j', y_i)$ which, generally speaking, depend on previously transmitted and received symbols.

We will consider various sequences Y^n of symbols arriving at a channel input. Each such sequence Y_i^n can become various sequences $Y_j'^n$ at the output of the channel. The amount of information contained in such a received sequence with respect to that transmitted, according to (1.30), is equal to

$$I(Y_i^n, Y_j'^n) = \log \frac{P(Y_i^n/Y_j'^n)}{P(Y_i^n)}, \qquad (2.19)$$

and the average amount of information $I(Y^n, Y'^n)$ per sequence consisting of n symbols transmitted over a noisy channel is defined as a mathematical expectation (2.19) for all possible transmitted and received sequences and for all states of a channel, if it exists. This amount of information depends on the properties of the channel as well as on the distribution of symbol probabilities at the input of the channel.

Let v symbols per unit time arrive at the input of a discrete channel. If with a certain distribution of symbol probabilities at the input there exists

$$I(y, y') = \lim_{n\to\infty} \frac{v}{n} I(Y^n, Y'^n), \qquad (2.20)$$

it represents the rate of transmission of information over the channel. The upper limit of $I(y, y')$ is the carrying capacity of the channel for all possible probability distributions at the input.

Specifically, for a constant channel

$$I(y, y') = v \sum_i \sum_j p(y_i)\, p(y_j'/y_i) \log \frac{p(y_j'/y_i)}{p(y_j')} = v[H(y) - H(y/y')] = v[H(y') - H(y'/y)]. \qquad (2.21)$$

Calculating the carrying capacity even of constant channels in the general case is a rather difficult task. As far as channels with a memory are concerned, their carrying capacities in general can far from always be determined, inasmuch as a mathematical expectation of expression (2.19) or a limit (2.20) does not always exist. Nevertheless, for discrete representations of actual communication channels it is usually possible to construct mathematical models in informationally stable form, i.e., having a particular carrying capacity of discrete channels. Specifically, channels with a limited memory in which conversion probabilities depend only on a finite segment of the preceding sequence of symbols [7] or channels which are described by a finite number of states serve as such models if a current state can be determined from the preceding state and the last symbol transmitted [8].

According to Shannon's theorem, messages from a source with a controllable rate can be encoded so as to transmit them as accurately as desired over a discrete channel at a rate of $I'(x,y')$ which is less than the carrying capacity C. Messages from a source with a fixed rate can be transmitted as accurately as desired if the productivity of the source $H'(x)$ is less than the carrying capacity of the channel C.

We will note that the carrying capacity of a discrete channel C may not be greater than the carrying capacity C_n of a continuous channel included in it. This is so because the rate of information transmission cannot increase with transformation of a received continuous signal $z'(t)$ into a code symbol y'. Usually $C < C_n$ since the transformation of $z'(t)$ into y' is irreversible. If the modulator and the demodulator are not considered given, it is possible in principle to encode a message for a continuous channel and to transmit it as accurately as desired at a rate which is as close as desired to C_n, i.e., greater than C. Assuming the modulator and demodulator to be given and performing encoding in a discrete channel, we lose this capability and are obliged to transmit a message at a rate less than C. Nevertheless, we willingly accept this sacrifice inasmuch as the operations of encoding and decoding in a discrete channel lend themselves more readily to mathematical analysis and are usually simpler to implement technologically than in the case of a continuous channel. Here the words "more readily" and "simpler" should not be understood in an absolute sense. The theory of coding in a discrete channel is based on a rather complex mathematical apparatus and practical implementation of encoders and decoders often runs into technologically insurmountable obstacles.

The theory of interference-resistant codes finds its origin in the task of coding in a discrete channel. Not finding it possible to set forth this theory in detail, we will limit ourselves to its basic provisions and several results which will be offered without proof inasmuch as there now exist an extensive set of monographs pertaining to the two main lines--the probability theory of coding [9,10] and the algebraic [11,12] theory of coding.

We will demonstrate the essence of an interference-resistant code, using as an example block codes which are distinctive in that a block consisting of n symbols (code combination) is matched with every elementary message or with every sequence from a certain number of elementary messages. Let a certain value n be selected. It is possible to construct $N = m^n$ different sequences of code symbols with a base m. We will select from the N possible code combinations in a certain number $N_1 < N$ combinations which we will call permissible and we will match them with certain sequences of messages of the source. The remaining $N-N_0$ combinations are forbidden and are not used to transmit messages.

In transmission over a noisy channel some code symbols will be received incorrectly, as a result of which a code combination received will differ from that transmitted. If a code combination received with errors proves to be one of those permissible, the errors will not be detected and the message received, as a result of decoding, will differ from that transmitted. However if $N_0 \ll N$, the permissible combinations can be selected in light of the channel properties so that the probability of receiving an incorrect permissible combination is very small. As a rule, as a result of incorrect reception of separate symbols a received code combination is forbidden and this indicates the presence of errors.

Two methods of decoding when using interference-resistant codes, decoding with the detection of errors and decoding with correction of errors, are possible. With the first method a received forbidden code combination is not transformed into a message and the information included in it is either lost or is regenerated by a repeated transmission or by some other method. In many communication systems, because of the conditions under which they are used, it is very important not to send false messages to the recipient while possible loss of certain messages is not very serious. By using decoding with the detection of errors it is possible to employ rather simple means to decrease the probability of receiving a false message to any given magnitude and to achieve practically complete fidelity of received messages at the cost of rejecting a large number of code combinations containing detected errors. Decoding with detection of errors has found extensive application in communications systems providing the retransmission in which the existence of feedback permits regeneration of the information contained in received forbidden combinations. These systems will be examined in detail in Chapter XI.

In decoding with correction of errors, received forbidden code combinations are transformed by a decoder (second decision system) into a message in accordance with several rules established for a given communications system. These rules are determined in accordance with a selected statistical criterion. Specifically, if the criterion of the ideal observer lies at the bases of the decoding rules, they provide for the least possible probability of incorrect decoding under the given conditions. Most frequently the rules for decoding are based on the criterion of maximal likelihood which coincides with the criterion of the ideal observer if all permissible code combinations arrive at the input of the channel with the same probabilities and independently of one another. If this condition is not met, the probability of incorrect decoding when the criterion of maximal likelihood is applied will be somewhat greater than in the case of the criterion of the ideal observer. Still it can in principle be made as small as desired with a sufficiently long block n if the number of permissible code combinations N_0 satisfies the condition

$$\nu \log N_0 < nC \qquad (2.22)$$

We will note that the permissible code combinations become equiprobable and independent if prior to the receipt of the correcting code the redundancy of the message is eliminated using the methods described in Section 2.3. In this case the entropy per code combination is equal to $\log N_0$ and inasmuch as ν/n code combinations are transmitted per unit time, the amount of information arriving at the input of the channel in the unit time is equal to $\nu/n \log N_0$ and inequality (2.22) indicates that the rate of information transmission must be less than the carrying capacity of the channel.

Mixed methods of decoding are also possible with shich in some cases errors are corrected and in others they are only detected.

The application of a correcting code means the introduction of redundancy into a sequence of code symbols used to increase the fidelity of reception. The magnitude of a redundancy can be computed if it is borne in mind that the maximal entropy of a code combination containing n symbols with a code base of m (i.e., the entropy which would prevail if all m^n code combinations were permissible and were transmitted with equal probabilities and independently), is equal to n log m. This redundancy, in accordance with (1.12), is equal to

$$r_k = 1 - \frac{\log N_0}{n \log m}. \qquad (2.23)$$

The magnitude $1 - r_k = \frac{\log N_0}{n \log m}$ is called the effectiveness of the code.

The required redundancy of a correcting code can be found from condition (2.22) in light of (2.23):

$$r_k > 1 - \frac{C}{C_0}, \qquad (2.24)$$

where $C_0 = \nu \log m$ is the carrying capacity of a noiseless channel. Thus, r_k is defined by the relationship C/C_0. As far as the probability of incorrect decoding is concerned, it depends not only on r_k but also on the length of the code combination n and on the selection of permissible code combinations. In meeting condition (2.24) n must be increased in order to decrease the probability of incorrect decoding.

The circumstance that the complexity of an encoder and a decoder increases rapidly with an increase in n is very important. This pertains particularly to the decoder. We will consider, for example, the following universal (suitable for any codes) method of encoding and decoding. A transmitted message is analyzed in an encoder and replaced in accordance with the code used by one of the permissible code combinations. For this purpose all N_0 permissible combinations containing N_0 n log m bits must be stored in the memory of the encoder.

Remembering that from (2.23) $N_0 = m^{(1-r_y)n}$ holds, the size of the encoder memory must be equal to $m^{(1-r_y)n} n \log m$ bits. The received code combination is compared with combinations stored in the memory of the decoder, to each of which a certain decision corresponds. Obviously, with such a method of decoding, the decoder must store all possible code combinations, permissible as well as forbidden, i.e., have a memory size equal to $m^n n \log m$ bits. Thus, the sizes of the memories of encoder and decoder increases more rapidly with an increase in n than they would if the law were exponential. As a result, with a value of n equal to 30, the required size of decoder memory for such a universal method (assuming m = 2) reaches a magnitude on the order of 10^{10} bits, i.e., many times greater than that technologically attainable.

On the other hand, to obtain high fidelity in reception using correcting codes, it is often essential to use a value of n on the order of hundreds or even higher. Therefore, a principal task in modern coding theory is to find codes which permit detection and correction of errors, not by means of comparison with code combinations stored in a memory, but by using relatively simple operations performed on a received code combination. Several achievements have already been scored along this line. Codes have been suggested in the application of which the complexity of the encoder and decoder increases with an increase in n, not exponentially but in proportion to a rather small power of n [10,11,13]. Brief information about these codes will be presented in the following paragraphs. A detailed classification of suggested correcting codes is given in [illegible].

As was shown at the end of Section 2.7, message source redundancy also permits error correction in a received code sequence. Nevertheless, it often proves advisable to use a coding method which first [illegible] message redundancy (by methods of decorrelation and subsequent use of nonuniform codes), after which the code symbol sequences formed, containing no redundancy, are recoded by one of the methods of correcting codes. Thus, redundancy is introduced which is used to raise [illegible] fidelity.

The following arguments may be adduced in favor of [illegible]. Message source redundancy by no means always corresponds to the channel characteristics and may not be completely utilized to increase fidelity, whereas it is always possible to choose the correcting code which best matches a given channel. Most of the proposed correcting codes permit the detection and correction of erroneously received sequences by means of relatively simple rules before decoding, while message source redundancy is often stipulated by very complicated probabilistic relationships (e.g., those inherent in grammatical rules) and permits error detection and correction after decoding. The main role is played by intuition and experience, so that it is impossible to formalize the process of error detection and correction.

It is natural that in cases where the direct recipient of a message is not a man but a machine (e.g., in automated control systems), message source redundancy is difficult to use for increasing reception fidelity. This redundancy is only harmful because it uselessly occupies a portion of the carrying capacity of the channel. Contrariwise, redundancy con[illegible] introduced during encoding, if it correctly matches the channel characteristics, permits improving communication fidelity.

Incidentally, in some cases the redundancy of a message may even be useful in transmission from machine to machine. A system transmitting to a computer the values of the coordinates of a moving object at [illegible] instants of time may serve as an example. Let adjacent readings be [illegible] related with each other. This correlation causes redundancy [illegible] and may be useful in increasing fidelity. [illegible] simple interference-resistant code is used and [illegible] detection of errors. Readings with detected errors [illegible] stored by interpolation between adjacent readings.

2.5. Constant Symmetrical Channel [illegible]

A constant symmetrical channel [illegible] base m, the technical rate [illegible] p of incorrect symbol reception [illegible] bability of error):

$$p = \sum_{j \neq i} p(u'_j | u_i) \quad [illegible]$$

Its carrying capacity [illegible]

$$C = \max\{H [illegible]\}$$

[illegible]

Only $H(y')$ in this expression depends on the distribution of probabilities $p(y_i)$. Hence, in order to determine the carrying capacity of a constant symmetrical channel we must find such a probability distribution $p(y_i)$ as provides the maximum value of $H(y')$. It is obvious that $H(y')$ adopts a maximum value of log m in the case where the received symbol probabilities $p(y_i')$ are equal and do not depend on other symbols received. But from the property of a symmetrical channel it is easy to show that for the foregoing it is necessary and sufficient to impose the same condition on the probability distribution $p(y_i)$ of the transmitted symbols. Under this condition the rate of information transmission $I'(y,y')$ reaches a maximum which equals the channel carrying capacity:

$$C = v\left\{\log m + p\log\frac{p}{m-1} + (1-p)\log(1-p)\right\}. \quad [illegible]$$

In the particular case where m = 2,

$$C = v[1 + p\log p + (1-p)\log(1-p)] \text{ bits/sec.} \quad [illegible]$$

Let us observe that when p = (m-1)/m the carrying capacity is [illegible]. Therefore, without impairing generality, it may be assumed in what follows that

$$p < \frac{m-1}{m} \ [illegible] \ \frac{1}{2}.$$

[illegible] message from [illegible] source [illegible] transmitted [illegible] constant symmetrical channel [illegible] with [illegible] uniform [illegible], the probability [illegible] transmitted letter [illegible] [illegible]

$$q = (1-p) \quad [illegible]$$

In general, the probability that [illegible] message [illegible] sequence of n [illegible] will be correctly received is

$$q = (1-p) \quad [illegible]$$

By using redundant coding for sources with [illegible], we can substantially increase probability of correct reception.

In a constant symmetrical channel the probability of incorrect reception of a certain symbol does not depend on how other symbols were received. Therefore, the probability that in a transmitted code combination [illegible] symbols a certain r symbols were somehow distorted and the remaining n-r received correctly, according to (2.25) is

$$p_n(r) = \left(\frac{p}{m-1}\right)^r (1-p)^{n-r}. \tag{2.31}$$

It can easily be seen that $p_n(r)$ is a monotonic decreasing function of r inasmuch as $p/m-1 < 1/m \leq 1-p$ and does not depend on what symbols were received incorrectly.[1] But $p_n(r)$ can be viewed as a function of the likelihood of a transmitted code combination if it differs from the one received in certain r digits.

From this it follows that in a constant symmetrical channel the function of likelihood assumes a maximal value for that permissible code combination which differs from that received (forbidden) combination in the least number of digits. (If there are several such combinations, they all have the same values of likelihood function).

In coding theory the number of digits in which code combination A differs from code combination B is called the Hamming distance [13], d_{AB}, between these combinations. It can easily be seen that the Hamming distance satisfies the usual conditions of metrics, and namely: 1, when and only when combinations A and B are identically equal; 2, $d_{AB} = d_{BA}$, and 3, for any three combinations the axiom of a triangle holds

$$d_{AC} \leq d_{AB} + d_{BC} \tag{2.32}$$

And so, in decoding according to the criterion of maximal likelihood a received forbidden combination must be interpreted as the permissible combination nearest to it (in the sense of Hamming distance). Incorrect decoding in this case will obtain only when, as a result of the action of noise, the received code combination is farther from the one actually transmitted than any other permissible combination.

If between two permissible code combinations A and B the Hamming distance is equal to d_{AB}, in order for the transmitted combination A to be decoded as B, it is necessary that in the received combination A' there be not less than $1/2\, d_{AB}$ incorrectly received symbols. Therefore, the greater d_{AB}, the less will be the probability that in the given channel the code combination A will become B in decoding with correction of errors based on the criterion of maximal likelihood.

---

[1] We will note that the probability of incorrect reception of any r symbols of n follows a binomial distribution (which is a distinguishing feature of a constant symmetrical channel) and is equal to $p_n(r) = C_n^r p^r (1-p)^{n-r}$. This function has a maximum when $r \approx np$.

In constructing a correcting code it is, of course, desirable to choose permissible code combinations so that the Hamming distances between them are as great as possible. Let the Hamming distance between two permissible code combinations be d_{min}. Then the error detecting device will detect any erroneously received code combinations in which the number of erroneously received symbols does not exceed $d_{min}-1$. Actually, if r symbols in the received code combination are erroneous, its Hamming distance from the transmitted combinations is, by definition, r. Therefore when $r \leq d_{min}-1$, the received code combination cannot be one of the permissible ones.

If d_{min} is an odd number the error correcting device will correctly decode all received code combinations on condition that the number of erroneously received symbols (or the so-called multiplicity of the errors) does not exceed $(d_{min}-1)/2$. Actually, when the number of erroneously received symbols $r \leq (d_{min}-1)/2$ the Hamming distance between the received combination and any of the permissible combinations (except the one actually transmitted) may, according to (2.32), not be less than $d_{min}-r \geq (d_{min}-1)/2 \geq r$, and erroneous decoding does not occur. When d_{min} is even it is possible, as may be easily ascertained, to correct all errors of a multiplicity not exceeding $(d_{min}/2)-1$ and also at the same time to detect errors of multiplicity $d_{min}/2$, since, generally speaking, in the latter case there may be two or several permissible combinations at the same distance from the received one. Correction of all errors of multiplicity not exceeding $(d_{min}-1)/2$ and a certain probability of correction of errors of multiplicity $d_{min}/2$ may also be provided.

Therefore, the task of constructing a correcting code reduces to the choise of N_0 n-digit combinations having a maximum possible distance of d_{min}.

In this connection a question arises as to the greatest possible value of d_{min} with given N_0 and n, or as to the greatest possible value of N_0 with given n and d_{min}. There is no exact expression which yields an answer to this question, however, there are several known ways of estimating.

We will present without proof two of the most important estimations which pertain to binary correcting codes (m = 2). As Hamming [15] showed, for any binary code

$$N_0 \leq \frac{2^n}{\sum_{i=0}^{\left[\frac{d_{min}-1}{2}\right]} C_n^i} \qquad (2.33)$$

These codes for which in (2.33) there is equality are called densely packed. Along with primary codes there are known several densely packed correcting codes, for example, the code consisting of the two combinations 000 and 111, for which $N_0 = 2$, $n = 3$, and $d_{min} = 3$.

On the other hand R. R. Varshamov showed [11] that with any n and d_{min} there are codes for which

$$N_0 \geq \frac{2^n}{\sum_{i=0}^{d_{min}-2} C_{n-1}^i} \qquad (2.34)$$

The task of constructing an optimal (for a constant symmetrical channel) code having, with given n and N_0, the greatest possible value of d_{min} has not been solved in its general form. One method of approaching the solution of this problem (with a sufficiently large n), as paradoxical as it may seem, is the random selection of permissible code combinations of all m^n possible symbol blocks of length n. We will explain this using binary codes as an example (m = 2).

Let there be randomly selected N_0 sequences of symbols "0" and "1" of length n. This can be done, for example, by flipping a coin nN_0 times and writing a "1" when heads comes up and "0" at other times. We will consider any pair of combinations constructed in this way. The distance d between these combinations is a random magnitude. We will determine its mathematical expectation and dispersion.

Obviously, the probability that in a certain digit the combinations under consideration will have different symbols is equal to 0.5. Inasmuch as the events including the appearance of the symbols in different digits of the given two code combinations are independent, the problem reduces to a Bernoulli pattern (sequence of n independent trials with a probability of positive outcomes equal to 0.5). As is known, when this is so the mathematical expectation of distance d is

$$\bar{d} = 0.5n,$$

and its dispersion is

$$\sigma^2(d) = 0.25n.$$

When n ≫ 1, by using the integral limiting theorem of Moivre-Laplace (see, for example, [16]), it is possible to evaluate the probability that the Hamming distance between two randomly selected code combinations will be less than $(0.5 - \varepsilon)n$, where ε is a certain small positive number:

$$P\{d < (0.5 - \delta)n\} \sim$$

$$\sim \frac{1}{\sqrt{2\pi}} \int_{\infty}^{-2\delta\sqrt{n}} e^{-\frac{z^2}{2}} dz = F(-2\delta\sqrt{n}). \qquad (2.35)$$

With an increase in n the Laplace function $F(-2\delta\sqrt{n})$ rapidly approaches zero. Therefore, for any positive δ and ε, it is possible to find an n such that with a probability greater than 1-ε the Hamming distance between any pair of randomly-selected code combinations will be greater than (0.5 - δ)n.

The discussion presented yields only a qualitative confirmation of the [illegible] of random selection when it is possible with a probability close to unity to obtain a large Hamming distance between permissible code combinations. For this purpose in order to determine the rate of transmission of information in random coding we will evaluate the probability of incorrect decoding when n ≫ 1.

We will assume that N_0 permissible code combinations, which are known at the transmitting as well as receiving ends, are randomly selected. Let [illegible] combination M be sent over a communication channel. Generally [illegible], the combination received M' will contain incorrect symbols. [illegible] mathematical expectation of the number of incorrect symbols [illegible] the probability of error in the channel. [illegible] arbitrarily selected positive magnitudes as small as desired then, [illegible] law of large numbers, there is a magnitude n_0 such that [illegible] will not [illegible] with a probability of [illegible].

Incorrect decoding occurs in that case if there is [illegible] one permissible combination [illegible] the distance from which [illegible] the received combination M' is [illegible]. We will evaluate the [illegible] general number [illegible] combinations located at a distance not greater than [illegible] from combination M':

$$L = \sum_{i=1}^{r} C_n^i < r C_n^r \; [illegible] \qquad (2.36)$$

Inasmuch as the magnitude of n is very great, it is possible, by using Stirling's approximation formula, to replace the right side of (2.36) in the following way:

$$L < \sqrt{\frac{r}{2\pi\, [illegible]}} \left(\frac{r}{n}\right)^{-r} \left(\frac{n-r}{n}\right)^{r-n}. \qquad (2.37)$$

Inequality (2.36) holds when r < n/2, inasmuch as C_n^i increases with an increase in i from 1 to n/2. Since p < 1/2, it is always possible to select [illegible] that this condition is met.

But the ratio r/n is less than p + ε with a probability of 1−α. We will designate p + ε = p*. Then with a probability of 1−α

$$k < \sqrt{\frac{p^*n}{2\pi(1-p^*)}}\, p^{*-p^*n}(1-p^*)^{-(1-p^*)n}. \qquad (2.38)$$

Any of these k combinations can be permissible with a probability of $N_0/2^n$ and the probability that not one of the k combinations (except combination A which was known beforehand) does not belong to those permissible is equal to

$$\left(1-\frac{N_0}{2^n}\right)^k > 1-k\frac{N_0}{2^n}. \qquad (2.39)$$

Thus, the probability Q(n) of correct decoding of a received combination A' is evaluated by the following expression:

$$Q(n) > (1-\alpha)\left(1-k\frac{N_0}{2^n}\right) > 1-\alpha-k\frac{N_0}{2^n} >$$

$$> 1-\alpha-N_0 2^{-n}\sqrt{\frac{p^*n}{2\pi(1-p^*)}}\, p^{*-p^*n}(1-p^*)^{-(1-p^*)n} = \qquad (2.40)$$

$$= 1-\alpha-N_0\sqrt{\frac{p^*n}{2\pi(1-p^*)}}\, 2^{-n[1+p^*\log p^*+(1-p^*)\log(1-p^*)]},$$

where the logarithm base is 2.

We will introduce still another designation:

$$C^* = v[1+p^*\log p^*+(1-p^*)\log(1-p^*)]. \qquad (2.41)$$

It can easily be seen that C* is less than the carrying capacity of the channel:

$$C = v[1+p\log p+(1-p)\log(1-p)] \qquad (2.42)$$

and approaches it as ε approaches zero. Rewriting (2.40) as follows:

$$Q(n) > 1-\alpha-\sqrt{\frac{p^*n}{2\pi(1-p^*)}}\, N_0 2^{-\frac{n}{v}C^*}, \qquad (2.43)$$

we see that with an increase in n the probability of correct decoding approaches 1−α on condition that N_0 satisfies the inequality

$$N_0 < n^{-\beta}2^{\frac{n}{v}C^*}, \qquad (2.44)$$

where β > 1/2.

Specifically, this will be true if $N_0 = \frac{1}{n} 2^{\frac{n}{v} C^*}$. Considering that with a sufficiently large n the magnitude ε may be as small as desired and the magnitude C^* as close as desired to C, it can be asserted that with a randomly selected code the probability of correct decoding of a received combination is as close as desired to unity (for a sufficiently large n) if the number of permissible combinations is

$$N_0 = \frac{1}{n} 2^{\frac{n}{v} C}. \tag{2.45}$$

Finally, we will determine the rate of information transmission with such coding. Since the probability of correct decoding is as close as desired to unity, the amount of information transmitted over a channel is equal to the entropy of the code combination. If all the permissible combinations are selected independently and equiprobabily, the entropy of a code combination is equal to $\log N_0$ and the entropy for each transmitted symbol is

$$H(y) = \frac{1}{n} \log N_0. \tag{2.46}$$

It follows, that the rate of information transmission is

$$I'(y, y') = vH(y) = \frac{v}{n} \log N_0$$

or, substituting (2.45)

$$I'(y, y') = C - v \frac{\log n}{n}. \tag{2.47}$$

With an increase in n the second term approaches zero, i.e., the rate of information transmission can be as close as desired to the carrying capacity of the channel.

Understandably, with the described random selection of code there is a certain probability of selecting a "bad" code. For example, it may happen, although with very slight probability, that two combinations of A and B will coincide with one another or differ in only one digit. With transmission of combination A it will be decoded as B with great possibility, and vice versa. If this occurs for several pairs of combinations, such a code will not provide for high fidelity in decoding.

The probability of correct reception, evaluation of which is given in (2.45), is in essence a joint probability of two events, i.e., the selection of a "good" code and correct decoding using this code. A more detailed analysis [9] leads to the following result. Among randomly selected codes there are "good" codes for which the probability of incorrect decoding $P(n) = 1-Q(n)$, with a sufficiently large n, follows the inequality

$$P(n) < Ae^{-\alpha n}, \tag{2.48}$$

where A is a certain coefficient which changes slowly with an increase in n; E(R) is a function of the rate of information transmission $R = v/n \log N_0$ which is positive when $R < C$ and equal to zero when $R = C$.

As far as probability P_p of selecting a "bad" code, for which (2.48) does not hold, it also approaches zero with an increase in n and much more rapidly than the probability of incorrect decoding. Thus, if n is so selected as to provide a sufficiently small probability of incorrect decoding (2.48), a randomly selected n-digit code will be good with a probability of practically 1.

It would seem that the results obtained solve completely the problem of error-free transmission of information at a rate close to the carrying capacity of a constant symmetrical binary channel. However, practical use of a purely random code encounters obstacles which are insurmountable at the present time, and, it seems, will remain insurmountable for scores of years. The fact is that storage in the memory of a decoder or encoder of at least all permissible combinations and comparison of them with the received combination is the only method of encoding and decoding (when using a random code). With values of n which provide for sufficiently small probabilities of incorrect decoding, the required size of memory greatly exceeds that achievable by modern technology.

It is for just this reason that purely random coding has not found practical application and efforts of researchers have been directed toward regular or semi-regular methods of coding for which it is possible to formulate certain rules for conversion of messages into code combinations and also rules for decoding with correction and detection of errors and also to construct, in accordance with these rules, relatively simple logic circuits.

2.6. Constant Symmetrical Channel. Regular Coding

The most widespread among regular codes are systematical codes, problems in the construction of which are considered in the algebraic theory of coding using the apparatus of modern algebra [11,12,13].

A systematic (n,k) code amounts to a set of n-digit code combinations of which k digits (usually the first ones) represent the result of primary coding of a message. They are called information digits. The remaining n-k digits are formed in accordance with certain rules from the information digits. They are called the checking (correcting) digits and they serve the purpose of detecting and correcting errors. For example, the code (7,4) is a code in which a seven-element code combination contains four information symbols.

In other words, the process of encoding a message may be imagined as a succession of two procedures--first, encoding in a uniform k-digit nonredundant code; and second, assigning n-k corrective digits formulated by certain rules to each of the code combinations.

The number of permissible code combinations in a systematic code is $N_0 = m^k$. From this the code redundancy may be easily determined:

$$r_0 = 1 - \frac{H(A)}{H_{max}(A)} = 1 - \frac{\log N_0}{\log N} = 1 - \frac{k}{n} = \frac{n-k}{n}. \qquad (2.29)$$

The correcting symbols are formed by linear operations, determined over a finite field and producible over the information symbols. In the particular case where m is a prime these operations are congruent with comparison modulo m [1]. We would remind the reader that the whole numbers A and B are called comparable modulo m [written $A \equiv B \pmod{m}$] if both of these numbers give the same remainder when divided by m. For example, $5 \equiv 11 \pmod{3}$ or $8 \equiv 1 \pmod{7}$. To form correcting symbols certain information symbols (represented by the numbers from 0 to m-1) are added modulo m. This means that after arithmetic addition of these numbers their sum is replaced by the least whole non-negative number comparable to this sum modulo m (or the "least remainder modulo m"). Thus, for example,

$$1 + 2 \equiv 0 \pmod{3},$$

$$2 + 2 \equiv 1 \pmod{3},$$

$$1 + 1 \equiv 0 \pmod{2}, \text{ etc.}$$

It is obvious that these operations produce numbers from 0 to m-1 which may represent correction symbols.[1] By appropriate selection of the equations for forming the correcting symbols we may construct a code with a given minimum Hamming distance d_{min}.

Without going into the details of the theory of correcting codes, which is very thoroughly set forth in a monograph [11], we will give an example of constructing code (7,4) when m = 2. As the designation shows, four digits of the code combination are occupied by information symbols. We will designate time a_1, a_2, a_3, a_4. The remaining three digits are occupied by correction symbols, which we will denote by b_1, b_2, b_3. The symbols a_i may adopt values "0" or "1", to be determined by the coded message. The symbols b_i are determined by the equations

$$\left.\begin{aligned} a_1 + a_2 + a_3 + b_1 &= 0, \\ a_2 + a_3 + a_4 + b_2 &= 0, \\ a_1 + a_2 + a_4 + b_3 &= 0, \end{aligned}\right\} \qquad (2.30)$$

where all addition is modulo 2. For example, if the information symbols are 1001, the correction symbols must be 110. In this code $d_{min} = 3$. We easily satisfy ourselves of this if we note that reversing the value of one of the information symbols results in value change of at least two correction symbols, while value reversal of any two information symbols leads to value change in at least one correction symbol. Therefore, any two permissible code combinations (i.e., those satisfying (2.50)) differ from each other in no less than three digits. Hence it follows that employment of this code can detect an error if no more than two symbols in a combination are erroneously received or can correct an error if one symbol has been erroneously received.

---

[1] We would note that in everyday life we often employ addition modulo 24 (or 12) when figuring time. Thus, if it is now 19 o'clock, in ten hours it will be 5:00 AM. Actually, $19 + 10 \equiv 5 \pmod{24}$.

The received code combination is checked to see whether it satisfies the equations used for forming the correcting symbols, the purpose of this operation being to detect errors.[*] If at least one of these equations is not satisfied the received combination is not a permissible one and consequently an error occurred in transmission.

Error correction must take into consideration which of the equations are not satisfied and be guided by special rules which are easily established for the specific code. If, for example, two equations of (2.50) are true and one is unsatisfied, then one of the correction symbols must be deemed to have been erroneously received and the received combination may be decoded from the information symbols without any correction. If the first two equations are not satisfied, symbol a_5 which is common to both, is liable to correction (changing "0" to "1" or "1" to "0"). The first and third equations, if unsatisfied, subject symbol a_1 to correction. If the second and third equations are unsatisfied symbol a_1 is to be corrected. Finally, if all three equations are unsatisfied, symbol a, is liable to correction. Of course, if two or more symbols of the code combination are erroneously received that combination will not be correctly decoded.

We will now consider several algebraic properties of binary systematic codes which permit examining detail their detecting and correcting capabilities.

A binary systematic code containing a combination consisting of zeroes only forms an Abelian group with respect to the operation of digit modulo 2. This means that after adding modulo 2 symbols found in the same digit spaces of two permissible code combinations, we obtain a combination which is also permissible. Therefore, such codes are also called group codes.

A group code may be uniquely determined by giving only k linear independent combinations included in it. They form a generating matrix G having k rows and n columns. It can be used to construct all remaining code combinations, adding (by digit modulo 2) in pairs, by threes, by fours, etc., rows of a generating matrix. Specifically, by adding any line to itself, we will obtain a zero combination (consisting of n zeroes).

---

[*] In binary correcting codes these checks are called "parity checks" since the expression $\sum x_i = 0 \pmod{2}$ simply means that this sum is an even number.

For the code considered above (7,4) the generating matrix can be written, for example, in the following form:

$$G = \begin{Vmatrix} 1\,0\,0\,0\,1\,0\,1 \\ 0\,1\,0\,0\,1\,1\,1 \\ 0\,0\,1\,0\,1\,1\,0 \\ 0\,0\,0\,1\,0\,1\,1 \end{Vmatrix}.$$

It is sufficient to have a generating matrix in the memory of the encoder. By using a device for digit-by-digit summation it is possible to obtain any code combination.

For decoding with detection or correction of errors it is sufficient to have in the memory of the decoder a checking matrix H containing n-k rows and n columns. In each line of this matrix ones are found for those digits which enter into the corresponding checking equation[1] (2.50). In our example,

$$H = \begin{Vmatrix} 1\,1\,1\,0\,1\,0\,0 \\ 0\,1\,1\,1\,0\,1\,0 \\ 1\,1\,0\,1\,0\,0\,1 \end{Vmatrix}$$

Thus, with a systematic code the size of the memory of the encoder and decoder increases not exponentially with an increase in n but only proportionally to n² (considering that k is proportional to n). Nevertheless, the number of operations which must be performed for decoding with the correction of errors and consequently, the complexity of the decoder increases approximately exponentially although with a smaller exponent than in the case of random coding.

For the past years particular attention has been devoted to a variety of systematic code which is called cyclical. This type of code is distinctive in that any cyclical rearrangement of symbols of a permissible code combination also leads to a permissible combination. This distinguishing feature and also several algebraic properties of cyclical codes makes it possible to greatly simplify encoding and decoding systems [11,14,18].

---

[1] The product of a generating matrix and transpositioned checking matrix is equal to zero.

For several cyclical codes there may be a relatively simple (although not always optimal) procedure for decoding which is called majority or threshold decoding [13,19].

Among the cyclical codes which are intended for a constant symmetrical channel the best are codes with a particular algebraic structure which are called Bouz-Chodkhuri codes. For any whole s and t there is a Bouz-Chodkhuri code which corrects all errors divisible by t (i.e., having $d_{min} = 2t + 1$) when $n = 2^s - 1$ and $k \geq 2^s - 1 - st$. For example, for $s = 6$ and $t = 3$ we obtain a code with $n = 63$, $k \geq 45$, and $d_{min} = 7$.

The introduction of cyclical codes has made it possible to construct encoders and decoders for n on the order of several tens in the correction of errors and on the order of hundreds in the detection of errors. Such codes with varying redundancy permit a rather high level of fidelity in reception in actual channels. Nevertheless, in many cases it is desirable to use even longer codes with a decoder whose design is as simple as possible. Therefore, efforts to find new methods of constructing codes never cease. Here mention should be made of the work of R. Callagher [20] who suggested a systematic code, the construction of the check matrix for which contains a random selection in one of the stages. Therefore such a code cannot be considered completely regular. Thanks to the fact that the rows and columns of the check matrix contain far fewer ones than zeroes, this code permits a relatively simple procedure in decoding.

Extremely promising are the package codes for which a method of sequential decoding has been developed [10]. They usually pertain to random codes, however, in their construction there is an element of regularity which permits simplifying the process of decoding.

All the codes which have been discussed above are block codes. There also exist continuous codes in which the sequence of the transmitted symbols cannot be subdivided into blocks.

By way of example we will describe one of the recurrent codes, called a chain code [14,21]. It is distinctive in its extremely simple methods of encoding and decoding. Specifically, it permits majority decoding.

As already indicated, this code does not divide the code symbol sequence up into separate code combinations. Correction symbols are included in the stream of information symbols so that between every two information symbols there is one correction symbol. Denoting as before the information symbols by a_i and the correction symbols by b_i we obtain a symbol sequence like

$$a_1b_1a_2b_2a_3b_3 \ldots a_ib_ia_{i+1} \ldots$$

The information symbols are determined by the transmitted message and the correction symbols by the following rule:

$$b_k = a_{k-s} + a_{k+s+1} \pmod 2. \qquad (2.51)$$

where s is an arbitrary whole number called the code step.

It is apparent that when some correction symbol b_i is incorrectly received correlation (2.51) will not be observed in the received sequence for $i = k$.

If, however, information symbol a_i is incorrectly received correlation (2.51) will not be observed for two values of k, namely $k = i-s-1$ and $k = i + s$. From this it is easy to deduce the rule for correcting errors in decoding. Correlation (2.51) is checked for every b_k in the code sequence received. If it proves to be unfulfilled for the two values of k ($k = k_1$ and $k = k_2$) and moreover $k_2 - k_1 = 2s + 1$, then information element a_{k_1+s+1} must be replaced by the opposite one.

The redundancy of such a code is obviously 1/2. It permits correcting all incorrectly received signals if they occur sufficiently infrequently. Thus, if $s = 0$, it gives correct decoding when between two incorrectly received symbols there are no less than three (and in some cases, two) correctly received symbols (this refers to both types of symbol). The factors stimulating the choice of step s will be explained in Section 2.8.

2.7. Coding in a Constant Symmetrical Erasure Channel

In an erasure channel the symbol alphabet used contains an extra $(m + 1)$-th erasure symbol besides the m transmitted symbols. In an actual channel whose discrete representation is an erasure channel we can always obtain an arbitrarily small probability of erroneous symbol reception $p = p(y'_j|y_i)$ ($i = j \neq m + 1$) at the cost of increasing the probability of erasure $p_c = p(y'_{m+1}|y_i)$, if we properly choose the first decision [illegible].

If we disregard the probability of erroneous symbol reception the carrying capacity of the erasure channel may be expressed by the erasure probability [22]:

$$C = \max\{H(y) - H(y|y')\} = \max\{H(y) - p_c H(y)\}$$

Here $H(y|y') = p_c H(y)$, since when a symbol is correctly received the residual entropy of y is zero and when a symbol is erased it equals initial entropy $H(y)$. Therefore

[illegible]

The same correcting codes as are used in the stationary symmetrical channel may be used to restore erased symbols. If a systematic correcting code with minimum Hamming distance of d_{min} is employed, then any received code combination with an erased symbol quantity of $k_e \leq d_{min} - 1$ may be correctly decoded. In fact, the unerased symbols which remain in this process differ in at least one of their digits from the symbols of other permissible code combinations besides the one actually transmitted. Of course, in isolated cases combinations, the number of erased symbols in which exceeds $d_{min} - 1$, may be correctly decoded since even then such combinations are possible when the symbols kept in at least one digit differ from the symbols of other permissible combinations. It is apparent, however, that there can be no code which would correct any received combinations with a number of erased symbols $k = d_{min}$. In actual fact, by definition of the Hamming distance between permissible combinations, there exist at least two combinations differing by only d_{min} digits. If one of these combinations is transmitted and those digits in it are erased which differentiate it from the other combination, it will be impossible to distinguish them from each other.

In the case where the code employed does not permit correction of more than $d_{min} - 1$ erased symbols, the probability of incorrect decoding when $p_e \ll 1$ may be approximately defined as the probability that of n symbols any d_{min} of them are erased (disregarding the probability that a much larger number of symbols may be erased):

$$\approx C_n^{d_{min}} p_e^{d_{min}} (1 - p_e)^{n - d_{min}} \approx C_n^{d_{min}} p_e^{d_{min}} \qquad (2.53)$$

We will note that in the use of codes with a great Hamming distance it is possible to decode received code combinations correctly even in those cases when along with the erased code symbols there also are incorrectly received symbols. For this purpose it is sufficient (but not always necessary) to meet the condition

$$k_e + 2k_0 \leq d_{min} - 1, \qquad (2.54)$$

where k_e is the number of erased symbols and k_0 is the number of incorrectly received symbols in the code combination.

Indeed, let k_e symbols located in certain digits of a received combination be erased. We will eliminate these digits in all allowable code combinations. Then we will obtain a new set of combinations (codes) in which the minimal Hamming distance is $d'_{min} \geq d_{min} - k_e$. If among the erased symbols there are those received in an amount not exceeding $(d'_{min} - 1)/2$, in principle correct decoding is always possible as was shown in Section 2.7, (2.54) follows from this.

Based on this it is possible to use the first decision pattern which permits representing an actual channel as a discrete erasure channel with a probability of erasure, is not negligibly small. Incidentally, the decoding system in such a channel must be much more complex than in a channel where only erasure of symbols is in practice possible.

2.8. Coding in Asymmetrical Channels and Channels with Memory

Asymmetrical uniform channels are comparatively rarely encountered in practice. This is not surprising because coding theory for these channels has been only slightly worked out. With nonredundant coding the maximum rate of information transmission in a non-symmetrical channel occurs with nonuniform distribution of a priori probabilities when use is most often made of those symbols which are received more correctly.

We shall limit ourselves to a brief description of an asymmetrical constant channel. Let the two symbols "0" and "1" be employed, and $p(y' = 0 \mid y = 1) = p_1$ and $p(y' = 1 \mid y = 0) = p_2$, $p_2 \neq p_1$. We will designate the a priori probabilities of these symbols by $P(0)$ and $P(1) = 1 - P(0)$, respectively. Then the average quantity of transmitted information per symbol is

$$\begin{aligned} I(y',y) = H(y') - H(y'|y) = -[P(0)(1-p_2) + \\ + P(1)p_1]\log[P(0)(1-p_2) + P(1)p_1] - [P(0)p_2 + \\ + P(1)(1-p_1)]\log[P(0)p_2 + P(1)(1-p_1)] + \\ + P(0)(1-p_2)\log(1-p_2) + P(0)p_2\log p_2 + \\ + P(1)p_1\log p_1 + P(1)(1-p_1)\log(1-p_1). \end{aligned} \qquad (2.55)$$

Differentiating this expression over P(0) with regard to P(1) = 1 - P(0) and setting the derivative equal to zero, we may find the optimum a priori symbol probability distributions which provide the maximum of transmittable information. The optimum value of P(0) proves to be

$$P_{\mathrm{opt}}(0) = \frac{1}{1-p_1-p_2}\left(\frac{1}{1+\exp\left(\frac{A}{1-p_1-p_2}\right)} - p_1\right), \qquad (2.56)$$

where

$$\begin{aligned} A = p_2\log p_2 + (1-p_2)\log(1-p_2) - \\ - p_1\log p_1 - (1-p_1)\log(1-p_1). \end{aligned}$$

Finding $P_{\mathrm{opt}}(0)$ and substituting it in (2.55) we may find the maximum amount of information transmittable in this channel $I_{\max}(y',y)$ and its carrying capacity $C = vI_{\max}(y',y)$.

In the particular case of a symmetrical channel $p_1 = p_2$ the value of A is zero and the optimum value of P(0), as was to be expected, is 0.5.

In the limiting case, where one of the symbols, "1", for instance, is always correctly received ($p_1 = 0$), while the other symbol "0" may be received as "1" with probability p_2, expression (2.56) is simplified

$$P_{opt}(0) = \frac{1}{1 - p_2 + \left(\frac{1}{p_2}\right)^{\frac{p_2}{1-p_2}}} \qquad (2.56a)$$

Let us note that in this case the symbol "1" is the "reliable" transmitted symbol since it is always received correctly. The "certain" received symbol, however, is "0" since when it is received it can be asserted with complete certainty that it was just this symbol which was transmitted.

In the particular case where $p_1 = 0$ and $p_2 = 0.5$ the optimum symbol probability distribution is P(0) = 0.4 and P(1) = 0.6. The carrying capacity of such a channel is 0.322v bits/sec. Let us observe that this carrying capacity is considerably higher than that of a symmetrical channel with the same average error probability ($p_1 = p_2 = 0.25$), where it is 0.199v bits/sec.

The carrying capacity of a binary asymmetric channel is zero when $p_1 + p_2 = 1$. In this case the received symbols contain no information at all about the transmitted symbols because the a posteriori and a priori probabilities of the "0" and "1" symbols coincide.

Efficient error-correcting-and-detecting codes [14] may be used in an asymmetrical channel in which $p_1 = 0$ (or $p_1 \ll p_2$, so that p_1 may be practically disregarded). The theory of these codes has, however, been little elaborated and differs essentially from the theory of coding in symmetrical channels. For example, a code consisting of the two code combinations 00 and 11 allows one error to be corrected (change of "0" into "1") if it is stipulated that the received code combinations 01 and 10 be decoded as 00. At the same time a code consisting of the combinations 01 and 10 do not afford the possibility of correcting the error, but only of detecting it, although both of these codes are characterized by the same Hamming distance of 2. Let us note that in a symmetrical channel both of these codes permit only detection of a single error.

Of considerably more practical interest are nonconstant channels or channels with memory. Included in them are most channels which are found in communications equipment. Symmetrical channels with memory differ from symmetrical constant channels in that the distribution of the number of errors over the length of a certain block of symbols with any length n does not always follow a binomial distribution. If in a constant channel the conditional probability of incorrect reception of the (i + 1)-th symbol, on condition that the i-th symbol is received incorrectly, is equal to the unconditional probability of error

$$p(i + 1|i) = p \qquad (2.57)$$

then in a channel with memory it may be greater or less than this magnitude.

There are several reasons for the deviation of error distribution from the binomial in actual channels. Thus, a channel with memory proves to be a discrete representation of most radio channels, usually because of the fading which occurs and this will be considered in Chapter V. Another reason may be atmospheric and mutual interference. Sometimes deviation from a binomial distribution is caused by peculiarities in the methods used for modulation and demodulation. In multiplexed cable communication lines the presence of commutation interference, which occurs when switching separate elements of the channel and, in essence, temporarily putting the channel out of order, is usually considered to be the cause of "memory."

Studying channels with memory, developing correcting codes for them, and evaluating their effectiveness is made difficult by the fact that to describe such a channel it is insufficient to know one parameter (such as the probability of error in a constant symmetrical channel). For this purpose it is necessary to be able to define the probabilities of any combinations of errors within the limits of a block of any length n. For the purpose of obtaining such data, resort is had to experimental research into various actual channels. However, generalization of the experimental results obtained is made difficult by the fact that it is not always possible to select a convenient analytical representation and, furthermore, different channels behave differently. Therefore, researchers are trying to construct mathematical models of a discrete channel with memory which will be determined by only a small number of parameters, an approximate selection of which would at least permit, in general terms, describing the behavior of actual channels.

We will take note first of all of the principal peculiarities in accordance with which it is possible to classify channels with memory. Most channels encountered in practice satisfy the condition

$$p(i + r|i) > p \qquad (2.58)$$

This means that, in comparison with a constant channel, in such a channel errors have a tendency to group. With an increase in r, inequality (2.58) usually approaches an equality. Such channels will be called channels with error grouping.

In most channels with error grouping $p(i + r|i) < (m - 1)/m$ and, specifically, in a binary channel $p(i + r|i) < 1/2$. Such channels can be called normal channels with error grouping in distinction from anomolous channels in which $p(i + r|i)$ can exceed $(m - 1)/m$.

Channels with dispersed errors in which

$$p(r) < p \tag{2.59}$$

are encountered much more infrequently. An example is provided by a channel in which impulse interference is the cause of error if each impulse destroys only one symbol and the source of interference has the property that the probability of occurrence of the following impulse immediately following a preceding one is very small and increases with time.

There also may be channels with memory for which with some values of r (2.58) holds and for other values (2.59) holds. Thus, if (2.58) is met with odd r and (2.59) with even r, there is a tendency toward doubling of errors in the channel. An example of such a channel will be presented later on.

All known mathematical models of channels with memory are constructed almost exclusively for the description of normal channels with error grouping. A Markov model is the simplest model of a channel with memory in that it represents sequences of errors in the form of a simple Markov chain [2]. In this case the probability that a given symbol will be received incorrectly is equal to a certain magnitude p_1, if the preceding symbol was received correctly, and equal to a certain other magnitude p_2 if the preceding signal was received incorrectly.

When $p_2 > p_1$ a Markov model is a normal channel with error grouping and when $p_2 < p_1$ it represents a channel with dispersed errors. The unconditional (average) probability of error p in such a channel must satisfy the equation

$$p = pp_2 + (1 - p)p_1,$$

whence

$$p = \frac{p_1}{1 + p_1 - p_2}. \tag{2.60}$$

With such a model it is exceedingly simple to calculate the probability of any combination of errors and easily evaluate the effectiveness of any code. Unfortunately, this model only very roughly reproduces the properties of actual channels with error grouping. Therefore, it is not used at the present time.

Attempts to describe a channel by using a Markov chain of a higher order (i.e., to consider that the probability of incorrect reception of a symbol is determined unambiguously by how the preceding k symbols were received) have not met with success either. With a small k such a model agrees with experiments very poorly and with a large k it is inconvenient for purposes of calculation.

The Gilbert model (or more exactly, the Dzhil'bert) [23] has been used somewhat more successfully. In accordance with this model a channel may be in two states, S_1 and S_2. In state S_1 there are no errors and in state S_2 errors occur independently with a probability of p_2. The probability of an α-shift (in transmission of the next symbol) from state S_1 to state S_2 and the probability of a β-shift from S_2 to S_1 are known. Thus, here, not a sequence of errors but a sequence of states forms a simple Markov chain.

The probabilities of existence of a channel in states S_1 and S_2, can easily be computed and are equal to

$$P_1 = \frac{\beta}{\alpha + \beta},$$

$$P_2 = \frac{\alpha}{\alpha + \beta},$$

and the unconditional probability of error is

$$p = p_2 \frac{\alpha}{\alpha + \beta}.$$

Most frequently in using a Gilbert model for a binary channel $p_2 = 1/2$ is assumed. In other words, state S_2 is considered as a complete break in communication while in state S_1 there is no noise in the channel. This agrees rather well with ideas about a channel in which only commutation interference is found.

The Bennet-Froelich model is more general but less convenient in making calculations [24]. In accordance with this model errors occur in the form of more or less long-lasting surges or packets. By packet is understood a sequence of signals in which the first and last are received correctly and between them there may be symbols which are received either correctly or incorrectly. It is assumed that packets occur independently of one another with a probability of p_p. Besides this probability this channel is characterized by a probability of p_2 of errors within a packet and a distribution of $p(l)$ of probabilities of length (number of symbols) of packet l. By selecting the values of p_p and p_2 and also of the form of the function $p(l)$, it is in many cases possible to obtain a description of a channel agreeing with experimental results. Computation of the probabilities of various combinations of errors and the result of their correction by correcting codes in accordance with a Bennet-Froelich model is rather complex and is usually replaced by modeling in digital computers.

We will note that the concept of a packet of errors does not coincide with the concept of state S_2 in a Gilbert model. State S_2, as in the case of a packet, is characterized by a non-zero probability of error p_2, but in distinction from a packet the condition that state S_2 begin and end with incorrectly received symbols is not stipulated.

A Bennet-Froelich model is more flexible than a Gilbert model since it permits a very wide selection of function p(L) on which only the usual condition of standardization is imposed since in a Gilbert model the distribution of probabilities of the duration of state S_2 is always expressed by the formula $p(L_2) = [illegible](1-[illegible])^{L_2-1}$, i.e., the magnitude of [illegible] is uniquely determined. Nevertheless, for many experimental research channels it is not possible to select the parameters of a Bennet-Froelich model satisfactorily. This is especially true of a Gilbert model. In view of this, O. V. Popov suggested [25] a more complex model of a discrete channel differing from a Bennet-Froelich model in that packets of errors are considered to be not independent. In accordance with this model a channel can have two states in the first of which errors do not occur and in the second of which, with a certain probability, packets of errors do occur. The probabilities of shift from one state to the other, the probability of occurrence of a packet in the second state, the probability of an error within a packet (which is usually equal to 0.5), and the distribution of probabilities of packet length are the parameters. In most cases it is possible to describe actual channels sufficiently well with these parameters.

An attempt to describe a binary channel with error grouping using only two parameters, i.e., the probability of error p and the grouping indicator α has been made in [37]. For this purpose the conditional mathematical expectation $[illegible]_r(n)$ of the number of errors in a block of length n on condition that no less than r errors have occurred is considered. The magnitude of $v(n) = [illegible]_r(n)/n$ with $p \ll 1$, according to experiments which have been conducted, is approximated sufficiently well for several channels by the empirical expressions

$$v_r(n) = \left(\frac{r}{n}\right)^{1-\alpha} \quad \text{with } \frac{r}{n} < 0.5,$$

$$v_r(n) = \frac{r}{n} \quad \text{with } \frac{r}{n} \approx 1,$$

where α is a parameter which depends on the characteristics of the channel. For constant channels α = 0. The more the errors are grouped, the greater is α. When α = 1 the errors follow in a continuous stream. We will note that v(n) = p according to definition. Knowing p and α it is possible to calculate the probabilities of a different number of errors in blocks of any length without giving any thought to the mechanism causing the grouping.

All described models of a discrete channel with memory are also in large measure formal. In the construction of them no attention is paid to the causes of error grouping and a probability system is simply selected which must describe observed facts. It is true that for several models, for example, the Bennet-Froelich, a "physical base" is mentioned to show that only commutation interference or surges of impulse interference, occurring independently (in the Popov model dependently) of one another and more or less destroying a long segment of a signal, is the source of error. But these models are used, and rather successfully, also for channels in which it is known that other types of interference exist [44].

Lately attention has been devoted to the construction of physical models in distinction from formal mathematical models of discrete channels. In these models a discrete channel is viewed as a reflection of a continuous channel and the distribution of errors is deduced from the probability properties of the signal and the interference in a continuous channel. For example, V. I. Korzhik [26] viewed the distribution of errors in a channel with fluctuation interference when the signal is subjected to relay fading (see Chapter V). We will learn about some of these models in subsequent chapters.

Calculation of the carrying capacity of various models of a discrete channel with memory is a complex process. It is solved in [25] for a Gilbert model. Incidentally, in that case when the states of a channel change very rarely, it is possible to determine the carrying capacity very approximately by knowing the carrying capacity of constant channels corresponding to these states. For example, we will consider a generalization of a Gilbert model, assuming that the channel may be in state S_1 with a probability of error of p_1 and in state S_2 with a probability of error of p_2 in which the probabilities of α- and β-shifts from one state to the other are very small as a consequence of which the states change rarely. We will stipulate that $p_2 > p_1$.

The carrying capacity of such a channel may be approximately determined by averaging the "partial" carrying capacities with respect to states S_1 and S_2.[1]

$$C = P_1C_1 + P_2C_2, \qquad (2.61)$$

where P_1 and P_2 are the respective probabilities of states S_1 and S_2; and C_1 and C_2 are the carrying capacities of symmetrical channels with error probabilities p_1 and p_2.

If the state of the channel were known at every moment to both correspondents formula (2.61) would then be exact and each state could then use its own correcting code adapted to the given value of the error probability. But that would require that information, from which the channel state could be judged, would have to come to the transmitter from the receiving device. This case will be examined in Chapter XI.

It is obvious that it would serve no purpose to use a code designed for a constant symmetrical channel with error probability p equal to the average error probability in a channel with memory

$$p_{av} = P_1p_1 + P_2p_2,$$

---

[1] See Remark 3 to Chapter II.

Indeed, let an n-digit correcting code for k errors in a code combination be used. If $k > np_{av}$ in a constant channel, then the probability that there will be more than k erroneously received symbols in a code combination may be very small. This type of code gives a very high received message fidelity in a constant channel. But in the channel under investigation this code, if k is larger than np_{av} but smaller than np_2, does not provide fidelity since the combinations which are transmitted under the poorer conditions (in state S_2) will with great probability be received with a number of errors which is more tean k, and will therefore not be corrected. At the same time the combinations which are transmitted in state S_1 will generally have been substantially fewer than k incorrectly received symbols (since $k \gg np_1$) and for them the corrective capacity of the code is extraordinarily great, i.e., the code has extremely high redundancy. In other words, although the average number of errors in a code combination is np_{av}, these errors are nonuniformly distributed and appear most often in packets when the channel is in state S_2.

It would, of course, in this case be possible to use a corrective code designed for the worst conditions (state S_2) and at the cost of redundancy (i.e., delay in transmission of information) secure the required fidelity. But for state S_1 such redundancy would be excessively great. In addition, the usable channel carrying capacity is reduced essentially to C_2, the carrying capacity under the worst conditions. This method of coding is therefore extremely disadvantageous. In some cases (e.g., in radio communication by reflection from meteor trails) such coding is generally impossible since in state S_2 the carrying capacity is reduced to practically zero.

The following is a possible solution. Let us employ a code with [illegible] such long combinations that it will be very probable that the channel will change state several times during each such combination. Under these conditions the expected number of errors in a combination is defined by the average probability p_{av}. If the number of errors k to be corrected in the combination substantially exceeds np_{av}, all the combinations will in this case most probably be correctly decoded; but even this method of coding has two essential drawbacks. First, in real channels (e.g., during fading) the length of the code combination must be so great (on the order of [illegible] of digits) that a practicable realization of encoding and decoding systems comes up against almost insurmountable difficulties. Second, this method of coding is essentially designed for a uniform symmetrical channel with error probability $p = p_{av}$. But the carrying capacity of such a channel is, as may be demonstrated, always less than the carrying capacity of a channel with memory with the same average error probability. Therefore there must in principle be more economical codes affording the same reliability of nonuniform channel reception with less redundancy.

The first of these drawbacks may be to a considerable degree overcome by using correcting codes with relatively short combinations in connection with a system of "error decorrelation." This system amounts to encoding messages in the usual way using, for example, a systematic code in which the combination length and the number of mistakes corrected (and hence the redundancy of the code) are selected starting from the condition of obtaining the required reliability in a uniform symmetrical channel with error probability p_{av}. The code combination symbols thus obtained are not transmitted into the channel one directly after another, but at [illegible] time intervals. During these intervals the symbols of other [illegible] combinations are transmitted. This process may be graphically [illegible] [27]. Let us write down m code combinations in the form of a [illegible]

$$
\begin{array}{c}
a_1^{(1)} a_2^{(1)} a_3^{(1)} \dots a_n^{(1)}, \\
a_1^{(2)} a_2^{(2)} a_3^{(2)} \dots a_n^{(2)}, \\
\dots \\
a_1^{(m)} a_2^{(m)} a_3^{(m)} \dots a_n^{(m)}
\end{array}
$$

[illegible] second digits, etc. If the cycle [illegible] average number of errors [illegible]

[illegible]

[illegible]

[illegible]

In this connection codes which correct [illegible] are of great interest. [illegible] constant symmetrical channel the [illegible] in a block of n symbols, [illegible] errors occur, [illegible] received symbols in the block [illegible], [illegible] with errors [illegible] is much more probable that [illegible] symbols [illegible] [illegible]

[illegible]

[illegible]

[illegible]

[illegible] channel [illegible] state [illegible]

[illegible] probability [illegible] error [illegible] the carrying [illegible] constant channel [illegible] zero, whereas the channel [illegible] relatively large carrying [illegible]

[illegible]

[illegible] discrete representation of a real [illegible] phase modulation [illegible] modulation, the [illegible] transmitted.

[illegible] sequence of binary symbols [illegible] symbol [illegible] the new sequence $b_1, b_2, b_3, \ldots$ [illegible] following [illegible]

$$b_1 = a_1$$
$$b_2 = a_2 + b_1 \pmod 2$$
$$b_3 = a_3 + b_2 \pmod 2$$
$$\cdots\cdots\cdots$$
$$b_k = a_k + b_{k-1} \pmod 2$$

[illegible] occurred during their transmission the received [illegible] are simply converted into symbols according to the [illegible] $a_i = b_i + b_{i-1} \pmod 2$ (since subtraction modulo 2 coincides with addition).

Now let us suppose that while receiving symbol b_r a shift to negative occurred, [illegible] beginning with b_r all the remaining transmitted symbols b_i [illegible] $b_i + 1 \pmod 2$ until the next shift into positive. Then instead of symbol a_r will be received symbol $a_r' = b_r' + b_{r-1} \pmod 2$ [illegible] and instead of symbol a_{r+1}, symbol $a_{r+1}' = b_{r+1}' + b_r'$ [illegible] $b_{r+1} + b_r + 1 + 1 \pmod 2 = a_{r+1}$. It is easy to see that all the following symbols (until the channel shifts to positive) will also be correctly restored, but at the moment of shift into positive one symbol will be incorrectly restored. Thus, every time the channel changes from one state into the other one of the transmitted symbols will be received with an error. The result is that the recoding employed transforms this Markovian

---

*This method will be examined more detail in Section 4.6.

channel into a uniform channel with an error probability of $p \approx p_1$. Great reception fidelity may be obtained in this channel if a single-error correcting code is used in the initial encoding of the message into the symbols a_k.

It should be observed that if, instead of the channel described, we examine a more complex channel (and one closer to a representation of a real channel with phase modulation) which has along with probability p_1 of shift from one state into the other another probability p'_1 of erroneous reception of a symbol without changing channel state, then the described method of recoding will have as its result that two successive symbols will be incorrectly restored with probability p'_1. Actually, if symbol $b'_r = b_r + 1 \pmod 2$ is received instead of symbol b_r and the following symbols are received correctly, then when restoring symbols a_k we will obtain

$$a'_r = b'_r + b_{r-1} = a_r + 1 \pmod 2,$$

$$a'_{r+1} = b'_{r+1} + b'_r = a_{r+1} + 1 \pmod 2,$$

$$a'_{r+2} = b_{r+2} + b_{r+1} \pmod 2 = a_{r+2}$$

etc.

In order to provide high reception fidelity in this case we must employ a code which corrects error packets two digits in length when compiling the symbol sequence a_k [28].

To illustrate a channel which is not symmetrical but symmetrical on the average let us cite a discrete representation of an actual channel with frequency modulation (ChT) and narrow-band interference which may enter the compression channel, the release channel, or not enter the passband of the receiving device at all. This channel has three states. In state S_1 the channel is symmetrical and the error probability is $p_1 \ll 1$. In state S_2 the probability $p(0|1)$ that symbol "0" will be received for transmitted symbol "1" is negligibly small, but the probability $p(1|0)$ of receiving "1" when "0" was transmitted is $p_2 \approx p_1$. In state S_3, on the contrary, probability $p(1|0)$ may be disregarded and probability $p(0|1) = p_3$. It is apparent that the received code sequences in such a channel will contain bunches of one-sided errors, either replacements of zeros by ones or of ones by zeros. The probability that in a single code combination a symbol "0" will be received as "1" and a symbol "1" as "0" (the type of error that telegraphers call "shift errors") is very small. In fact, in state S_1 the probability of two errors in a code combination which is not very long is generally small since $p_1 \ll 1$, while in states S_2 and S_3 shift errors are practically impossible.

[illegible] state S_1 (or in the other direction) [illegible] transmission of a code combination is also slight if the states [illegible] comparatively slowly.

There are methods for coding such binary channels which make it possible [illegible] errors, if not correct them. One of these methods is [illegible] using a nonsystematic code with a constant weight. The number of nonzero symbols in a code combination is called its weight. [illegible] the number of ones. If in an n-digit code all combinations have [illegible] it can easily be seen that the number of [illegible]

[illegible]

[illegible] 10 [illegible] 7 and [illegible]

[illegible] weight [illegible] error.

[illegible]

[illegible]

[illegible]

The channels [illegible] Furthermore, [illegible] in general form [illegible] channels, [illegible]

[illegible] Methods of Comparing Correcting Codes

[illegible] correcting [illegible] code selection [illegible] problem, of course, it is necessary to bear many different factors in mind, specifically, the [illegible] decoding which affects the

---

[illegible] inverted [illegible] replacing zeroes with ones and ones with zeroes, [illegible] cost of [illegible] if one ignores purely technological and [illegible] only fidelity and the amount of information [illegible] selecting a code still remains a [illegible].

We will [illegible] codes. We will first note that, as [illegible] paragraphs, it is impossible to compare [illegible] characteristics of the channel for which they are intended [illegible] assume that the characteristics of a channel are [illegible] of a code determines, on one hand, [illegible] channel is used, and, on the other hand, [illegible] decoded message.

[illegible] these aspects of a correcting code [illegible] the rate of information transmission [illegible] much as it is associated with the [illegible] the probabilities of correct and incorrect [illegible] this direction does not lead us to our goal. [illegible] that the application of a correcting code [illegible] information transmission over a channel [illegible].

In [illegible] assume that a primary code is used [illegible] information transmission will [illegible] distribution of the [illegible] metrical channel with an [illegible] symmetrical channel with the [illegible] maximal rate by definition is [illegible]

[illegible] transmission with as high a [illegible] does not exceed the [illegible] ideal code which ensures [illegible] rate equal to the carrying [illegible] in transmission rate [illegible] of loss in rate is [illegible] of a coding block [illegible] random coding as discussed [illegible] concerned, for them always [illegible] always less than the carrying [illegible] rate attainable with primary [illegible] errors. Thus, increasing the fidelity [illegible] of information transmission, [illegible] in comparison with the [illegible] As a rule, this loss is [illegible]

[illegible] useful analogy with the [illegible] technological process will enable us to obtain more metal than the initial content [illegible] Any actual process will lead to certain losses [illegible] nificant, the greater are the demands made [illegible] very same way [illegible] contained in a received signal [illegible] at the price of certain losses.

In order to evaluate [illegible] information transmission rate, [illegible] Indeed, the rate [illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible] 0.0001 = 0.073. If the second code is used, the entire message will be contained in two code combinations and will be received correctly with a probability of $0.99^2 = 0.98$. Therefore, the probability that the message will be received correctly is greater in the case of the second code than in the case of the first, although the probability of correct decoding of the code combination is greater for the first code than for the second.

[illegible]

[illegible]

This seeming contra [illegible] with
the second code uncorrected [illegible] with the
first. Therefore, although [illegible]
corrected error is greater [illegible] chances
of avoiding the formation [illegible] same.
We will further note that [illegible]
is decoded incorrectly, [illegible]
contain 2-3 incorrectly [illegible]
the number of incorrect [illegible]
we may conclude that [illegible]
is better to use the [illegible]
all errors are equally [illegible]

We will note that [illegible]
errors are equally import [illegible]
codes. Therefore, if [illegible]
that neither the probabil [illegible]
the probability of an [illegible]
systems, as we have seen, [illegible]

In the preceding [illegible]
of a message containing [illegible]
selected entirely arbi [illegible]
probability that it w [illegible]
is easy to understand [illegible]
bility of error-free [illegible]
with an increase in leng [illegible]
of a code combination [illegible]
[illegible] then the pr [illegible]
[illegible]

In this equation [illegible]
[illegible] combination [illegible]
message. It is assumed [illegible]
during encoding.

[illegible] with any [illegible]
[illegible] zero. However [illegible]
[illegible] of error-free [illegible]
system it is possible [illegible]
system can be considered [illegible]
[illegible] is determined [illegible]
the course of a day and [illegible]
once every 3000 years the [illegible]
incorrectly due to the [illegible]

almost all applications this communication system can be considered faultless. We will note that there are no obstacles in principle to the task of achieving the indicated degree of fidelity (if the carrying capacity of the channel is sufficient) and the technological difficulties are no greater than the difficulty of obtaining apparatus reliability which guarantees an operating time per one failure on the order of [illegible] years.

Formula (2.62) holds also for primary coding when the number of symbols n in a code combination coincides with the number of information symbols k. For a binary constant symmetrical channel with error [illegible] n = k = 1 and $Q_n = (1 - p)^n = (1 - p)^{ri}$ and, consequently,

$$Q_c(ri) = (1 - p)^{ri} \qquad [illegible]$$

From the point of view of fidelity in decoding a long message two communication systems can be considered equivalent if with sufficient large ri $Q_{c1}(ri) = Q_{c2}(ri)$ at least in an asymptotic sense, i.e.,

$$\lim_{ri \to \infty} \frac{Q_{c1}(ri)}{Q_{c2}(ri)} = 1 \qquad [illegible]$$

Here the indices "1" and "2" pertain to the systems being compared. From here flows the possibility of characterizing any communication system with any code using the probability of error in a binary communication system with a primary code equivalent to it. We will call the [illegible] probability of error in a constant symmetrical binary channel, in which a system with primary coding is equivalent to the system being considered, the equivalent probability of error p_e of the communication system [illegible].

The equivalent probability of error [illegible]

[illegible]

where [illegible] = 1 - Q(n) is the probability of incorrect decoding of a code combination.

The magnitude $q_e = 1 - p_e = (1 - [illegible])^{1/[illegible]}$ is called the equivalent probability of correct reception.

Where [illegible] = 1,

$$p_e [illegible] \tag{2.66}$$

An analogous magnitude which is called the relative probability of decoding was also introduced in V. I. Siforov's work [32].

In the case of continuous codes in which it is impossible to subdivide the sequence of symbols into separate combinations (e.g., in the case of a chain code) the equivalent probability of error should be determined using the limiting transition

$$P_e = 1 - q_e = 1 - \lim_{i \to \infty} [1 - \alpha(i)]^{1/i}, \tag{[illegible]}$$

where α(i) is the probability of incorrect reception of a message containing i bits of information.

We will find the equivalent probability of error for a symmetrical constant channel when coding without redundancy using a code with a base of m = 2 [33,34]. If the probability of incorrect reception of a symbol in the channel being considered is equal to p, the probability of error-free reception of a sequence consisting of k symbols is equal to $(1 - p)^k$. Such a sequence contains $i = k \log_2 m$ bits of information. Thus, the equivalent probability of correct reception is equal to

$$q_e = (1 - p)^{\frac{k}{i}} = (1 - p)^{\frac{1}{\log_2 m}} \approx 1 - \frac{p}{\log_2 m}$$

(the approximate equality holds when p << 1), whence

$$p_e \approx \frac{p}{\log_2 m}. \tag{[illegible]}$$

From this it follows, for example, that [illegible] 10^{-3} [illegible] equivalent [illegible] error of [illegible]$\cdot 10^{-4}$.

[illegible]

$$(1-\varepsilon)=(1-p)^7+7p(1-p)^6.$$

Here the first term indicates the probability that all symbols are received correctly and the second term the probability that one term out of seven is distorted. The equivalent probability of correct reception is equal to

$$q_e=(1-\varepsilon)^{1/4}=[(1-p)^7+7p(1-p)^6]^{1/4}.$$

When p ≪ 1 this expression can be written approximately, ignoring high powers of p as follows:

$$q_e\approx[1-7p+21p^2-\dots+7p-42p^2+\dots]^{1/4}=$$
$$=[1-21p^2]^{1/4}\approx 1-5.25p^2,$$

whence the equivalent probability of error is equal to

$$p_e\approx 5.25p^2. \qquad (2.69)$$

This expression can be obtained more simply if it is considered that when pr ≪ 1 the probability that three or more symbols in a combination will be distorted in a symmetrical channel is much less than the probability of incorrect reception of two symbols. Therefore, the probability of incorrect decoding of a code combination ε in code (7, 4) in the first approximation is equal to the probability of incorrect reception of any two symbols out of seven:

$$\varepsilon\approx C_7^2\,p^2(1-p)^5\approx C_7^2\,p^2=21p^2,$$

whence, in light of (2.66) (2.69) immediately follows.

Generally speaking, if a binary correcting code is densely packed, reasoning similarly, we may show that when pr ≪ 1

$$p_e\approx\frac{1}{i}\,C_n^{r+1}\,p^{r+1}. \qquad (2.70)$$

For other than binary codes which correct errors in symbols divisible by r and not correcting errors divisible by a greater number, the approximate relationship shown in (2.70) also holds in a symmetrical channel. It should only be remembered that by i is understood the amount of information in a code combination expressed in bits. Expression (2.68) is a particular case of (2.70) when r = 0, since with any n

$$\frac{1}{i}\,C_n^{1}=\frac{n}{i}=\frac{1}{k_{\text{B},m}}.$$

If a systematic code being used provides for the possibility of correcting an error divisible by r in all cases and also in several cases of an error divisible by a higher number, the probability of incorrect decoding of a code combination ε when p ≪ 1 is approximately equal to the probability that there will be incorrect reception of the r + 1-th symbol in one of the uncorrected combinations.

$$\varepsilon \approx (C_n^{r+1} - a_{r+1})\,p^{r+1},$$

where a_{r+1} is the number of corrected combinations of errors divisible by $r + 1$, whence

$$P \approx \frac{1}{k}(C_n^{r+1} - a_{r+1})\,p^{r+1} \qquad (2.71)$$

From the expressions obtained it is apparent that the effectiveness of correcting codes is greater the less is the probability of error p in a stationary symmetrical channel. For example, when p = 0.1, the equivalent probability of error for code (7, 4), according to (2.69), is equal approximately to 0.05, i.e., only half that in the case of primary coding, while with $p = 10^{-3}$ the equivalent probability of error is on the order of $5 \cdot 10^{-6}$, i.e., in this case the increase in fidelity is very great.

In summing up results, we may note that for an objective comparison of codes (or communication systems) we may use either the probability of an uncorrected error or the equivalent probability of error. The first of these measures is suitable for communication systems transmitting messages, the value of which is not lost when there is a small number of errors and decreases only monotonically. The second measure is convenient in those cases when every finished message must be received without error, i.e., when even a single error ruins it completely.

Notes

1. (See Section 2.1). The term "channel with memory" indicates that the probability of error in such a channel depends on how the preceding symbols were received, i.e., the channel as it stores the preceding events in its memory.

In speaking of the dependence of errors in a channel with memory it must be borne in mind that what is meant here is not a causal dependence but a static dependence. Two events A and B are considered independent in a probability sense when and only when p(A/B) = p(A). Otherwise, they are dependent if it is even known that they are caused differently.

For example, let errors occur in a binary discrete channel due only to the influence of a certain source of interference S, the interference occurring at random instants of time with a probability of [illegible] and there existing a certain probability [illegible] that if the i-th symbol is transmitted in the presence of interference, at the instant of transmission of the (i + 1)-th symbol the interference will cease (Gilbert's model). If the conditional probability of error during the action of interference is equal to p_2, as can easily be calculated, the average unconditional probability of error is equal to

$$p = p_2 \ [illegible]$$

We will compute the conditional probability $p(i+1|i)$ of incorrect reception of the $(i+1)$-th symbol on condition that the i-th symbol is received incorrectly. Inasmuch as the i-th symbol was received at an instant when there was interference, the $(i+1)$-th symbol will be received with a probability of γ in the absence of interference, i.e., without error, and the interference will continue with a probability of 1-γ with the reception of the $(i+1)$-th symbol. In the latter case this symbol may be received incorrectly with a probability of p_e. Thus

$$p(i+1|i) = (1-\gamma)p_e.$$

Thus, errors in such a channel are dependent although an error in the reception of a preceding symbol is not the cause of incorrect reception of the $(i+1)$-th symbol.

In a constant symmetrical channel errors are independent. As is known, it follows from this that the occurrence of r errors in a block consisting of n symbols is determined by the binomial distribution

$$P_n(r) = C_n^r p^r (1-p)^{n-r}.$$

Therefore, any symmetrical channel in which the number of errors in a block does not follow a binomial distribution is a channel with memory.

2. (See Section 2.3). The condition for unambiguous decodability of a nonuniform code symbol sequence may be formulated. For the existence of a code containing l combinations, in which n_i is the number of symbols in the i-th combination and which permits unique decoding, it is necessary and sufficient that [1]

$$\sum_{i=1}^{l} m^{-n_i} \leq 1, \qquad (2.72)$$

where m is the code base.

The coding theorem for a noiseless channel may be deduced from condition (2.72) [7].

3. (See Section 2.3). The Morse telegraph code is usually given as an example of a practical coding method which approximates an economical one. This is not a very good example because the Morse code contains a substantial amount of redundancy, although the principle of the shortest combinations for the most frequently encountered letters was used in developing it.

Telegraphic practice afford another instructive example of the utilization of the statistical properties of a source for curtailing the average number of symbols in a code sequence which uses not so much the univariate probability distribution of the letters as the probabilistic connections between the transmitted signs. This is the example of various uniform codes which are used in teleprinting by the means of telegraph apparatus equipped with "registers."

The number of letters in the alphabet of most languages does not exceed thirty-two. Therefore, any telegram written with letters can be transmitted with a five-digit binary code (since $2^5 = 32$). In addition to letters, however, telegrams sometimes contain numerals and punctuation marks, so that the total number of signs in the alphabet of the telegraph apparatus is considerably more than thirty-two. Therefore, primary encoding in a uniform binary code would require six symbols in the code combination.

The average number of symbols per sign is substantially reduced by using a system of registers. In the simplest case a telegraph apparatus has two registers, i.e., two possible states. In the first state (letter register) five-digit code combinations corresponding to the usual letters of the alphabet and the space between words are sent and received. In the second state (numeral register) the same code combinations correspond to numerals and punctuation marks. Two code combinations are used to transfer the receiving apparatus from one register to the other.[1]

It is easy to ascertain that if all the signs (including numerals and punctuation marks) appeared in telegrams with the same probability the register system would not be economical, but, on the contrary, would give a certain loss in comparison with use of a primary six-digit code. In fact, if the probability that the next transmitted sign belongs to the letter register is 1/2, then the probability that exactly k signs in succession will belong to the letter register, after which there will appear a sign belonging to the numeral register, is $(1/2)^k (1/2) = 2^{-(k+1)}$. The same thing is also true of the probability that k signs in a row will belong to the numeral register.

The mathematical expectation of $\bar{k}$ signs coming in a row before a change of register is

$$\sum_{k} k\,2^{-(k+1)}$$ [illegible]

Therefore the sign for change of register will be, on the average, transmitted after every two information letters. Transmission of N signs requires an average of 5N/2 more symbols for register change combinations besides the 5N symbols for the code combinations themselves. The average number of symbols per sign will be $5(1 + 1/2) = 7.5$, i.e., more than with primary encoding in a six-digit code.

In reality, numerals and punctuation marks occur considerably less often in telegram texts than do letters. Furthermore, numerals often follow one after the other. As a result the average length of the sequence of signs belonging to one register is substantially more than two and ordinarily reaches several tens. By virtue of this the average number of symbols per sign proves to be but little more than five, i.e., using registers yields considerable savings.

---

[1] The widely used Soviet telegraph apparatus ST-35 uses three registers--Russian letters, Latin letters and numerals (including punctuation marks).

Let us note that the employment of registers, like other methods of reducing redundancy, decreases resistance to interference. This shows up in the fact that incorrect reception of a register-shift combination as a sign, or vice versa, causes errors in a series of following signs and even changes the number of signs received. Therefore, register coding is generally not used in modern systems of discrete information transmission of which high reliability is demanded.

4. (See Sections 2.7 and 2.8). The simplest method corrective coding is repetition of information symbols. If every k-digit combination of a primary code is repeated d times a correcting code with $n = dk$ and with a Hamming distance of d will be obtained. It is interesting to note that this code will be systematic and cyclical. According to the general rule it will permit detecting errors if their number does not exceed $d - 1$ or correcting errors if they are divisible (with an odd d) by a number not greater than $1/2(d - 1)$. Correction of errors is here performed by the rule of the majority which in a symmetrical constant channel corresponds to the criterion of maximal likelihood. The redundancy of such a code $r_v = 1 - 1/d$ is much greater than in more complex codes with the same Hamming distance. This shortcoming is made up for by simplicity of decoding only in rare cases.

In a particular case if a combination of a primary code is transmitted twice, a code with $d_{min} = 2$ is obtained which permits only the detection of single errors (and also other errors with odd divisibility) with a redundancy of $r_v = 1/2$. It is a simple matter to construct a binary code with such a redundancy but with $d_{min} = 4$ which permits correcting single errors and, furthermore, correcting double errors. For this purpose it is sufficient in repetition of a code combination, if the initial combination has an odd weight, to invert the symbols (i.e., to replace a "0" by a "1" and vice versa). The length of a resulting combination must be not less than four. Decoding is done in almost the same way as with ordinary repetition.

If combinations of a redundant code with a minimal Hamming distance d_1 are repeated several times, a code with $d_{min} = d_1 d_2$, where d_2 is the number of repetitions, is obtained. Thus, by repeating combinations with $d_1 = 2$ twice, we obtain a code with $d_{min} = 4$ and we are in this way able to correct single errors and to detect double errors.

The essence of such methods of coding does not change if the length of the repeated block is increased to the duration of an entire finished message. In this process the effectiveness of the code may be increased thanks to error correlation. Therefore, it is impossible, as some authors do, to contrast systems with repetition with systems with a correcting code. The repetition of a message is also a correcting code which is little effective but, on the other hand, easily realizable.

5. (See Sections 2.7 and 2.8). In speaking of the limited possibilities of guaranteeing a high degree of fidelity in noisy channels by using correcting codes with a long block length n, we had in mind, firstly, the difficulties inherent in encoding and decoding. In some cases, however, obstacles of another nature arise which limit the magnitude of n and thereby the achievable fidelity.

When n ≫ 1 a significant amount of time elapses between the start and end of reception of a block. But prior to the termination of reception of a block it is impermissible to proceed to the decoding of it. Thus, between the start of reception of a message and issuance of it to the recipient there is an inevitable time delay which is proportional to n. If messages are created by a source with a fixed rate, almost the same delay occurs during coding inasmuch as a code combination may be formed only after the source has emitted a sufficient volume of message. In some systems a long delay is impermissible inasmuch as the transmitted information rapidly loses its value.

6. (See Section 2.8). Formula (2.61) is accurate if the channel state is known to both correspondents. In the general case for a nonuniform channel the rate of information transmission [illegible] may be found from the "conditional information" formula derived by A. N. Kolmogorov [33]:

$$I(u, S) = I(u, \omega) \text{ [illegible]} \quad (2.73)$$

$$+ I(S, \omega) \text{ [illegible]}$$

where $I(u, S)$ is the rate of transmission of information contained [illegible] sequence [illegible] sequence [illegible] channel state [illegible],

[illegible] is the average, with respect to all the values [illegible] rate of transmission of information [illegible] sequence [illegible] the channel state when the values [illegible]

[illegible] is the average, with respect to all states, rate of transmission [illegible] sequence when the channel states [illegible]

[illegible] quence v, hence [illegible] (2.73) that

$$I(u \text{ [illegible]}) \quad (2.74)$$

Since $\overline{[I'(S, y'y')]_{y'}} \geq 0$ (because the average rate of information transmission cannot be negative), we will obtain

$$I'(w', w) \geq [I'(w', w/S)]_S$$

On the other hand, in formation theory it is proved that

$$[I'(S, y'y')]_{y'} \leq I'(S, S) = H'(S)$$

where $H'(S)$ is the entropy of the channel state, whence it follows that

$$I'(w', w) \geq [I'(w', w/S)]_S - H'(S)$$

Combining these inequalities we find

$$[I'(w', w/S)]_S - H'(S) \leq I'(w', w) \leq [I'(w', w/S)]_S \qquad (2.75)$$

Let $p(v)$ be the distribution of symbol probabilities maximizing $I''(v', v)$. Then the carrying capacity of a channel with memory is

$$C = \max_{p(v)} I'(w', w) = I'(w', w)_{\bar{p}(v)} \qquad (2.76)$$

and since inequality (2.75) holds with any distribution of $p(v)$, then

$$C \leq [I'(w', w/S)]_{\bar{p}(v)} \leq \max_{p(v)} [I'(w', w/S)]_S \qquad (2.77)$$

But the maximum of the average value of several magnitudes does not exceed the average value of the maximums, whence

$$C \leq \max_{p(v)} [I'(w', w/S)]_S \leq [\max_{p(v)} I'(w', w/S)]_S = \bar{C}_S \qquad (2.78)$$

where $\bar{C}_S$ is the carrying capacity averaged over states S computed on the assumption that the states are known.

If the number of states is not great and replacement occurs rarely, then $H'(S) \ll I'(v', v)$ and from (2.75) follows the approximate equality

$$C \approx \bar{C}_S$$

whence for the case of two states (2.58) results.

In [41] it is shown that with certain additional conditions (2.79) becomes an exact equality.

BIBLIOGRAPHY

1. Shannon, C., *Matematicheskaya teoriya svyazi* [Mathematical Theory of Communication]. In the collection: Works in the Theory of Information and Cybernetics, Foreign Literature Press, 1963.

2. Romanovskiy, V. I., *Diskretnyye tsepi Markova* [Discrete Markov Chains], Gostekhizdat, 1949.

3. Fano, R., *Peredacha informatsii. Statisticheskaya teoriya svyazi*, Information Transmission. Statistical Communication Theory], "Mir" Press, 1965.

4. Huffman, D. A., "A Method for Constructing Minimum Redundancy Codes," PIRE, No. 9, 1952.

5. Oliver, B. M., "Efficient Coding," BSTJ, No. 3, 1952.

6. Siforov, V. I. and I. F. Borodin *Primeneniye korrektiruyushchikh kodov v vedomstvennoy svyazi* [Use of Correcting Codes in Departmental Communication]. In the collection: 100 Years from the Date of Birth of A. S. Popov, Press of the Academy of Sciences of the USSR, 1960.

7. Faynsteyn, A., *Osnovy teorii informatsii* [Fundamentals of Information Theory], Foreign Literature Press, 1960.

8. Shannon, C., *Nekotoryye rezul'taty teorii kodirovaniya dlya kanalov s shumami* [Some Results of Coding Theory for Noisy Channels]. In the collection: Works on Information Theory and Cybernetics, Foreign Literature Press, 1963.

9. Fleyshman, B. S., *Konstruktivnyye metody optimal'nogo kodirovaniya dlya kanalov s shumami* [Constructive Methods of Optimal Coding for Noisy Channels], Press of the Academy of Sciences of the USSR, 1963.

10. Vozenkraft, Dzh. and B. Rayffen, *Posledovatel'noye dekodirovaniye* [Sequential Decoding], Foreign Literature Press, 1963.

11. Piterson, U., *Kody, ispravlyayushchiye oshibki* [Error Correcting Codes], Foreign Literature Press, 1964.

12. Udalov, A. P. and B. A. Suprun, *Izbytochnoye kodirovaniye pri peredache dvoichnoy informatsii* [Redundancy Coding in the Transmission of Binary Information], "Svyaz" Press, 1964.

13. Massey, Dzh., *Porogovoye dekodirovaniye* [Threshold Decoding] "Mir" Press, 1966.

14. Kharkevich, A. A., *Bor'ba s pomekhami* [Combatting Interference], Fizmatgiz, 1963.

15. Hamming R. V., *Kody s obnaruzheniyem i ispravleniyem oshibok* [Codes with the Detection and Correction of Errors]. In the collection: Theory of Information Transmission, Foreign Literature Press, 1956.

16. Levin, B. R., *Teoreticheskiye osnovy statisticheskoy radiotekhniki* [Theoretical Fundamentals of Statistical Radio Technology], Vol. I, "Sovetskoye radio" Press, 1966.

17. Vinogradov, I. M., *Osnovy teorii chisel* [Fundamentals of Number Theory], Fizmatgiz, 1959.

18, Peterson, W. W. and D. T. Brown, "Cyclic Codes for Error Detection," PIRE, No. 1, 1961.

19. Folesnik, V. D. and Ye. T. Mironchikov, *Dekodirovaniye tsiklicheskikh kodov* [Decoding of Cyclical Codes], "Svyaz'" Press, 1968.

20. Gallager, R. Dzh., *Kody s maloy plotnost'yu proverok na chetnost'* [Codes with a Low Density of Parity Checks], "Mir" Press, 1966.

21. Hagelbarger, D. W., "Recurrent Codes," BSTY, No. 4, 1959.

22. Elayas, P., *Kodirovaniye dlya dvukh kanalov s shumami* [Coding for Two Noisy Channels]. In the collection: Theory of Information Transmission, Foreign Literature Press, 1957.

23. Gil'bert, E. N., *Propusknaya sposobnost' kanala s paketami oshibok* [Carrying Capacity of a Channel with Error Packets], Kiberneticheskiy sbornik, "Mir" Press, No. 9, 1964.

24. Bennet, W. R. and F. S. Froelich, "Some Results on the Effectiveness of Error Control Procedures in Digital Data Transmission," IRE Translation, No. 1, CS-9, 1961.

25. Popov, O. V. and V. Ya. Turin, *Zakon raspredeleniya veroyatnostey peremennogo chisla oshibok v kombinatsii* [Law Governing Distribution of Probabilities of a Varying Number of Errors in a Combination], "Elektrosvyaz'", No. 5, 1967.

26. Korzhik, V. I., *Raspredeleniye oshibok v kanale s releyevskimi zamiraniyami* [Error Distribution in a Channel with Relay Fading], Works of the Second All-Union Conference on Coding Theory, Baku, 1965.

27. Blokh, E. L. and A. A. Kharkevich, *Kodirovaniye ustoychivoye po otnosheniyu k zamiraniyu* [Fade-Resistant Coding], "Elektrosvyaz'", No. 4, 1960.

28. Abramson, N. M., "A Class of Systematic Codes for Non-Independent Errors," IRE Translation, IT-5, 1959.

49. [illegible], R. R., "A Class of Multiple-Error-Correcting Binary Codes for Non-Independent Errors," Sylvania Report, RSL-E-2.

50. Berger, J. M., "A Note on Error Detecting Codes for Asymmetric Channels," Inf. and Contr., No. 1, 1961.

51. Fink, L. M., [illegible] [Applicability of Binary Correcting Code in a Channel for Transmitting Discrete Information], "Radiotekhnika", No. 10, 1961.

52. Siforov, V. I., [illegible] [Resistance to Interference in a System with Correcting Codes], Radiotekhnika i elektronika, No. 5, 1956.

53. Helstrom, C. W., "The Comparison of Digital Communication Systems," IRE Trans., CS-8, No. 3, 1960.

54. Wolf, J. K., "On Comparing N-ary Systems," IRE Trans., CS-10, No. 2, 1962.

55. Kolmogorov, A. N., [illegible] [Theory of Information Transmission], Press of the Academy of Sciences of the USSR, 1956.

56. Dobrushin, R. L., [illegible] [Transmission of Information over a Channel with Feedback], In the collection: Probability Theory and Its Application, Vol. III, Issue 4, 1958.

57. Purtov, L. P., A. S. Zamrii and A. I. Zakharov, [illegible] [Fundamental Laws Governing the Distribution of Errors in Discrete Communication Channels], "Elektrosvyaz'", No. 2, 1967.

58. [illegible], [illegible] [Effective Coding], "Energiya" Press, Moscow-Leningrad, 1965.

59. Gilbert, E. N. and E. F. Moore, "Variable-Length Binary Encoding," BSTJ, No. 4, 1959.

60. McMillan, B., "Two Inequalities Implied by Unique Decipherability," IRE Trans., IT-2, No. 4, 1956.

61. Ovseyevich, I. A., [illegible] [Carrying Capacity and Rate of Transmission in Channels with Random Parameters]. In the collection: Problems in Information Transmission, Issue 4, 1968.

62. [illegible], S. I., [illegible] [Interference-Resistant Coding], "Nauka" Press, 1966.

13. Borodin, L. F., [illegible] [Introduction to Theory of Interference-Resistant Coding], "Sovetskoye Radio" Press, 1968.

14. [illegible] [Statistics of Errors in the Transmission of Digital Information], Collection of translations edited by S. I. Samoylenko, "Mir" Press, 1966.

CHAPTER III

CONSTANT-PARAMETER CHANNEL WITH ADDITIVE FLUCTUATION INTERFERENCE

3.1. Statement of the Problem

This chapter examines the transmission of discrete messages with element-by-element reception in a constant-parameter channel. Each symbol of the code sequence is represented as some segment (element) of a signal sent from the transmitting to the receiving device. The receiving device determines from the received signal plus noise and the chosen statistical criterion what symbol was transmitted and then determines the transmitted message from the restored sequence of the code symbols.

In an analysis of the transmission and reception of signals bearing discrete information discloses the following main questions arise:

a) what should be the design of the first decision system which determines from the received signal element what symbol was transmitted;

b) what is the maximum probability of correct element reception with given parameters of the signal and interference and with the use of an optimum decision system;

c) how does this probability change if the actual system of the receiving device deviates from the optimum decision system;

d) what signal shape should be chosen in order to produce the greatest fidelity when different restrictions are imposed (e.g., given signal strength, etc.); and

e) what is the carrying capacity of the channel under examination with different restrictions imposed on the signal.

These questions have, to a significant degree, been answered for a channel with additive fluctuation interference by V. A. Kotel'nikov [1], C. E. Shannon [2], and other authors.

3.2. Representation of Signal and Interference Using Expansion Into a Fourier Series

Let each of m symbols of a code be transmitted as some element of signal $z_i(t)$ $(i = 1,\dots,m)$ given over time interval T_e. Thus, the signal represents a sequence of elements of equal length. We will study the conditions of reception (discrimination) of a single element of the signal. A signal plus additive arrives at the input of the receiving device, the signal being expressed by:

$$z'(t) = \mu z_i(t - t_p) + n(t) \quad (t_0 \le t \le t_0 + T_c). \tag{3.1}$$

where μ is the channel transmission factor; t_p, the passage time of the signal in the channel; $n(t)$, the additive interference; and t_0, the instant of start of transmission.

In this chapter we will regard μ and t_p as constants. In addition, it is assumed that the value of μ is known when the signal is received,[1] just as is the shape itself of each element of signal $z_i(t)$. Moreover, the instant $t_0 - t_p$ of the start of reception of a given element is known, i.e., the transmission system is considered to be synchronous. Taking this moment as the beginning of the time count, we may write the expression for the received signal as

$$z'(t) = \mu z_i(t) + n(t) \quad (0 \le t \le T_c). \tag{3.1a}$$

The first decision system must determine the index of the transmitted element of $z_i(t)$, i.e., analysis of the received signal $z'(t)$ must be used to decide what code symbol y_i was transmitted.

Let us represent the functions $z_i(t)$, $n(t)$ and $z'(t)$ in the interval $0 \le t \le T_c$ as a Fourier series:[2]

$$\left.\begin{aligned} z_i(t) &= \sum_{k=0}^{\infty} (a_{ik} \cos k\omega_0 t + b_{ik} \sin k\omega_0 t), \\ n(t) &= \sum_{k=0}^{\infty} (\alpha_k \cos k\omega_0 t + \beta_k \sin k\omega_0 t), \\ z'(t) &= \sum_{k=0}^{\infty} (A_k \cos k\omega_0 t + B_k \sin k\omega_0 t). \end{aligned}\right\} \tag{3.2}$$

---

[1] It will be shown below that in a number of cases this restriction may be removed.

[2] In some cases, as will be shown below, it is convenient to study the signal in the interval $0 \le t \le T$, where $T < T_c$. Here information is lost which is contained in the discarded portion of the signal element in interval $T \le t \le T_c$. But if $T_c - T \ll T$, this lost information has little effect on reception fidelity. At the same time, proper selection of T substantially simplifies the decision system.

See Remark 2 to Chapter III for the meaning of the expansion of a random process into a series.

where

$$\omega_0 = 2\pi / T, \quad T \geq T_0.$$

It is obvious that

$$A_k = \mu a_{ik} + \alpha_k,$$
$$B_k = \mu b_{ik} + \beta_k \tag{3.3}$$

The Fourier coefficients A_k, B_k contain all the information about the transmitt4d signal that there is in the received signal $z'(t)$, since $z'(t)$ may be uniquely restored from these coefficients.

In the vast majority of cases signals $z_i(t)$ are so chosen that only a finite number of coefficients a_{ik} and b_{ik} differ from zero. If this condition is not observed a value $T \geq T_0$ may usually be selected such that when the signal is represented as a Fourier series in the interval $0 \leq t \leq T$ (as stated in the footnote) a finite and, in most cases, a small number of coefficients a_{ik} and b_{ik} may be derived which are different from zero.

Let k_{1i} and k_{2i} represent, respectively, the least and the greatest subscripts of the Fourier coefficients a_{ik} and b_{ik} which are different from zero. Then k_{1i}/T and k_{2i}/T are the least and greatest frequencies in the Fourier series expression of signal $z_i(t)$. Let's agree to call the magnitude $F = (k_{2i} - k_{1i} + 1)/T$ the frequency band occupied by signal $z_i(t)$. If by k_1 and k_2 are understood, respectively, the least and the greatest values of k with respect to the whole signal set used in the given communications system, the magnitude

$$F = \frac{k_2 - k_1 + 1}{T} \tag{3.4}$$

represents the frequency band occupied by the system.

Obviously $F_i \leq F$. We will call dense these systems for which $F_i = F$ for all i, i.e., when all realizations of the signal occupy the same frequency band.

It should be noted that the magnitude F is not the signal spectrum width in the ordinary sense. As a rule, the signal has a continuous, not a discrete, power spectrum, and furthermore it is unlimited.[1] The idea of a signal with an unlimited spectrum, [illegible] works on general communication theory [illegible] at times leads to fundamental errors [3]. [illegible] occupied by the signal or the system [illegible] on total duration on signal spectrum. [illegible] signal consisting of sequences of elements [illegible] monic oscillations of the [illegible]

---

[1] See Remark 1 to Chapter III.

plitude of their a_{ii} and b_{ii} remain constant [illegible] of one element and, generally speaking, [illegible] ment changes. Each such segment [illegible] spectrum.

Nevertheless, as will be [illegible] concept of frequency band F (3,4) is very [illegible] very real. The total number of Fourier [illegible] may differ from zero with a given frequency band F [illegible]

$$B = 2(F_2 - F_1)T = 2FT$$

We will call the value B = 2FT the base of the signal [illegible] at some elementary examples.

Example 1. Let the signals

$$z_1(t) = a \cos \omega t$$
$$z_2(t) = 0$$

be used with code base m = 2 (amplitude modulation).

If $\omega = n\omega_0 = 2\pi n/T_c$, where n is a whole number, [illegible] for $z_1(t)$ contains a single term differing from zero [illegible] $a_n = a$. But if ω is not divisible by ω_0, [illegible] into a Fourier series in interval $(0, T_c)$, all [illegible] from zero, namely

a_k [illegible]

b_k [illegible]

In this case, however a value T_c may be so chosen that when [illegible] is expanded into a Fourier series in interval [illegible] the series will differ from zero. For this it suffices [illegible] We will derive the greatest possible value [illegible] determined from the condition

$$n \le \frac{\omega}{2\pi} T_c < n + 1$$

For signal $s_2(t)$ in the given case all the Fourier coefficients are identically equal to zero. The base of the system is obviously two.

We will agree to call simple systems those systems of communication which are distinguished by the capacity of every signal element to be represented by a Fourier series in which all the coefficients, except the coefficients with one subscript k, are zero. In other words, in simple systems every signal element represents a segment of a harmonic.

Example 2. Let the signals

$$s_i(t) = a\cos(\omega t + \varphi_i) \quad (i = 1, \ldots, m) \qquad (3.9)$$

be used, where a and ω are identical for all signals and φ_i is uniquely determined by the code symbol (phase modulation with code base m). If $\omega = 2\pi n/T_e$, where n is a whole number, every signal element may be represented by a Fourier series in interval $(0, T_e)$ in which only coefficients a_n and b_n differ from zero

$$a_n = a\cos\varphi_i, \quad b_n = [illegible]\sin\varphi_i. \qquad (3.10)$$

When $\omega \neq 2\pi n/T_e$ a suitable expansion interval $(0, T)$ may, as in the preceding case, be selected for which (3.10) is true. This system is just as simple and its base is also two. This is the only dense simple system.

Example 3. Let

$$s_i(t) = a\cos\omega_i t \quad (i = 1, \ldots, m) \qquad (3.11)$$

where frequency ω_i is determined by the transmitted code symbol (frequency modulation).

If every frequency ω_i is divisible by $2\pi/T$, where $T \geq T_e$, this system is also simple but not dense. Every signal element may be represented in interval $(0, T)$ by a Fourier series in which only coefficients with subscript $k_i = \omega_i T/2\pi$ differ from zero. The base of this system is $2(k_{max} - k_{min} + 1) \geq 2m$. If, however, this condition of divisibility is not fulfilled a system with such a signal is not simple. This divisibility stipulation is observed in most modern systems of discrete information transmission.

Example 4. Let

$$s_i(t) = \sum_{k=k_1}^{k_2} (a_{ik}\cos k\Omega t + b_{ik}\sin k\Omega t) \quad (i = 1, \ldots, m) \qquad (3.12)$$

where all a_{ik} and b_{ik}, generally speaking, differ from zero when $k_1 \leq k \leq k_2$.

If, furthermore, the average values of a_i and b_i, taken with respect to it, are identical and not dependent on i,

$$[illegible] \sum [illegible] = [illegible] \sum [illegible] , \qquad [illegible]$$

then this signal is to a certain degree similar to the segments which give noise with a uniform spectral density. Its base is equal to [illegible].

We would note that the values of the Fourier coefficients a_{1i}, b_{1i} may be experimentally determined with a fundamentally very simple system (Figure 3.1). Signal $s_i(t)$ at the initial instant t = 0 of transmitting an element is fed to a multiplying device[1] to which is also applied an auxiliary voltage with the unit amplitude cos $\lambda_0 t$, goes to an integrating device and at instant t = T the result of integration measured.

$$I = \int_0^T [illegible] \qquad [illegible]$$

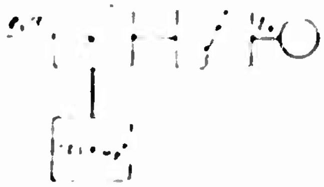


Figure 3.1. System for Determining Values of Fourier Coefficients.

Substituting the expression for [illegible] as a Fourier series (3.[illegible]) in expression (3.13) and taking into account the known property of orthogonality of trigonometric functions [3] it is easy to derive

$$I = \int_0^T [illegible] = a_{1i} \frac{T}{2}, \qquad (3.14)$$

and, knowing I, a_{1i} may be found. Similarly, feeding auxiliary voltage

---

[1] Within certain limits a ring diode mixer, as well as many other mixer circuits, may be used as this multiplier. The most accurate multiplication of two voltages may be accomplished with a mixer based on the Hall effect. A nonleaking capacitor whose charge at the moment of signal transmission is zero may be used as the integrator here.

[illegible] to the multiplier, we may first [illegible]

The coefficients [illegible] are random variables throughout the examined signal element. If the instantaneous interference values have a normal probability distribution, then the probability distribution of the Fourier coefficients [illegible] are also known to be normal with uniform expectation $\bar{a}_k = \bar{b}_k = 0$ and dispersion [illegible].

If, furthermore, the interference is white noise, i.e., has uniform power spectral density, then the random values of [illegible] are not correlated [4]. We may in practice consider interference to be white if its power spectrum is uniform in a frequency band which is substantially broader than frequency band occupied by the signal. For white noise for all values of k the magnitude [illegible]. Parseval's theorem [5] states that for any function u(t) given in interval (0, T) the following equation holds:

$$\frac{1}{T}\int_0^T u^2(t)\,dt = \frac{1}{2}\sum_k (a_k^2 + b_k^2)$$

It expresses by Fourier coefficients the power expended by the voltage of u(t) per unit of resistance. Therefore the quantity $(a_k^2 + b_k^2)/2$ may be regarded as the part of this power taken over by a frequency band of width 1/T around frequency $f_k = k/T$. For the interference the mathematical expectation of this portion of power is $\overline{(a_k^2 + b_k^2)}/2 = \sigma_k^2$, and the relationship of σ_k^2 to frequency band 1/T is the spectral interference density at frequency k:

$$S^2(f_k) = \sigma_k^2 T$$

For white noise the spectral density $S^2(f_k) = \sigma_0^2$ does not, by definition depend on frequency.

3.3. Decision Principle and Decision System

Let a signal plus interference z'(t) enter the receiver input. We will study the case where the interference is normal white noise. On reception the transmitted signals $z_i(t)$ (i = 1,...,m) corresponding to every symbol are exactly known, as well as the a priori probabilities of transmission of each symbol.[2] According to the ideal observer criterion the first decision

---

[2] The probability connections between the symbols in a code sequence are not taken into consideration here since we are dealing with element-by-element reception in which the first decision system determines the symbol transmitted with respect to the received signal element without consideration of the values of other symbols.

[illegible]

$$[illegible] \qquad (3.18)$$

[illegible]

$$[illegible] \qquad (3.19)$$

The a posteriori probability of transmitting symbol v_r (or signal $z_r(t)$) is, after (1.18),

$$[illegible] \qquad (3.20)$$

A symbol v_r which has the greatest a posteriori probability must be chosen according to the ideal observer criterion. Since the denominator in (3.20) does not depend on r it suffices to compare the numerators of this expression for all possible signals $z_r(t)$. Consequently, a decision system constructed in accordance with the ideal observer criterion and analyzing the Fourier coefficients of a received signal with frequencies to k_n must select symbol A_0 if, for all $r \neq 0$, for the given realization of $z'(t)$,

$$p_0 W_0[z'(t)]_n > p_r W_r[z'(t)]_n \qquad (3.21)$$

or

$$p_0 \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^{2n} \exp\left\{-\frac{1}{2\sigma^2}\sum_{k=0}^{n}[(a_k-\mu a_{0k})^2+(b_k-\mu b_{0k})^2]\right\} >$$

$$> p_r \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^{2n} \exp\left\{-\frac{1}{2\sigma^2}\sum_{k=0}^{n}[(a_k-\mu a_{rk})^2+(b_k-\mu b_{rk})^2]\right\} \qquad (3.22)$$

Reducing and taking the logarithms we derive an equivalent inequality expressing the decision principle when all $r \neq 0$.

$$\sum_{k=0}^{n}[(a_k-\mu a_{0k})^2+(b_k-\mu b_{0k})^2]+2\sigma^2\ln\frac{1}{p_0} <$$

$$< \sum_{k=0}^{n}[(a_k-\mu a_{rk})^2+(b_k-\mu b_{rk})^2]+2\sigma^2\ln\frac{1}{p_r} \qquad (3.23)$$

For a decision system which analyzes the received signal completely the decision principle may be derived from expression (3.23) by passing to the limit when $n \to \infty$

$$\sum_{k=0}^{\infty}[(a_k-\mu a_{0k})^2+(b_k-\mu b_{0k})^2]+2\sigma_0^2\ln\frac{1}{p_0} <$$

$$< \sum_{k=0}^{\infty}[(a_k-\mu a_{rk})^2+(b_k-\mu b_{rk})^2]+2\sigma_0^2\ln\frac{1}{p_r} \qquad (3.23a)$$

when all $r \neq 0$.

The same rule may be presented in integral form

$$\frac{1}{T}\int_0^T[z'(t)-\mu z_0(t)]^2\,dt+\sigma_0^2\ln\frac{1}{p_0} <$$

$$< \frac{1}{T}\int_0^T[z'(t)-\mu z_r(t)]^2\,dt+\sigma_0^2\ln\frac{1}{p_r}. \qquad (3.24)$$

It is easy to ascertain this by substituting the values of $z'(t)$ and $z_r(t)$ expressed as series (3.2). Taking into account the orthogonality of the trigonometric functions

$$\int_0^T [z'(t) - \mu z_r(t)]^2 dt = \int_0^T \sum_{k=1}^{\infty} [(A_k - \mu a_{rk})^2 \cos^2 k\omega t \, dt +$$
$$+ \int_0^T \sum_{k=1}^{\infty} (B_k - \mu b_{rk})^2 \sin^2 k\omega t \, dt = \frac{T}{2} \sum_{k=1}^{\infty} [(A_k - \mu a_{rk})^2 +$$
$$+ (B_k - \mu b_{rk})^2], \qquad (3.25)$$

whence, after simple transformations, follows the equivalence of inequalities (3.24) and (3.23a).

The decision principle in integral form (3.24) was first derived by V. A. Kotel'nikov [1]. In the particular case where the a priori probabilities of the symbols are identical, $p_r = p_i$, this principle adopts the simple form

$$\frac{1}{T} \int_0^T [z'(t) - \mu z_r(t)]^2 dt < \frac{1}{T} \int_0^T [z'(t) - \mu z_i(t)]^2 dt, \qquad (3.24a)$$

signifying that the decision system must select the expected signal $\mu z_r(t)$ which has the least mean-square deviation from received signal $z'(t)$. In this case inequality (3.23a) adopts the form

$$\sum_{k=1}^{\infty} [(A_k - \mu a_{rk})^2 + (B_k - \mu b_{rk})^2] < \sum_{k=1}^{\infty} [(A_k - \mu a_{ik})^2 +$$
$$+ (B_k - \mu b_{ik})^2]. \qquad (3.23b)$$

Let us observe that in expressions (3.23a) and (3.23b) it is in fact sufficient to take into account only the terms for which a_{rk} or b_{rk} are not identically equal to zero (at all values of r), because the remaining terms are identical on both sides of the inequalities. In other words, the number of coefficients A_k, B_k of the received signal which are significant when making a decision about the transmitted symbol is equal to the base of the system (3.5).

It is apparent that the principle expressed by (3.23b) or by (3.24a), which is equivalent to it, can be obtained with arbitrary values of the a priori probabilities of the symbols if the criterion of maximal likelihood is used instead of the criterion of the ideal observer. We will draw attention to the fact that this principle, in distinction from (3.24), does not require knowledge of the intensity of the interference which determines the dispersion σ_0^2. In this lies one more merit of the criterion of maximal likelihood.

Figure 3.2 shows the functional diagram of a device which operates in accordance with decision principle (3.24). Received signal z'(t) goes to r subtracting units, each of which is fed current $z_r(t)$ from one of the simulators of the expected signals to serve as the subtrahend. These $z_r(t)$ voltages must exactly reproduce the shape and size of the received signals and exactly coincide with them in time. The voltages from the subtracting devices are squared in corresponding nonlinear circuits with quadratic characteristics and are integrated, for example, by charging over great resistances of capacitors without leakage.[1] At the instant t = T the voltages pass from the capacitors to a comparator circuit so arranged as to present at its output the number of the capacitor with the lowest potential. It is apparent that the result of these operations is to determine the l-th symbol which satisfies inequality (3.24a). After this the capacitors discharge by instantaneous short circuit and the circuit is ready to receive the next signal element. In that case when the symbols are not equiprobable, the capacitors, instead of discharging, must charge to a potential numerically equal to $\sigma_0^2 \ln 1/p_r$. Here, as may easily be ascertained, the circuit will operate in conformity with the principle expressed by (3.24).

The circuit examined is, of course, not suited to practical use. In particular, it is very difficult to square accurately with a nonlinear circuit. This difficulty may, however, be circumvented by transforming the reception principle in expression (3.24a) or (3.23b). Opening the parentheses, reducing, and introducing the designation

$$\left.\begin{aligned} \frac{\mu^2}{T}\int_0^T z_r^2(t)\,dt &= \frac{\mu^2}{T}\sum_{k=1}^{n}(a_{rk}^2+b_{rk}^2) = P_r \\ \frac{2\mu}{T}\int_0^T z_r(t)\,x'(t)\,dt &= \mu\sum_{k=1}^{n}(A_k a_{rk}+B_k b_{rk}) = X_r \end{aligned}\right\} \qquad (3.26)$$

we derive the equivalent inequality

$$X_l - P_l > X_r - P_r \qquad (3.27)$$

when all r ≠ l.

---

[1] It is presumed that the resistors through which the capacitors discharge are so large that the discharge current is strictly proportional to the voltage at the output of the nonlinear circuit independently of the size of charge on the capacitor.

[2] In the following material decision principles and functional diagrams will be given for systems with equiprobable symbols. They also meet the criterion of maximal likelihood in the case of arbitrary a priori probability symbols.

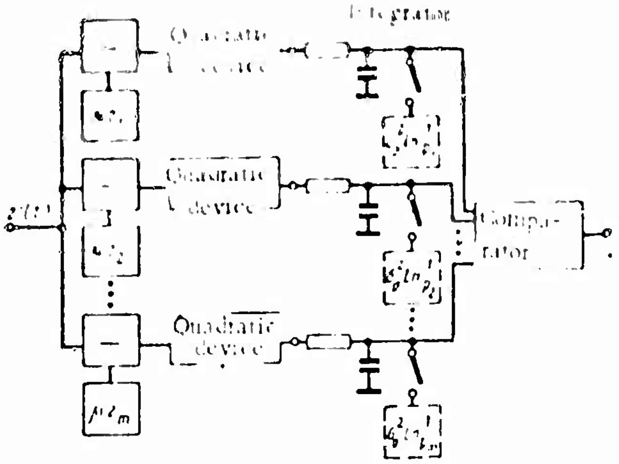


Figure 3.2. Decision System Realizing the Ideal Observer Criterion (Kotel'nikov's Criterion).

Here P_r is the average power of signal $z_r(t)$ at the input of the receiver, while X_r is the scalar product of received signal $z'(t)$ and expected signal $z_r(t)$.

The functional diagram (Figure 3.3), designed from inequality (3.27), contains m multipliers[1] to which come the received signal $z'(t)$ and the voltages $z_r(t)$ $(r = 1,\ldots,m)$ from signal simulators. The voltages from the output of each multiplier are integrated and the result of integration is fed to a subtracting device in which magnitude P_r is subtracted from it. At instant $t = T$ the voltages from all the subtracting devices are compared to each other in the comparator, which emits the number of the symbol for which voltage $X_i - P_i$ exceeds the other potentials $X_r - P_r$. Thereafter the potentials in the integrators are cut off and the circuit is ready to receive the next signal element.

The decision principle and its functional diagram may be greatly simplified if the signals $z_r(t)$ are so chosen that their power (or average power) is identical ($P_r = P_i$ = const). Then inequality (3.27) takes on this simple form

$$X_i > X \qquad (3.28)$$

---

[1] See first footnote on p. 148. In the given circuit the multiplier is often called the synchronous detector.

when all $r \neq l$, and the subtracting devices indicatated by the broken lines in Figure 3.3 may be omitted. But the simplification which can be obtained is not limited to this. Inequality (3.28) differs from (3.27) in that it does not depend on transmission coefficient μ and, consequently, when the signals are of equal power, it does not require a priori knowledge of the "scale" of the expected signals, but only of their shape, to bring about optimum reception in accordance with the ideal observer criterion. The signals generated by the simulators must coincide with the expected signals $z_r(t)$ in shape and, of course, must be strictly synchronized. As for the "scale" of the simulating signals, it may be arbitrary and the most convenient for practical realization, as long as it is the same in all simulators. In fact, when we increase the voltage of all the simulators by a factor of n we increase the voltage of all the simulators by a factor of n we increase X_r and X_l by the same number of times and hence do not affect the observance of inequality (3.28)

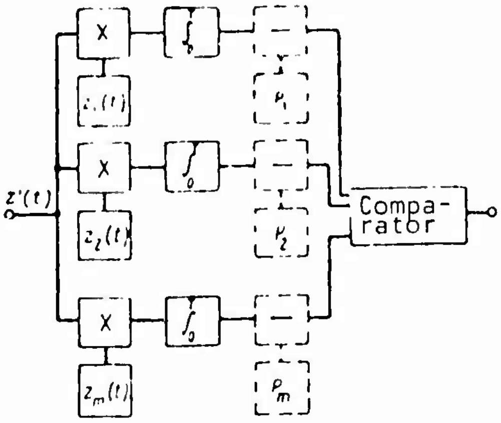


Figure 3.3. Variation of Decision System Realizing the Ideal Observer Criterion.

As will be apparent from the following, this important property of systems in which the power of the signal element does not depend on the transmitted symbol (feasibility of optimum reception without a priori knowledge of the propagation coefficient or even of the power of the emitted signal) is also maintained for channels with variable parameters. We will agree to call such systems "active-interval systems."

The circuits examined contain elements with variable parameters (short-circuited capacitors). They may, however, be converted so that they contain only elements with constant parameters and at the same time function in conformity with the reception principle of inequality (3.27) or (3.28) (for active-pause systems). This variant differs from those examined in that at the output of every multiplier is connected a linear pulsed filter instead of an integrating capacitance. This filter's pulse response is

$$g(t) = \begin{cases} 1 \text{ when } 0 < t < T. \\ 0 \text{ when } 0 < 0 \text{ and } t > T. \end{cases} \qquad (3.29)$$

This filter is physically realizable. The voltage at the filter's output at instant $t = t_1$ will, according to Duhamel's theorem [6], be

$$\begin{aligned} u_{\text{out}} &= \int_{-\infty}^{t_1} \mu z_r(x) z'(x) g(t_1 - x)\,dx \\ &= \mu \int_{t_1 - T}^{t_1} z_r(x) z'(x)\,dx, \end{aligned} \qquad (3.30)$$

where $z'(t)$ is the received signal and $z_r(t)$ is the voltage of the r-th local oscillator (signal simulator).

At the moment an element is finished the voltage at filter output represents the result of integration during the reception time of that element. At this instant the voltage at the outputs of the filters (or of the subtracting device are compared to each other and act on the recording device.

It must be remarked that with respect to the requirements for synchronization this variant has no advantages over the system in Figure 3.3. While the capacitance-integrating system needs to cut in the integrating circuits at certain instants the system with optimum filters behind the multipliers needs to feed voltage at certain instants to the recording device, and the requirements for accuracy of synchronization are identical in both cases. The optimum filter of (3.29) may in principle be realized with a delay line figured for time T, e.g., as shown in Figure 3.4.

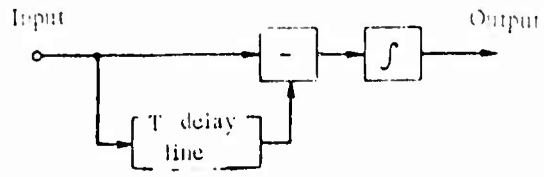


Figure 3.4. Diagram of Integrating Filter.

Let us adduce still another decision system design variant (Figure 3.5) which likewise contains no components with variable parameters and, furthermore, requires no signal simulators, but uses optimum matched filters instead.

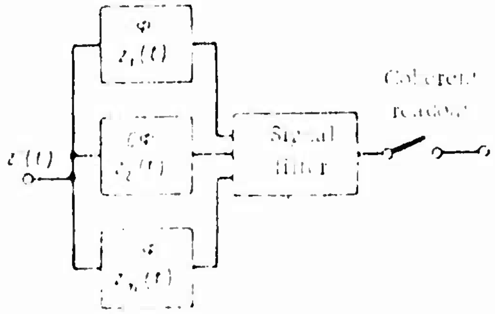


Figure 3.5. Decision System with Matched Filters and Coherent Reading.

Here, the received signal z'(t) goes to m filters matched respectively to the expected shapes of signal $z_r(t)$ ($r = 1,\ldots,m$). A filter matched with signal $z_r(t)$ means a filter whose pulse response satisfies the condition

$$g(t) = az(t_0 - t), \tag{3.31}$$

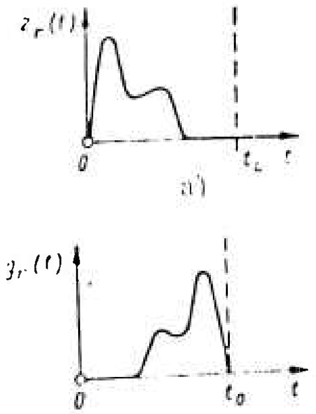
i.e., represents a mirror image of signal z(t) with respect to the ordinate axis shifted to the right by an amount equal to t_0 (Figure 3.6). This same condition can also be written in spectral form

$$K_r(j\omega) = aS^*_r(j\omega)e^{-j\omega t_0}, \tag{3.32}$$

where $K_r(j\omega)$ is the transfer function of a matched filter; a, an ordinary constant; $S^*_r(j\omega)$, a function complex-conjugate to spectral density $S_r(j\omega)$ of signal $z_r(t)$.

Since when $t < 0$, signal $z(t) = 0$, it then follows that from (3.31) $g(t) = 0$ when $t > t_0$, provided that $t_0 \geq T$. As is known, under this condition the filter is physically realizable. In the following we will assume $t_0 = T$, which has no effect on the generality of the results.

It is well known that a matched filter enables us at instant $t = t_0$ to obtain the maximum ratio of instantaneous signal value at its output to the mean-square value of interference. We, however, will not be interested in this property of the filter, but in the feasibility of realizing by means of it a system for putting the optimum decision principle into practice and thus for providing the greatest possible probability of correct signal identification.



a)

b)

Figure 3.6. Signal (a) and Pulse Response (b) of Filter Matched to it.

The voltage at the output of the matched filter at some instant t is, according to Duhamel's theorem

$$u_{out}(t) = \int_0^t z'(x)g_r(t - x)dx. \qquad (3.33)$$

Since $g_r(t-x) = az_r(T - t + x)$, then, taking into account that when x > t and when x < t − T, function $g_r(t - x) = 0$, we derive

$$u_{out}(t) = a\int_{t-T}^{t} z'(x)z_r(T - t + x)dx$$
$$= a\int_{t-T}^{t} z'(x)z_r(x + \tau)dx,$$

where the designation τ is introduced for T − t. At instant t = T

$$u_{out}(T) = a\int_0^T z'(x)z_r(x)dx = \frac{a}{\mu}X_r. \qquad (3.34)$$

Feeding the voltage u_{out} from the outputs of all the filters at instant t = T to the comparator circuit which selects the symbol x_r for which the greatest voltage has been produced, we obtain the system which realizes reception in conformity with the principle of expression (3.28).[1]

[1] This system may be generalized for the case of the system with unequal power in signals $z_r(t)$ by adding the appropriate subtracting devices.

Let us observe than when $t \geq 2T$ the voltage at the output of the matched filter and caused by signal $z_r(t)$ acting in interval $(0,T)$ equals zero. From this it follows that the readout instant when the next element is being received there is no voltage at the output of the matched filter caused by the preceding signal elements.

A matched filter with a transfer function (3.32) amounts to a linear circuit with constant parameters. Sometimes it is convenient to refrain from the demands of constancy in a filter and this yield additional possibilities for designing different variants of an optimal decision system. The idea behind their design is based on the fact that equality (3.34) holds if the pulse response of the filter satisfies condition (3.31) only over the interval $0 < t < T$, and when $t > T$ it may have any value. If a received signal $z'(t)$ is delivered to such a filter at instant $t = 0$, at instant $t + T$ a reading equal to [illegible] X_r can be taken from it inasmuch as the values of $q_r(t)$ when $t > T$ do not take part in the limits of integration (3.34).

However, with such a filter the remark made above to the effect that its reaction to preceding elements of a signal will fade completely by the instant of reading of the next element does not hold. Thus, (3.34) holds only for the first received element. This shortcoming is eliminated completely if after each reading the filter is brought to zero initial conditions by damping the oscillations. This can be done by shorting all capacitances of the filter for a instant and opening its inductances. Thereby such a filter becomes a circuit with variable parameters, periodically rejecting the energy accumulated in its elements.

It is convenient to select a function $q_r(t)$ of such a filter such that over the interval $0 < t < t_0 = T$ it satisfies condition (3.31) and when $t > T$ it continues periodically. In other words, this filter can be matched in the sense of conditions (3.31) and (3.32) with a periodically extended signal $z_r(t)$.

In the particular case of a simple circuit when $z_r(t)$ represents a segment of a sinusoid, an ideal oscillating circuit without damping with a resonance frequency of ω_p coinciding with the frequency of signal $z_r(t)$ shorted for an instant after each reading constitutes such a filter. In practice a circuit which is damped much less than $\delta_r T$ is used. Such circuits with variable parameters (with a periodic reset) have been called commutated filters.

All the decision systems considered above permit, at instant of reading $t = T$, obtaining at the input of the circuit comparisons of potential equal to the magnitudes of X_r (with an accuracy to a common factor). However, when $t < T$ the potentials at the output of the matched or commutated filter in Figure 3.5 differ greatly from the potentials at the output of the integrator in Figure 3.3. Let us illustrate this by the example where the signal is a quasi-harmonic with relatively slowly changing amplitude and phase with respect to ω:

$$z_i(t) = A(t)\cos[\omega t + \Phi(t)],$$

and $\omega T \gg 1$. Let us assume that signal $z_i(t)$ is actually being transmitted and that interference is so small that it can be neglected, so that $z'(t) = z_i(t)$ with a common factor of accuracy. Then at instant $t \leq T$ the potential in the integrator in the system in Figure 3.3 (or in the optimum filter in Figure 3.4) is

$$u_{int}(t) = \int_0^t A^2(x)\cos^2[\omega x + \Phi(x)]\,dx = \frac{1}{2}\int_0^t A^2(x)\,dx +$$

$$+ \frac{1}{2}\int_0^t A^2(x)\cos 2[\omega x + \Phi(x)]\,dx.$$

Since the integral of a rapidly oscillating function is approximately equal to zero, the second term can be ignored. Therefore,

$$u_{int}(t) \approx \frac{1}{2}\int_0^t A^2(x)\,dx, \qquad ([illegible])$$

i.e., the voltage at the integrator's output is a non-decreasing function and gradually rises from zero to its value when $t = T$. In particular, if $A = \text{const}$, this voltage rises linearly (Figure 3.7a).

The voltage at the output of a matched filter in the system in Figure 3.5 is defined by formula (3.33). Substituting in it the value $z'(t) = z_i(t) = A(t)\cos[\omega t + \Phi(t)]$ (when $0 \leq t \leq T$) and setting [illegible], we obtain (to a common factor of accuracy)

$$u_{mf}(t) = \int_0^t A(x)\cos[\omega x + \Phi(x)]\,A(x + T - t)\cos[\omega(x + T - t) +$$

$$+ \Phi(x + T - t)]\,dx = \frac{1}{2}\int_0^t A(x)A(x + T - t)\{[illegible]\}$$

$$[illegible]$$

$$[illegible]\,dx.$$

---

[1] It is easy to ascertain that the second integral has an order of $1/\omega$, whereas the first integral, order t. Granting that $\omega T \gg 1$, the second integral may be disregarded for all values of t which are essentially larger than $1/\omega$.

where [illegible] designates [illegible] the first [illegible]

$$[illegible]$$

where [illegible]

This [illegible]

$$[illegible]$$

[illegible] example [illegible] the system [illegible] voltage from the [illegible] will result [illegible] (Figure [illegible]) [illegible] this amount [illegible]

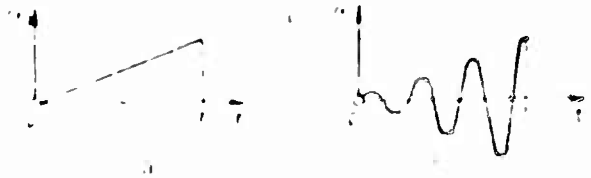


Figure [illegible] [illegible] Matched Filter [illegible]

[illegible] the [illegible] of the [illegible] systems [illegible] discussed in the next [illegible].

[illegible] Probability [illegible] Binary Code

We will first [illegible]

assuming that reception [illegible]

formed from the [illegible]

transmitted symbols are [illegible]

[illegible]

least possible [illegible]

[illegible]

[illegible]

[illegible]

and symbol [illegible]

[illegible]

[illegible]

[illegible]

The left side [illegible]
represented in integral form

$$\frac{\mu^2}{2}\sum_{k=1}^{n}[(a_{1k}-a_{2k})^2+(b_{1k}-b_{2k})^2]=\frac{\mu^2}{T}\int_0^T[z_1(t)-z_2(t)]^2dt=P_e, \qquad (3.38)$$

and it may be called the equivalent power of the signal pair $z_1(t)$ and $z_2(t)$. It is, in fact, the power of the difference between these signals.

The right side of inequality (3.37) represents the linear combination of the independent normally distributed random variables η_k, ξ_k and hence also has a normal probability distribution. Let us denote the right side by the letter ζ. Since all the terms on this side have a mathematical expectation of zero, then the mathematical expectation of ζ is also zero. The dispersion of the variable ζ equals the sum of the dispersions of the terms, i.e.,

$$D(\zeta)=\mu^2\sum_{k=1}^{n}[(a_{1k}-a_{2k})^2\sigma_k^2+(b_{1k}-b_{2k})^2\sigma_k^2]$$

$$=\mu^2\sigma^2\sum_{k=1}^{n}[(a_{1k}-a_{2k})^2+(b_{1k}-b_{2k})^2]$$

Considering (3.16) and (3.38) we get

$$D(\zeta)=2P_e\frac{\sigma^2}{T}. \qquad (3.39)$$

Knowing the mathematical expectation and dispersion of the normally distributed variable ζ, we can write the expression for its probability density,

$$[illegible] \qquad (3.40)$$

The probability of error is the probability that inequality (3.37) is [illegible], that is, inequality (3.37) is not fulfilled. In other words, error probability is the probability that random variable ζ will exceed P_e:

$$[illegible] \qquad (3.41)$$

where $\Phi(x)$ denotes a tabulated Kramp function.*

---

*Sometimes $\Phi(x)$ denotes other functions connected with the integral of the normal probability density, namely [illegible] and [illegible]

$$\Phi(x) = \sqrt{\frac{2}{\pi}}\int_0^x \exp\left(-\frac{t^2}{2}\right)dt \tag{3.42}$$

It is easy to ascertain that in the case where signal $z_2(t)$ is in fact transmitted the error probability also is determined from (3.41). Therefore, the discrete representation of such a channel is a uniform symmetrical binary channel, whatever signals $z_1(t)$ and $z_2(t)$ may be.

A very important conclusion follows from (3.41). The minimal probability of error is uniquely determined by the ratio between the equivalent power of a signal element P_eT and the spectral density of interference ν^2 (which also has power dimensions) and does not depend on other parameters, including the frequency band occupied by the signal.

The results derived may be simply interpreted in a geometrical way. We will regard the Fourier coefficients μa_k and μb_k of the incoming signals as rectangular coordinates in a B-dimensional space ($B = 2FT$ is the base of the system). Then each of the two signals $z_1(t)$ and $z_2(t)$ may be represented by a point (or vector) in this B-dimensional space. The received signal $z'(t)$ may also be represented by a point with coordinates A_k, B_k or by a vector which is the geometrical sum of the vector of the arrived signal $z(t)$ with components $\mu a_k, \mu b_k$ ($k = k_1, \ldots, k_2$) and of the interference vector with components ξ_k, η_k ($k = k_1, \ldots, k_2$). This presentation is implied by formula (3.3). It is apparent that the Fourier coefficients of the interference with frequencies lying outside of the F band cannot be taken into account since in the above-derived formulas for the reception principle and for the probability of error these coefficients are either absent or curtailed.

Decision principle (3.23) or (3.24) meanwhile acquire a very definite geometrical meaning. The optimum (in the sense of the maximal likelihood criterion) decision system must choose the possible signal whose point is nearer than the others to the point of received signal $z'(t)$. In the binary case this principle reduces to dividing the B-dimensional space into two semi-spaces by means of a hyperplane perpendicular to and bisecting the straight line connecting the points $z_1(t)$ and $z_2(t)$. If point $z'(t)$ lies in the same semi-space as $z_1(t)$ the decision system chooses symbol v_1, and vice versa. Figure 3.8 shows the plane drawn through points z_1, z_2, and z'. The straight line MN represents the intersection of this plane with the hyperplane dividing the space into two reception areas.

It is easy to see in this geometrical representation that the value of P_e (3.38) is half the square of the distance between the points representing signals $z_1(t)$ and $z_2(t)$. Denoting this distance[1] by D, we derive $D = \sqrt{2P_e}$

---

[1] Sometimes D is called the Kotel'nikov distance.

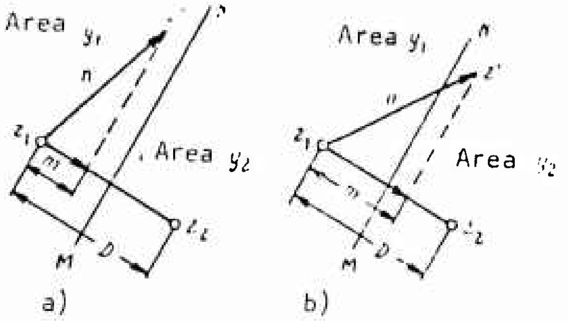


Figure 3.8. Geometrical Representation of Maximal Likelihood Criterion for a Binary System.

The variable [illegible] represents, as is apparent from the right side of inequality (3.37), the scalar product of the interference vector n(t) and the difference between the vectors of signals $z_1(t)$ and $z_2(t)$. But this scalar product may be represented by projecting m of interference vector n(t) on the straight line joining points z_1 and z_2 and multiplying by distance D between these points. Thus, inequality (3.37) may be written as

$$\frac{1}{2} \cdot D^2 > mD$$

or

$$m < \frac{D}{2} \cdot \tag{3.43}$$

This inequality is the condition for correct reception of a transmitted symbol. Therefore the error probability is the probability that inequality (3.43) will not be fulfilled, i.e., that the projection m of the interference vector will be more than half the distance between the points representing signals $z_1(t)$ and $z_2(t)$. It is apparent that under this condition the point representing received signal $z'(t)$ will not be in the semi-space in which is situated the point corresponding to the actually transmitted signal (Figure 3.8b), and an error will occur. It is not difficult to derive expression (3.41) starting from nonfulfillment of condition of inequality (3.43).

We would observe that error probability is affected only by the interference vector component which coincides in direction with the straight line connecting the points representing signals $z_1(t)$ and $z_2(t)$. The interference components perpendicular to this direction do not affect the error probability. This property is characteristic of optimum reception methods in the cases where the initial signal phases are known a priori. These reception methods are usually called coherent.

In order to secure the greatest potential resistance to interference in a binary system the signals $z_1(t)$ and $z_2(t)$ must be so chosen as to produce the greatest possible P_e. Usually the average signal power is prescribed, which with equiprobable a priori symbols y_1 and y_2 is $P_s = 1/2(P_1 + P_2)$. The expression for P_e (3.38) may be represented as:

$$P_e = \frac{\mu^2}{T}\left[\int_0^T z_1^2(t)\,dt + \int_0^T z_2^2(t)\,dt - 2\int_0^T z_1(t)\,z_2(t)\,dt\right] =$$
$$= P_1 + P_2 - \frac{2\mu^2}{T}\int_0^T z_1(t)\,z_2(t) \tag{3.44}$$

With given values of $P_1 + P_2$, μ, and T, a maximum of P_e is obtained if $\int_0^T z_1(t)\,z_2(t)\,dt$ is negative and maximum in absolute value. According to the Bunyakovskiy-Shvarts inequality

$$\left|\int_0^T z_1(t)\,z_2(t)\,dt\right| \leq \sqrt{\int_0^T z_1^2(t)\,dt\int_0^T z_2^2(t)\,dt} = \frac{T}{\mu^2}\sqrt{P_1P_2},$$

and equality is attained if $z_2(t) = Cz_1(t)$ where C is an arbitrary constant. On the other hand, average geometrical $\sqrt{P_1P_2}$ attains a maximal value equal to the arithmetic average $P_s = 1/2(P_1 + P_2)$ on condition that $P_1 = P_2$, i.e., when $|C| = 1$. Thus, to obtain a maximum of P_e, it is necessary to take $C = 1 -1$, i.e., $z_2(t) = -z_1(t)$. When this is so all terms in the Fourier series for $z_2(t)$ have the same amplitude as do terms of $z_1(t)$, but there phases are shifted 180°.

Thus, the binary system which is optimal with respect to resistance to interference is the one with an active pause and opposite signals. Substituting $z_2(t) = -z_1(t)$ in (3.44) we obtain for it $P_e = 4P_s$. Consequently, the probability of error according to (3.41) is

$$p = \frac{1}{2}\left[1 - \Phi\left(\sqrt{\frac{2P_sT}{\nu_0}}\right)\right] = \frac{1}{2}\left[1 - \Phi(\sqrt{2}h)\right], \tag{3.45}$$

where we introduce the designation

$$h^2 = \frac{P_sT}{\nu_0}. \tag{3.46}$$

The magnitude h^2 represents the ratio of average signal element power at receiver input to the spectral density of the interference. We will use this symbol throughout the whole book.

We are often interested not in the ratio of signal element power to spectral interference density, but in the ratio $(P_s/P_n)_F$ of the powers of signal to noise at receiver input in frequency band F or in the ratio of the spectral densities of the signal and interference. Since $P_n =$ [illegible] F, the following ratios occur:

$$h^2 = \left(\frac{P_s}{P_n}\right)_F FT = [illegible]\, FT, \qquad (3.47)$$

where [illegible] $= P_s/F$ is average spectral signal density.

From this it follows that a prescribed value of h may be produced with any arbitrarily small ratio $(P_s/P_n)_F$ or [illegible], if the signals have a large enough base $B = 2FT$.

For example, when $h = 3$ the probability of error in the opposite-signal system is $1/2[1 - \Phi(3\sqrt{2})] \approx 2\cdot10^{-}$[illegible]. If here $FT = 1$, the ratio $(P_s/P_n)_F$ needed to produce this reception reliability is $h^2 = 9$. But if the signals have a large base, e.g., if $FT = 100$, the same error probability will be obtained when $(P_s/P_n)_F = h^2/FT = 0.09$.

It must not be thought, however, that the employment of wideband signals (signals with large base) permits transmitter power reduction at a given reception fidelity. In fact, the interference power in frequency band F is proportional to this band and the transmitter power needed to secure a given value of h is $P_s/$[illegible]$^2 = h^2 P_n/$[illegible]$^2 FT = h^2$[illegible]$_0^2/$[illegible]$^2 T$. Therefore if the givens are signal element duration T, transmission factor [illegible], and spectral interference density [illegible], the transmission power needed is uniquely determined by the required value h and does not depend on frequency band F occupied by the signal. Reduction of required transmission power (or reduction of error probability at given transmitter power) can be accomplished only by increasing [illegible] (e.g., by using directional antennas), by decreasing [illegible] (e.g., if interference is caused by internal receiver noises, by lowering the noise factor), or, finally, by increasing T (decelerating transmission).

If signals z(t) are relatively narrow-band in the sense that frequency band F is considerably lower than the average frequency of the signal spectrum (which practically always occurs in radio channels), then, as is well known, any signal may be represented as the quasi-harmonic

$$z(t) = E(t)\cos[\omega_{av} t + \varphi + \Phi(t)], \qquad (3.48)$$

where E(t) and $\Phi(t)$ are relatively slowly changing time functions, and during one period of "high-frequency filling" $2\pi/\omega_{av}$ the values of E and Φ may with sufficient accuracy be considered constant.

Under this condition the average signal strength is

$$P_s = \frac{1}{2T}\int_0^T E^2(t)\,dt. \qquad (3.49)$$

and peak (maximum) power

$$P_{\max} = \frac{1}{2}E_{\max}^2, \qquad (3.50)$$

where $E_{\max}$ is the maximum value of envelope E(t).

From the above findings it is clear that peak power value does not directly affect error probability. If, as is often the case, transmitter peak power is limited, then in order to increase resistance to interference a signal shape must be chosen such that the given peak power $P_{\max}$ affords the greatest possible average power, P_s, i.e., signals with the least envelope peak factor must be used.[1] It is apparent that this condition will be fulfilled if E(t) = const, since in this case $P_s = P_{\max}$ and the peak factor is unity.

One simple binary system with opposite signals and with a peak factor of unity is the system with 180° phase keying (FT), in which these signals are used:

$$\left.\begin{aligned} z_1(t) &= a\cos(\omega t + \varphi), \\ z_2(t) &= -a\cos(\omega t + \varphi) = a\cos(\omega t + \varphi + \pi) \end{aligned}\right\} \qquad (3.51)$$

It is very simple to produce an optimum coherent decision system (Figure 3.3) for such signals, since a single generator of cos(ωt + φ) harmonics is used as the signal simulator whose voltage is connected to a single multiplier as shown in Figure 3.9a. This system records symbol v_1 if at the readout moment t = T the voltage at integrator output (or at optimum filter output) is positive, and symbol v_2 is this voltage is negative. It is easily ascertained that this system conforms to the principle of expression (3.28). Indeed, when $z_2(t) = -z_1(t)$, it follows from expression (3.26) that $X_2 = -X_1$ and the principle of expression (3.28) may be converted to the form

$$X_1 > 0. \qquad (3.52)$$

It can easily be seen that this system is suitable for any binary system with opposite signals if a voltage proportional to $z_1(t)$ is delivered as signal simulator.

A simpler decision system can be designed without a multiplier (Figure 3.9b) which is equivalent to the preceding one and which is called a phase detector. Here the sum and difference of the arriving signal z'(t) and the support voltage of a local oscillator $\cos(\omega t + \varphi)$ are formed and then detected separately by mean-square detectord KD and the difference of the detected voltages is integrated over an interval from 0 to T. As in the preceding system, a decision as to a transmitted symbol is reached based on the sign of voltage U at the output of the integrator at instant of reading t = T.

---

[1] The value $\sqrt{P_{\max}/P_s}$ is here called the peak factor.

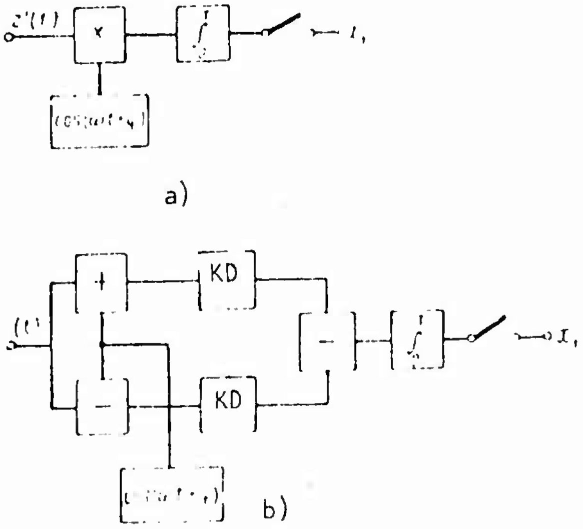


Figure 3.9. Functional Diagrams of Coherent Reception for an FT System: a, With a multiplier: b, With a phase detector.

In saying that these systems are equivalent, we assert that with delivery to them of the same signal z'(t) the signs of the voltages at their outputs will be the same. Consequently, incorrect decisions will arise simultaneously in both systems.

To prove this assertion we will write the values of the output voltages of both systems at the instant of reading:

$$X_1 = C\int_0^T z_1(t)\, z'(t)\, dt,$$

$$U = \int_0^T \{[Cz_1(t) + z'(t)]^2 - [Cz_1(t) - z'(t)]^2\}\, dt,$$

where C is an arbitrary constant.

Removing the parentheses we obtain

$$U = 4C\int_0^T z_1(t)\, z'(t)\, dt.$$

i.e., the magnitude of U is proportional to X_1 and, consequently, has the same sign.

In addition to the opposite-signal systems there are a number of other binary systems which are of interest. We will here mention two classes of systems--those with orthogonal signals and those with a passive interval.

In systems with orthogonal signals the functions $z_1(t)$ and $z_2(t)$ satisfy the condition of orthogonality in the interval (0,T):

$$\int_0^T z_1(t) z_2(t)\,dt = 0. \tag{3.53}$$

In this case it follows from expression (3.44) that $P_e = P_1 + P_2 = 2P_s$. Substituting this value in expression (3.41) we will find the error probability in a system with orthogonal signals:

$$p = \frac{1}{2}\left[1 - \Phi\left(\sqrt{\frac{PT}{N}}\right)\right] = \frac{1}{2}\left[1 - \Phi(h)\right] \tag{3.54}$$

Comparing expressions (3.54) and (3.45) we may observe that in order to obtain identical signal reception probabilities in an orthogonal system we must have a power $\sqrt{2}$ times greater than in a system with opposite signals. It is advantageous to have $P_1 = P_2$ to reduce required peak power, as well as to simplify the decision system.

Examples of simple orthogonal systems are the system with 90° phase keying with signals

$$\left.\begin{aligned} z_1(t) &= a\cos(\omega t + \varphi), \\ z_2(t) &= a\sin(\omega t + \varphi) = a\cos\left(\omega t + \varphi + \frac{\pi}{2}\right). \end{aligned}\right\} \tag{3.55}$$

and the system with frequency keying (ChT) with signals

$$\left.\begin{aligned} z_1(t) &= a\cos(k_1\omega t + \varphi), \\ z_2(t) &= a\cos(k_2\omega t + \varphi) \end{aligned}\right\} \tag{3.56}$$

Systems with signals of the following types may also serve as examples of orthogonal systems with an active interval:

$$\left.\begin{aligned} z_1(t) &= a\cos(k_1\omega t + \varphi_1) + a\cos(k_2\omega t + \varphi_2) \\ z_2(t) &= a\sin(k_1\omega t + \varphi_1) + a\cos(k_2\omega t + \varphi_2) \end{aligned}\right\} \tag{3.57}$$

when

$$\left.\begin{array}{l} z_1(t)=\begin{cases} a\cos(\omega t+\varphi) & \text{when } 0\leqslant t<T/2,\\ 0 & \text{when } T/2\leqslant t<T,\end{cases}\\ z_2(t)=\begin{cases} 0 & \text{when } 0\leqslant t<T/2,\\ a\cos(\omega t+\varphi) & \text{when } T/2\leqslant t<T.\end{cases}\end{array}\right\} \tag{3.58}$$

when

$$\left.\begin{array}{l} z_1(t)=\begin{cases} a\cos(k_1\omega_0 t+\varphi) & \text{when } 0\leqslant t<T/2,\\ a\cos(k_2\omega_0 t+\varphi) & \text{when } T/2\leqslant t<T,\end{cases}\\ z_2(t)=\begin{cases} a\cos(k_2\omega_0 t+\varphi) & \text{when } 0\leqslant t<T/2,\\ a\cos(k_1\omega_0 t+\varphi) & \text{when } T/2\leqslant t<T.\end{cases}\end{array}\right| \tag{3.59}$$

As many such examples as desired could be given.

As is apparent from (3.54), the error probabilities for all the systems given here are the same, provided that the average power of the incoming signals and the duration T of the signal elements are also identical and if reception occurs with an identical level of white noise. Some of the systems listed have an advantage over others with respect to the peak signal power required. These are systems (3.55), (3.56), and (3.59), whose peak factor is unity. Various additional considerations, some of which will be discussed below, prompt us to prefer one or another of these systems in various specific cases.

Systems with a passive interval are a particular case of orthogonal systems. One of the signals, e.g., $z_1(t)$, may be any time function, and $z_2(t) = 0$. It is apparent that here the condition for orthogonality (3.53) is fulfilled and the probability of error is defined by expression (3.54). It must, however, be borne in mind that in this case $P_s = (P_1 + P_2)/2 = P_1/2$, i.e., the power of signal $z_1(t)$ must be twice as great as the signal power in the equivalent orthogonal system with an active interval.

An example of a system with a passive interval is the simple system with multiple keying (AM) and signals

$$\begin{aligned} z_1(t) &= a\cos(\omega t+\varphi),\\ z_2(t) &= 0. \end{aligned} \tag{3.60}$$

Let us observe that the error probability in coherent reception for any binary system may, in conformity with expression (3.41), be written as

$$p=\frac{1}{2}\,[1-\Phi(\gamma h)], \tag{3.61}$$

where coefficient γ depends on the scalar product of the signals and is equal to

$$\gamma = \sqrt{\frac{P_1}{P_2}}\left(1 - \frac{\mu^2}{TP_2}\int_0^T z_1(t) z_2(t)\,dt\right)^{\frac{1}{2}}. \tag{3.61a}$$

Coefficient γ may take values from zero (when $z_1 = z_2$) to $\sqrt{2}$ (when $z_2 = -z_1$). When $\gamma = 0$ the probability of error, as should be expected, is equal to 1/2 and the carrying capacity of the channel in accordance with (2.28) is equal to zero. In actual fact, signals in this case are indistinguishable even when there is no interference. For orthogonal systems $\gamma = 1$.

The probability of correct reception of a symbol may be expressed as:

$$1 - p = \frac{1}{2}\left[1 + \Phi(\gamma h)\right] \tag{3.61b}$$

Figure 3.10 shows the dependence of p on h as determined from formula (3.61).

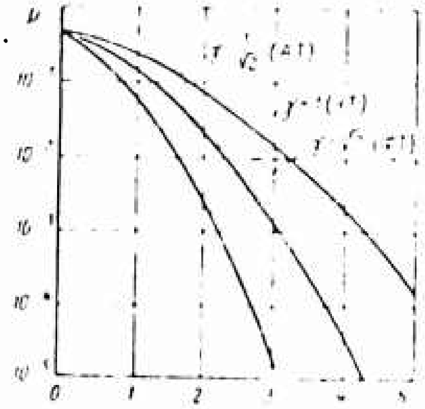


Figure 3.10. Error Probability with Coherent Reception in Binary Systems.

3.5. Error Probabilities and Potential Resistance to Interference with Code Base m 2

If a code with base m 2 is used the optimum decision principles of expression (3.28) or (3.27) (depending on whether the power of all m signals variants is equal) remain in force. An error in receiving symbol y_1 occurs in cases where inequality (3.28) or (3.27) is not fulfilled for at least one of the values of subscript r.

Just as in the binary case, it is convenient here to apply a geometrical interpretation, regarding the possible incoming signals as m points in a B-dimensional space with coordinates a_{rk}, rk. Errors will occur here every time that a point representing received signal $z'(t)$ is farther from point $z_1(t)$ representing the signal which actually arrived than it is from any other point $z_r(t)$.

The distance between the pair of signal points $s_r(t)$ and $s_s(t)$ in this model is

$$D_{rs} = \mu\sqrt{\sum_k [(a_{rk}-a_{sk})^2+(b_{rk}-b_{sk})^2]} = \sqrt{\frac{2\mu}{T}\int [s_r(t)-s_s(t)]^2\,dt} \qquad (3.62)$$

The mutual disposition of the signal points is uniquely determined by the aggregate of all the paired distances D_{rs}. If symbol x_r is transmitted, the point representing $z(t)$, the signal received, is closer to some signal point $s_s(t)$ than to $s_r(t)$ in the case when the projection of the interference vector onto the straight line connecting points $s_r(t)$ and $s_s(t)$ is positive and exceeds $D_{rs}/2$. For white noise the interference vector is spherically symmetric, with the result that such a probability does not depend on the direction of the indicated vectors but only on the distance between them. [illegible] ... the same for any pair of points [illegible] is determined by the mutual disposition of the signal points [illegible] the appropriate D_{rs} [illegible] systems with the same arrangement of signal points are called equivalent. In a number of cases the resistance to interference of a system with signals of complex structure may be most fully [illegible] with simpler signals.

[illegible]

[illegible]

[illegible] In general, [illegible]

The specific shapes of the signals in equidistant systems may be different. Thus, for example, when m = 3, the equidistant systems may contain signals like

a) [illegible]

[illegible]

[illegible]

(120° and 240° phase keying);

b) [illegible]

[illegible]

$s_3(t) = 0$;

c) [illegible]

[illegible]

[illegible]

d) [illegible]

[illegible]

[illegible]

e) [illegible]

[illegible]

[illegible]

(frequency keying, etc.).

The geometrical representation of these systems [illegible] are shown in Figure 5.11. [illegible] ascertain that for all the examples given [illegible] the signal power, however, in these cases varies. Examples a, b, [illegible] an active interval since for them [illegible]. In particular, for examples a and b, [illegible] 2, while for examples [illegible] 1. [illegible] and e are not systems with an active interval. [illegible] symbols [illegible] considered equiprobable [illegible]. In this case [illegible] for example [illegible] and [illegible] for example [illegible]. At the same time the error probabilities for all these systems are identical, [illegible] of d_r are identical.

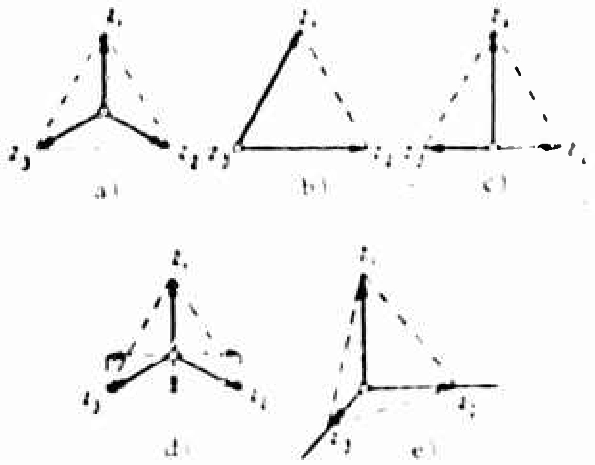


Figure 3.11. Geometrical Representation of Signals of Equidistant Ternary Systems.

Let us observe that the signals in example e are orthogonal. It is easy to ascertain that any system of m mutually orthogonal signals with equal power are equivalent. In fact, let $z_r(t)$ and $z_\ell(t)$ represent any two signals of this system. Then from (3.61)

$$D_{r\ell} = \frac{1}{T}\int_0^T [z_r(t) - z_\ell(t)]^2 dt =$$

$$= \frac{\mu^2}{T}\int_0^T [z_r^2(t) + z_\ell^2(t) - 2z_r(t)z_\ell(t)]dt = 2P,$$

since

$$\frac{\mu^2}{T}\int_0^T z_r^2(t)dt = \frac{\mu^2}{T}\int_0^T z_\ell^2(t)dt = P,$$

and $a\int_0^T z_r(t)z_\ell(t)dt = 0$ from the condition of orthogonality.

Therefore, $D_{r\ell}$ does not depend on subscripts r, ℓ = 1, 2, ..., m and is the same for any pair of signals. In other words, orthogonal systems with an active interval are a particular case of equidistant systems.

Let us compute the probability of correct reception for an orthogonal system with an active interval with an optimum decision system. Since all orthogonal systems with an active interval at a given signal power are isomorphic, let us select a simple system with frequency keying which contains signals

$$\left.\begin{array}{l} z_1(t) = a\cos k_1\omega_1 t, \\ z_2(t) = a\cos k_2\omega_1 t, \\ \dots\dots\dots\dots \\ z_m(t) = a\cos k_m\omega_1 t \end{array}\right\} \quad (3.63)$$

Let us, for the sake of definiteness, assume that signal [illegible] was transmitted.

The optimum decision system for the active orthogonal system operates in conformity with the principle of expression (3.58), which may be simplified for the given specific case. From expression (3.[illegible]) we find, by taking expression (3.[illegible]) into account, that

$$X_r \begin{cases} [illegible] & \text{when } r \neq [illegible] \\ [illegible] & \text{and when } r = [illegible] \end{cases}$$

Substituting these expressions into (3.58) and dividing both sides of the inequality by [illegible], we obtain the rule for selecting symbol x_r for the given orthogonal system:

$$[illegible] \qquad \text{when all } r \neq [illegible] \tag{3.64}$$

The probability of correct reception is the probability that with all $r \neq$ [illegible] inequality (3.64) will be fulfilled. But since all [illegible] represent independent random variables with the same normal probability distribution, the conditional probability of correct reception of symbol [illegible] with some [illegible] is [illegible]$]^{m-1}$. To find the total probability of correct reception, this expression must be averaged with respect to [illegible]:

$$[illegible] \tag{3.65}$$

where substitution of the variables [illegible] is made.

For a given orthogonal system [illegible]. Therefore, correct reception probability (3.65) may be expressed by [illegible] to obtain in this way the general formula for all isomorphic equidistant systems

$$q = [illegible] \int [1 - [illegible]]^{[illegible]} \exp\left[-\frac{1}{2}\left(u - [illegible]\right)^2\right] du \tag{3.66}$$

In the orthogonal system [illegible] (see section [illegible]) [illegible].

Taking into account that [illegible] we may express the probability of correct reception for the orthogonal system thus

$$[illegible]$$

The integral in expression [illegible] cannot be expressed [illegible] functions and numerical [illegible].

The expressions [illegible] show that correct reception probability, [illegible] in binary systems, is determined by the ratio [illegible] of signal element power to spectral noise density and does not depend on the frequency band occupied by the signal. When [illegible], expression (5.5[illegible]) may be derived from expression [illegible].

The orthogonal system, however, is not optimum. An isomorphic system may be constructed which secures the same error probability with less signal power. In order to find the least possible power of an equidistant system with identical a priori symbol probabilities the origin of the coordinates must, without change in the mutual arrangement of the signal points, be placed at such a point that the sum of the squares of the distance from it to the signal points be minimum. This is nothing but the problem of finding the center of gravity when identical masses are lumped at the signal points. In Figure 5.11 this dimension occurs in examples a and d.

Minimum signal power with given [illegible] in an equidistant system may also be found from geometrical considerations. It equals [illegible], where [illegible] is radius of an (m − 1)-dimensional hypersphere described about a simplex of side [illegible].

From geometry it is known that [illegible], whence [illegible]. Substituting the value of [illegible] in (5.6[illegible]) we will derive the greatest probability of correct reception in the equidistant system with signal power [illegible]

$$[illegible] \qquad (5.6[illegible])$$

In the case where the a priori symbol probabilities are not identical, minimum average signal power is also secured by putting the coordinate origin at the center of gravity, but in this case a mass proportional to the a priori probability of the corresponding symbol must be attributed to every signal point.

Neither can this integral be represented by elementary functions.

With equal a priori signal probabilities and optimum decision system all the probabilities of transitions in an equidistant system are identical and obviously are

$$p_{ik} = \frac{1 - q}{m - 1}, \quad i \neq k$$

whence it follows that a discrete representation of the channel in this case is symmetrical.

Nonequidistant systems do not preserve this property, since when they are used the discrete representation of a channel with constant parameters and additive white noise is not, generally speaking, symmetrical.

The class of so-called biorthogonal systems may be cited as an example of nonequidistant systems. In these systems m is always even and for every signal $z_1(t)$ there exists an opposite signal $-z_1(t)$, and the remaining (m - 2) signals are orthogonal to signal $z_1(t)$. The following system may serve as an example of a biorthogonal system with an active interval:

$$\left.\begin{aligned} z_1(t) &= a\cos k_1\omega_0 t,\\ z_2(t) &= a\cos k_2\omega_0 t,\\ &\ldots\ldots\ldots\ldots\\ z_{\frac{m}{2}}(t) &= a\cos k_{\frac{m}{2}}\omega_0 t,\\ z_{\frac{m}{2}+1}(t) &= -a\cos k_1\omega_0 t,\\ z_{\frac{m}{2}+2}(t) &= -a\cos k_2\omega_0 t,\\ &\ldots\ldots\ldots\ldots\\ z_m(t) &= -a\cos k_{\frac{m}{2}}\omega_0 t. \end{aligned}\right\} \qquad (3.69)$$

The biorthogonal system's probability of correct reception, computed by analogy with formula (3.65), is

$$q_{\text{biort}} = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} [2F(u) - 1]^{\frac{m}{2}-1} \times \exp\left[-\frac{1}{2}\left(u - \sqrt{2h^2}\right)^2\right] du \qquad (3.70)$$

This probability of correct reception is less than for an equidistant system with the same values of m and h. The biorthogonal systems, however, make it possible to obtain a given value of m with a smaller signal base than do the equidistant systems. This is sometimes an advantage.

Another example of a nonequidistant system is a phase keying (PK) system:

$$z_r(t) = a\cos\left(k\omega_0 t + \frac{2\pi(r-1)}{m}\right), \qquad (3.71)$$
$$r = 1, \dots, m.$$

In this system, regardless of m, the base is equal to 2 and the signals may be represented by vectors on a plane. When m = 2 and m = 3 this system is equidistant (for m = 3 its geometric representation is given in Figure 3.11a) and when m = 4 it is biorthogonal. Therefore, for these values of m the probability of correct reception can be found from formulas (3.61b), (3.68) and (3.70) respectively. Specifically, for m = 4 from (3.70) we may easily obtain by a substitution of variables

$$\xi = \frac{u+v}{\sqrt{2}}h, \quad \eta = \frac{u-v}{\sqrt{2}}h,$$

$$q_{ft_{m=4}} = \frac{1}{2\pi}\int_0^{\infty} du \int_{-u}^{u} \exp\left[-\frac{(u-\sqrt{2h})^2+v^2}{2}\right]dv + \frac{1}{2\pi}\int_{-h}^{\infty}\int_{-h}^{\infty}\exp\left[-\frac{\xi^2+\eta^2}{2}\right]d\xi\, d\eta = \frac{1}{4}\left[1+\Phi(h)\right]^2. \qquad (3.70a)$$

In the general case it follows from decision principle (3.24) that in a system with a matched filter a decision that signal $z_r(t)$ was transmitted must be reached if the initial phase of the voltage at the output of the filter is within the limits from $(2r-3)\pi/m$ to $(2r-1)\pi/m$. An error occurs if the phase lies outside these limits. By using as a probability distribution phases of a sinusoidal signal with gaussian noise [20], we can compute the probability:

$$P_{FT} = 1 - \frac{1}{2}\Phi\left(\sqrt{2h}\sin\frac{\pi}{m}\right) - \frac{1}{m} - 2V\left(\sqrt{2h}\sin\frac{\pi}{m}, \sqrt{2h}\cos\frac{\pi}{m}\right). \qquad (3.71a)$$

where $V(x, y) = \frac{1}{2\pi}\int_0^x\int_0^{\frac{y}{x}t} e^{-\frac{t^2+u^2}{2}}\,du\,dt$ is a Nicholson tabulated function.

With large values of m and h the following evaluation is quite accurate:

$$P_{FT} < 1 - \Phi\left(\sqrt{2h}\sin\frac{\pi}{m}\right). \qquad (3.71b)$$

Generally speaking, with an increase in the base of code m, if the power of the signal P_sT and the spectral density of the noise remain unchanged, the probability of correct reception of an element decreases. If the power of the signal and the rate of information transmission remain the same (in this case T and consequently h^2, increase proportionally to log m), the equivalent probability of correct reception may increase. An example of this is provided by the comparison of orthogonal systems with m = 2 and m = 32 provided in [1].

3.6. Decision System and Resistance to Interference

In the preceding sections it was assumed that the additive interference was normal white noise. Let us see how the findings change if the interference is normal, as before, but not white.

The problem of choosing the optimum decision system and figuring the probability of correct (or incorrect) symbol reception in normal noise with a nonuniform spectrum may be reduced to an analogous problem with white noise by using the following method which was first proposed by V. A. Kotel'nikov [1].

The problem of designing the optimum decision system[1] (optimum receiver) with any given signal parameters has been solved in the case of white noise (see Section 3.3). Now let a signal plus normal noise with a spectral power density of $G(\omega)$ be present at receiver input. If this mixture of signal and noise is passed through line filter Φ_1 with frequency characteristic $\Phi(j\omega)$ satisfying with accuracy to a constant factor the condition

$$|\Phi(j\omega)|^2 = \frac{1}{G(\omega)}, \qquad (3.72)$$

the noise at the filter output will remain normal (since filter Φ_1 is linear), but will prove to be white (since its power spectrum will be $G(\omega)|\Phi(j\omega)|^2 = 1$). The signals at the output of filter Φ_1 will, of course, be different from the signals at its input. Since we know the expected signals $z_1(t), z_2(t), \ldots, z_m(t)$ at filter input, however, and we determine filter characteristics $\Phi(j\omega)$, we can find signals $z_1(t), z_2(t), \ldots, z_m(t)$ at the output of filter Φ_1. Let us observe that condition (3.71) defines only the modulus of the frequency characteristic of filter Φ_1, while its phase characteristic may be chosen at will. The physical feasibility of such a "whitening" filter is assured if the spectral power density $G(\omega)$ satisfies certain conditions[2], in particular, if it does not take on the values of zero or infinity in a finite segment of frequencies ω.

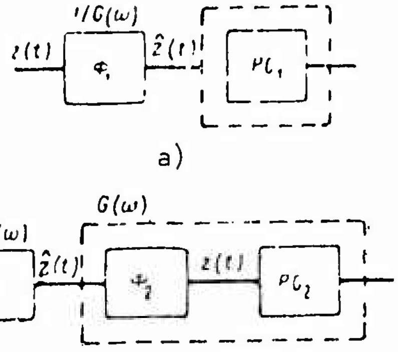
Let us now connect the output of filter Φ_1 to the optimum decision system PC_1 designed for the new set of signals $z(t)$ received against a background of white noise (Figure 3.12a). We will demonstrate that the combination of filter Φ_1 and decision system PC_1 is the optimum decision system for signals $z(t)$ received against a given background of colored noise.

---

[1] Although the present chapter deals with optimum decision systems principally from the angle of the maximal likelihood criterion the following discussion holds true for any criterion of optimality.

[2] The conditions for the realizability of filter Φ_1 coincide with those under which the noise with power spectrum $G(\omega)$ is indeterminate and these conditions are always fulfilled in practice [3].

Let us assume that our statement is untrue. Then there must exist some other decision system PC_2 which better satisfies the criterion of optimality than the system shown in Figure 3.12a. Let us connect two filters in series to decision system PC_2 input--filter Φ_1 which was mentioned above with frequency characteristic $\Phi(j\omega)$ and filter Φ_2 with frequency characteristic $1\Phi(j\omega)/\Phi(j\omega)$, (Figure 3.12b)[1]. Since the series connected filters Φ_1 and Φ_2 do not change the signal and interference entering them, the system of Figure 3.12b is entirely equivalent to decision system PC_2. White noise and signals of $\tilde{z}(t)$ are present at the output of filter Φ_1. Therefore, that part of the Figure 3.12b system contained in the broken line may be regarded as the decision system which signals $z(t)$ arrive against a background of white noise.



a)

b)

Figure 3.12. Decision System with Gaussian Noise and Nonuniform Spectrum.

According to the assumption made the decision system PC_2 satisfies the criterion of optimality for signals $z(t)$ against a background of white noise better than does the system of Figure 3.12a. Hence, correct decisions will be found more often at the output of the Figure 3.12b system that at the output of the Figure 3.12a system. If this is so, the part of the system in Figure 3.12b within the broken line better satisfies the criterion of optimality for signals $z(t)$ against a background of white noise than does decision system PC_1. But this contradicts the condition according to which PC_1 is the optimum decision system for signals $z(t)$ with white noise. This contradiction proves that the Figure 3.12a system is the optimum decision system for the initial $z(t)$ signals received against a background of normal noise with power spectrum $G(\omega)$.

---

[1] Physical realizability of filter Φ_2 is not required here, since this filter is needed only for the line of reasoning being pursued. It is alone sufficient that series connection of filter Φ_2 and decision system PC_2 be physically realizable [19]. It may be proved that this condition is always fulfilled if signals $z(t)$ are of finite duration and power.

A question is often raised as to the case in which maximum probability of reception of given signals will be higher--with white noise or with noise of the same strength but with a nonuniform spectrum. In the form stated this question is not definite enough, since how to compare noise intensity is not indicated. It is impossible to compare white and colored noise at full power because the power of ideal white noise is infinite. When they are compared with respect to spectral density the frequency at which it is measured must be indicated.

The author of work [9] examines the maximum interference resistance of simple signals with noise of a symmetrical frequency characteristic having a single maximum at the average signal frequency where the spectral density is also measured. Here it is not the maximum probability of correct signal reception characterizing potential interference resistance and realizable in an optimum decision system (e.g., that of Figure 3.12a), which is computed, but the error probability which may be obtained by using a system which is optimal for white noise (i.e., without use of a "whitening" filter). It is apparent that the thus computed probability of correct reception will, generally speaking, be less than maximum. Nevertheless, it is larger than with white noise and monotonically increases as the effective width of the interference spectrum is narrowed.

It appears that even with any normal noise frequency characteristic the probability of correct reception of given signals will be no less than with white noise, if the comparison is made with respect to the same values of maximum spectral density.

It must be emphasized that these findings are true in the case where the noise has a nonuniform spectrum at the very input of the receiving device where the signals have given shape $z(t)$. The relationships will be absolutely different if the noise spectrum becomes nonuniform because of having passed through a circuit with a nonuniform frequency characteristic connected between the receiver input and the decision system. Here it is obvious that even a signal passing through the selective circuit will change its shape, so that noise of irregular frequency characteristic and modified $z(t)$ signals will be present at the output of the circuit.

Of course, such a selective circuit is unnecessary with regard to protection against fluctuation interference. The optimum decision system with white noise may, as shown above, contains no line filters at all (for example, Figures 3.2, 3.3, 3.5, etc.) Nevertheless, in practical receiver circuits frequency selectivity is always provided for two reasons. First, as will be shown in Chapter VIII, frequency selectivity is often needed for protection against lumped noise. Second, even if there were only fluctuation interference its power in the absence of frequency selectivity could become so great that nonlinear phenomena would begin to exert a disturbing effect on the operation of the decision system.

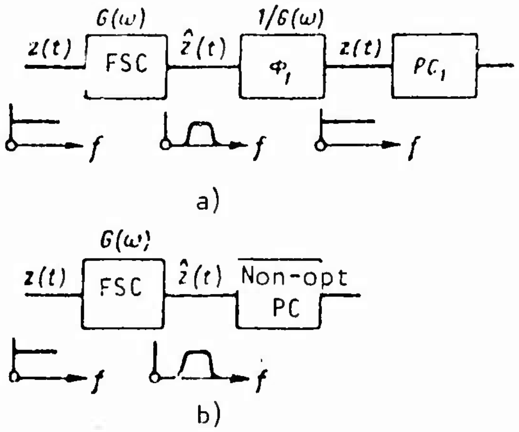


Figure 3.13. Decision System with Frequency-Selecting Circuit [FSC] at Input.

In conformity with the above finding the optimum method of reception in this case amounts to feeding the signal and noise which have passed through the frequency-selecting circuit to filter Φ_1 to convert the noise into white noise and then in feeding it to decision system PC_1 (Figure 3.13a). But it is easy to see that Φ_1 is a filter with frequency characteristics opposite to that of the selective circuit. Therefore at the output of filter Φ_1 will be signals of the original shape $z(t)$ against a background of white noise, and the decision system will operate just as if there were no selective circuits in the receiver. Consequently, the maximum probability of correct reception (potential interference resistance) does not change if the signal together with the interference is passed through a reversible frequency-selecting circuit.[1]

---

[1] This result is, of course, also true for any reversible circuit, including a nonlinear one. Hence it also follows that the carrying capacity of the channel does not change when any reversible circuit is connected into it after the interference sources.

The system in Figure 3.13a cannot, however, ordinarily be used in practice. The situation is that the employment of a frequency-selecting circuit in the receiving unit pursues the definite aims which were related above. The inclusion of filter F_1 behind the selective circuit, however, is equivalent to canceling frequency selectivity. Therefore in preference to the system in Figure 3.13a is to be used that in Figure 3.13b where after the selective circuit the signal plus interference enters a decision system containing no filter F_1 and not disturbing the frequency selectivity introduced. But since such a decision system differs from that in Figure 3.13a it is no longer optimum. Therefore, the use of linear selective circuits before the decision system should increase the probability of error caused by fluctuation interference (in particular, white noise).

The immediate factors causing the rise in probability of error in this case are, first, depressed signal power (from suppression of a portion of the signal spectrum in the selective circuit) and, second, expansion of the signal in time during transit through the selective circuit. The second reason plays the greater role and is the reason that reception of a given signal is disturbed not only by fluctuation interference, but also by the noise resulting from transient processes in the selective circuit which were in turn provoked by the preceding elements of the signal.

In order that a linear selective circuit in the receiver not result in perceptible increase in error probability, the requirement must obviously be made that its effective passband be of adequate size in comparison with the effective width of the signal spectrum and that the transient processes in the prescribed passband be damped as quickly as possible. This last requirement is known to be met if the resonance response diagram of the selective circuit is close to gaussian (bell-shaped) [9]. If the effective passband of a gaussian filter exceeds $F + 5/T$ to $F + 10/T$, where F is the conditional frequency band occupied by the signal (3.4) and T is the duration of a signal element, then, as many examples show, it is possible to ignore the decrease in signal power and the effects of transient processes. By using such a filter as a selecting circuit, it is possible to not consider it in calculating the probability of errors caused by fluctuation interference.[1]

---

[1] It should be observed that so-called "quasi-optimum" filters or selective filters with a given characteristic performance curve and optimum passband determined with regard to obtaining the maximum signal voltage to noise voltage ratio (e.g., see [19]) cannot serve as examples of optimum selective circuits in the case under consideration. The fact is that the passband of a quasi-optimum filter is chosen without regard to the transient processes caused by the preceding elements and that its optimality occurs only during reception of single pulses separated by considerable intervals during which the transient processes have time to damp.

Let a decision system PC$_1$ have matched filters (Figure 3.5). Then the "whitening" filter Φ_1 shown in Figure 3.12a, and filter C, matched with signal $z_r(t)$ which is received against a background of white noise, be connected in series. They can be considered to be one filter matched with signal $z_r(t)$ under conditions of gaussian noise with a nonuniform spectrum (Figure 3.13).

If $\Phi(j\omega)$ is the frequency characteristic (transfer function) of filter Φ_1 and $S_r(j\omega)$ is the complex spectral density of signal $z_r(t)$, then signal $z_r(t)$ at the output of filter Φ_1 has a spectral density of $\bar{S}_r(j\omega) = \Phi(j\omega)\,S_r(j\omega)$. A filter matched with $z_r(t)$ under conditions of white noise must have, according to (3.32), a transfer function of

$$K_r(j\omega) = a\Phi^*(j\omega)S^*_r(j\omega)e^{-j\omega T}.$$

The series connection of filter Φ_1 and the filter matched with $z_r(t)$, i.e., the filter matched with signal $z_r(t)$ when there is interference with a power spectral density of $G(\omega)$, has the following transfer function

$$K_{rc}(j\omega) = K_r(j\omega)\Phi(j\omega) = a|\Phi(j\omega)|^2 S^*_r(j\omega)e^{-j\omega T}$$

or, in light of (3.73)

$$K_{rc}(j\omega) = a\frac{S^*_r(j\omega)}{G(\omega)}e^{-j\omega T}.$$

Thus the transfer function of a matched filter when there is noise with a nonuniform spectrum differs from (3.32) only in the multiplier $1/G(\omega)$. It is determined unambiguously although function $\Phi(j\omega)$ which was used in the derivation was determined only by modulus. Inasmuch as the spectral density of noise behind the whitening filter Φ_1 is equal to one and the power of signal $z_1(t)$ is equal to

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}|\bar{S}_1(j\omega)|^2 d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{|S_1(j\omega)|^2}{G(\omega)}d\omega,$$

then for a binary system with opposite signals the probability of error can be determined from formula (3.15), substituting in it

$$h = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{|S_1(j\omega)|^2}{G(\omega)}d\omega} \qquad (3.74)$$

The probability of error for other systems can be computed similarly.

There has been one point on which agreement is lacking in all the above discussion. All the derivations indisputably hold when we are talking about the reception of a single isolated signal element. Let there be received a sequence of elements $z_r^{(i)}(t)$ $(i = 1, 2, \ldots)$, each of which has a duration of T and which, therefore, do not overlap in time. In case of white noise, all the variations of decision system considered permit separate processing of each signal element without mutual (interelement) interference. Specifically, as was noted, the reaction of a matched filter on proceding elements fades completely by the instant of reading for a current element.

The situation is otherwise when there is interference with a nonuniform spectrum. If signal elements $z_r^{(i)}(t)$ do not mutually overlap, after passage through filter K_1 (Figure 3.12) the signal elements converted by it $z_r^{(i)}(t)$, as a rule, are stretched out and are mutually overlapping to one degree or another. In other words, at the instant of a voltage reading, the voltage at the output of a filter with a transfer function (3.72) will be determined not only by the last signal element received but by a series of preceding elements, i.e., there will occur interelement interference, increasing (sometimes very significantly) the probability of error.

If in designing a communication system the spectral density of interference $G(\omega)$ is known ahead of time, it is possible to form signals $z_r(t)$ such that the transformed signals $z_r(t)$ do not mutually overlap. This can be done, for example, through "preselection" in the following way. The transmitter forms selected signals $z_r(t)$ with a duration of T. Prior to sending them to the channel they are sent through a special filter with a transfer function $D(i\omega)$, so selected that $D(i\omega)$ [illegible] $G(\omega)$. The signals $z_r(t)$ received at the output are sent to the channel. Obviously, by passing through the whitening filter K_1 of the receiver, they again are converted into nonoverlapping signals $z_r(t)$ which are received against a background of white noise.

One important distinction in the transmission of messages in a channel with a nonuniform interference spectrum, in comparison with the case of white noise, should be stressed. With white noise the probability of error depends on the ratio h between signal power and the spectral density of noise and on the mutual ratio of signals which is expressed in the binary case by coefficient ρ in formula (3.61) but does not depend on the "fine structure" of the signal. Thus, with opposing signals the probability of error is expressed by formula (3.65) and with known signal power does not depend on its shape, spectrum, etc.

This property is not retained with a nonuniform interference spectrum. In this case various values of h may correspond to signals with the same power $\frac{1}{2\pi}\int_{-\infty}^{\infty}|S(j\omega)|^2 d\omega$. From (3.73) it is apparent that the largest h is provided by those signals for which the modulus of spectral density differs from zero only in that band of frequencies where the spectral noise density $G(\omega)$ is sufficiently small. This conclusion, incidentally, is rather trivial. A more detailed theory permits determining the shape of an optimal signal depending on $G(\omega)$ if it exists [8]. In some cases there is no such optimal signal. Thus, if $G(\omega)$ decreases monotonically with an increase in frequency (a rather frequent event), obviously, the higher the frequency range in which most of the signal energy is concentrated, the less will be the probability of error. Usually, however, additional limitations are imposed on the range of frequencies used which do not permit increasing the resistance to interference as much as desired through such a method.

3.7. Carrying Capacity of Constant Parameter Channel with Additive Noise

Let us compute the carrying capacity of the channel in question. To define the carrying capacity we must first of all come to agreement about the restrictions imposed on the signal. It is most natural to limit the average power of the incoming signal and its base. Let us therefore first calculate the carrying capacity of a channel in which signals of power P_s and element duration T may be transmitted with given base $B = 2FT$, if white noise of spectral density [illegible] is assumed to accompany the signal. We impose no restrictions on m, the code base.

Every signal over time T may be represented by a finite Fourier series (3.2) in which 2FT coefficients are not identically equal to zero. The information carried by a signal is confined in the values of these coefficients, each of which bears its share of the information. If the values of the individual coefficients are statistically independent, then the whole quantity of information in the signal is the sum of the partial quantities of information carried by each coefficient. Statistical dependence between these coefficients can by its presence only decrease the total amount of information.

To find the carrying capacity we must determine the signal structure which provides the maximum transmittable information. Therefore, we must assume that all the Fourier coefficients are statistically independent random variables. Signal power must be somehow distributed among the Fourier coefficients. Let us suppose that this distribution has been performed in such a way that the powers[1] $\sigma_{ak}^2, \sigma_{ak}^2$ fall to the lot of a_k, b_k, respectively:

---

[1] Since a_k, b_k are random quantities (determined by signal selection in the transmitting unit) the powers σ_{ak}^2, σ_{ak}^2 represent the mathematical expectations of the squares of these coefficients--$\overline{a_k^2}$, $\overline{b_k^2}$.

$$\sum_{k_1}^{k_2} (\sigma_{ak}^2 + \sigma_{bk}^2) = 2P_S. \qquad (3.73)$$

The number of terms in this sum is $2(k_2 - k_1 + 1) =$ [illegible]. Let us find the maximum quantity of information transmitted over time T, i.e., by coefficient a_k with power σ_{ak}^2, and then let us determine how best to distribute the highest total rate of information transmission, which, obviously, will equal the channel carrying capacity under the given conditions.

The quantity of information contained in random variable A_k (the Fourier coefficient of the received signal) with respect to random variable a_k (the Fourier coefficient of the transmitted signal) may be represented by the different entropy from formula (1.50):

$$I_T(A_k, a_k) = h(A_k) - h(A_k/a_k). \qquad (3.75)$$

But $A_k = a_k + \xi_k$, from which it is easy to show that

$$h(A_k/a_k) = h(\xi_k) \qquad (3.76)$$

Actually, the conditional probabilities of getting a received quantity A_k when the transmitted quantity a_k is shown are nothing else but the probability of additive interference ξ_k, and since entropy is uniquely determined by the probability distribution, from this follows (3.76).

The first point of our planned program is therefore fulfilled by finding the maximum possible value of the variable

$$I_T(A_k, a_k) = h(A_k) - h(\xi_k) \qquad (3.77)$$

The magnitude $h(\xi_k)$ is determined by interference and does not depend on the signal. Therefore, the problem reduces to finding the maximum differential entropy of received signal $h(A_k)$. The power of A_k obviously is the sum of the power of a_k and ξ_k, since these magnitudes are statistically independent.

In information theory it is proved (e.g., [10]) that with a given dispersion of a random variable the greatest differential entropy is secured when its probabilities are normally distributed. Consequently $I_T(A_k, a_k)$ has the greatest value if $A_k = a_k + \xi_k$ is a random quantity with normal probability distribution. Since ξ_k has normal distribution, it is necessary and sufficient that a_k also have normal distribution.

Up until now we have examined only discrete collections of signals, and a_k is, generally speaking, a discrete random variable, the number of [illegible] without [illegible]

[illegible] the most [illegible] measured from [illegible] we assume that [illegible] average value [illegible]

[illegible]

$$\Delta E \; [illegible]$$

[illegible]

---

[1] This follows from the well-known theorem of the arithmetic mean and the geometric mean.

to be reached when all these variables are equal to each other. Therefore the condition for obtaining a maximum (3.81) is $\overline{a_k^2} = \overline{b_k^2}$ = const, whence

$$\overline{a_k^2} = \overline{b_k^2} = \text{const} = \frac{2P_s}{F}.$$

Substituting this result in (3.80) and taking into account that [illegible] and B = 2FT, we get

$$J(F,\nu) = \frac{1}{2}B\ln\left(1+\frac{[illegible]}{[illegible]}\right) = FT\ln\left(1+\frac{P_s}{P_n}\right)$$

Here $P_n = \nu_0^2 F$ is noise power in frequency band F.

The carrying capacity of the channel is

$$C = \frac{J(F,\nu)}{T} = F\ln\left(1+\frac{P_s}{P_n}\right)$$

The formula derived coincides completely with Shannon's [2] well-known expression for the carrying capacity of a channel with white noise, but its meaning is somewhat different. Shannon studied a channel with [illegible] of only a limited spectrum in a band of width F. Here, however, [illegible] is used to mean the conventional band of frequencies determined by the [illegible] of Fourier coefficients which are not identically equal to zero. The formula derived thus refers to a signal whose spectrum is not limited in the strict sense. Indeed, with sufficiently large values of [illegible] (if F is given) the difference between the signals examined here and [illegible] signals may be made as small as desired.

Let us now discard the stipulation that the signal band be limited and compute what carrying capacity a channel with additive white noise of spectral density ν^2 has, if signal power is P_s. To settle this question we may proceed from expression (3.84) and seek its maximum as F changes, [illegible], as F changes, since C does not depend on T. For this purpose we write expression (3.84) as

$$C = F\ln\left(1+\frac{P_s}{\nu^2 F}\right)\frac{\text{натур. ед.}}{\text{сек}}.$$

It is easy to satisfy ourselves that as F increases, carrying capacity increases, and that as F → ∞, carrying capacity tends toward the value

$$C_\infty = \frac{P_s}{\nu^2}\,\frac{\text{натур. ед.}}{\text{сек}}.$$

With use of the symbol $h^2 = P_s T/\nu^2$ which was introduced before, we may write the result also in this way:

$$C_\infty = h^2/T.$$

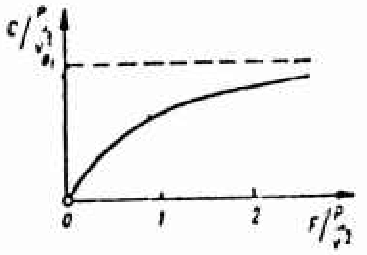


Figure 3.14. Dependence of Carrying Capacity on Value of Conditional Frequency Band of Signal.

Figure 3.14 shows how carrying capacity C rises as conditional frequency band F increases in accordance with formula (3.84a). Even when $F = P_s$ carrying capacity reached 70% of its maximum C_∞ and rises very slowly as F increases further.

Here it is useful to recall the coding theorem (Section 1.8) which explains the actual meaning of the concept of "carrying capacity" and to give another formulation to the proved relationships (3.84) and (3.85). Assume that in the channel under discussion it is possible to transmit any signals having an accepted frequency band of F and an average power (at the receiver input) not exceeding P_s. We will assign various values to the duration of signal T and for each of them we will construct in accordance with some rules or other (as yet not defined) a finite set containing m(T) signals satisfying the conditions imposed. When this is done

$$m(T) = 2^{H'T}, \tag{3.86}$$

where H' is a certain given value.

If a certain source with a fixed rate has a productivity of H' natural units per second, the number of different messages of duration T which the source can emit with a total probability as close to unity as desired with a sufficiently large T is equal to

$$N = 2^{H'T}. \tag{3.87}$$

Then, in light of (3.86) it is possible to compare one of the m signals with each of the messages of the source for transmission over the channel. Shannon's theorem asserts that with a proper selection of signals the probability of correct reception of such a signal can be less than any given $\varepsilon > 0$ if the value of T is sufficiently great and $H' < C$. In light of formula (3.84a) the latter condition can be written as:

$$H' < F\ln\left(1 + \frac{P_s}{\nu F}\right) \text{ natural units/sec} \tag{3.88}$$

or

$$P_s > \nu F\left(e^{H'/F} - 1\right) \tag{3.89}.$$

For that case when the frequency band F is unlimited, this condition becomes

$$H' < P_s/\nu^2 \qquad (3.90)$$

or

$$P_s > \nu^2 H'.$$

We will discuss selection of signals with a given T. From the proof given above it is apparent that they can be determined by performing 2mFT independent random selections of Fourier coefficients in accordance with the normal law of distribution with zero mathematical expectation and a dispersion equal to $2P_s/B = P_s/FT$. When this is so, it is possible only to assert that the mathematical expectation of signal power will be equal to P_s. As far as the power of each realization is concerned and even the average power for a finite number of m selected signals, it will differ from P_s in either direction.

The following questions are of interest. Is it possible to demonstrate a regular (not connected with a random selection) method of constructing m(T) signals possessing the property that when condition (3.89) or (3.90) is met, the probability of incorrect reception will approach zero with increasing T? Can these signals be so constructed that the power of each of them does not exceed P_s?

In the general case these questions remain open but for an unlimited band of frequencies F the answer to them may be affirmative. Furthermore, it is possible to demonstrate several methods of regular construction of such signals, specifically, they may form a simplex, biorthogonal or orthogonal system. By way of example we will show that for a system consisting of $m = e^{H'T}$ orthogonal signals with the same power of P_s the probability of error with a sufficiently large T is less than any given positive number if the condition in (3.90) is met.

The probability of correct reception for an orthogonal system is defined by formula (3.67). Performing the change of variable [illegible] (3.73), we obtain

$$q = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left\{\frac{1}{2}\left[1 + \Phi(\sqrt{2}h + u)\right]\right\}^{m-1} e^{-\frac{u^2}{2}} du \qquad (3.91)$$

With a given ε we determine the number a so that

$$\frac{1}{\sqrt{2\pi}} \int_{a}^{\infty} e^{-\frac{u^2}{2}} du = \frac{\varepsilon}{2}. \qquad (3.92)$$

As can easily be seen, $\left\{\frac{1}{2}\left[1 + \Phi(\sqrt{2}h + u)\right]\right\}^{m-1} < 1$ and is a nondiminishing function of v. Taking into consideration also that the integrand in (3.91) is not negative, we obtain

$$q \geq \frac{1}{\sqrt{2\pi}} \int_{-a}^{c} e^{-\frac{y^2}{2}} \left\{\frac{1}{2}[1 + \Phi(\sqrt{2}h + y)]\right\}^{m-1} dy \geq$$

$$\geq \frac{1}{\sqrt{2\pi}} \int_{-a}^{c} e^{-\frac{y^2}{2}} \left\{\frac{1}{2}[1 + \Phi(\sqrt{2}h - a)]\right\}^{m-1} dy =$$

$$= \left\{\frac{1}{2}[1 + \Phi(\sqrt{2}h - a)]\right\}^{m-1} \frac{1}{\sqrt{2\pi}} \int_{-a}^{c} e^{-\frac{u^2}{2}} du \geq \qquad (3.93)$$

$$\geq \left\{\frac{1}{2}[1 + \Phi(\sqrt{2}h - a)]\right\}^{m-1} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{u^2}{2}} du\right.$$

$$\left. - \frac{\epsilon}{2}\right] \geq \left\{\frac{1}{2}[1 + \Phi(\sqrt{2}h - a)]\right\}^{m-1} - \frac{\epsilon}{2}.$$

From condition (3.90) it follows that there exists a sufficiently small positive number δ with which

$$P_s > \nu^2(H + \delta)$$

or

$$\frac{\sqrt{P_s}}{\nu} > \sqrt{H + \delta} \qquad (3.94)$$

We will designate $\frac{\sqrt{P_s}}{\nu} - \sqrt{H + \delta} = \varepsilon$, where, according to (3.94), $\varepsilon > 0$.

Considering that by definition $h = \sqrt{P_s T}$ we have

$$\Phi(\sqrt{2}h - a) = \Phi\left(\sqrt{\frac{2P_sT}{\nu^2}} - a\right) = \Phi[\sqrt{2T(H + \delta)} +$$
$$+ \sqrt{2T}\,\varepsilon - a]$$

Let $T_1 = a^2/2\varepsilon^2$. Then in light of the fact that the Cramp function is nondiminishing, with $T > T_1$

$$\Phi(\sqrt{2}h - a) > \Phi[\sqrt{2T(H + \delta)}] \qquad (3.95)$$

From the known asymptotic expansion of an integral of probability with sufficiently large x

$$\Phi(x) \approx 1 - \sqrt{\frac{2}{\pi}} \frac{e^{-\frac{x^2}{2}}}{x}. \qquad (3.96)$$

Combining (3.93), (3.95), and (3.96), we obtain

$$q \geq \left[1 - \frac{e^{-T(H+\delta)}}{2\sqrt{\pi T(H + \delta)}}\right]^{m-1} - \frac{\epsilon}{2} \geq$$

$$\geq 1 - (m - 1)\frac{e^{-T(H+\delta)}}{2\sqrt{\pi T(H + \delta)}} - \frac{\epsilon}{2} >$$

$$> 1 - m\frac{e^{-T(H+\delta)}}{2\sqrt{\pi T(H + \delta)}} - \frac{\epsilon}{2}.$$

We will insert $m = e^{H'T}$. Then

$$q > 1 - \frac{e^{H}}{2\sqrt{\pi}(H' \ldots)} [illegible]$$

The last inequality follows from the fact that [illegible] when m [illegible]

Setting T_2 = 1/ ln 2 [illegible], we find that when T > max(T_1, [illegible]) [illegible] or p = 1 - q [illegible], and this is what we wanted to prove.

A more thorough analysis of expression (5,41) [10] shows that the probability of error with an increasing T approaches zero exponentially and the coefficient in the exponent decreases with an increase in the ratio H'/C and when H' = C becomes zero.

The result obtained shows that when condition (5,40) is met it is always possible to construct a communication system with orthogonal signals, selecting values for m and T so as to conduct transmission with as high fidelity as desired. Unfortunately, it is impossible to apply this result directly in practice inasmuch as with an increase in m, in the first place, the decision system becomes much more complex and, in the second place, the accepted frequency band widens.[1] An overwhelming majority of existing communication systems are binary, although the possibility of increasing reliability (with a given H') by increasing m has long been known.

Employing a binary system permits using the simplest first decision system and allocating the task of increasing fidelity to the second decision system (decoder) by applying a correcting code. This is based on the fact that even a complex decoder, inasmuch as it is based on discrete equipment, is simpler and more reliable than a system of matched filters of multipliers with integrators when m is large.

Inasmuch as the carrying capacity of a discrete channel [illegible] cannot exceed the carrying capacity of a continuous channel [illegible] it [illegible] be expected that with such [illegible] the [illegible] of a binary code the carrying capacity greatly decreases. We will find the carrying capacity of a channel in which there is normal [illegible] noise with a spectral density of [illegible] assuming that the power of the signal [illegible] must consist of a sequence of elements corresponding to a message [illegible] in the best possible way by a binary correcting code. [illegible] any limitations on the frequency band and, consequently, on the duration of the signal.

---

[1] In actual fact, the maximal number of orthogonal signals of duration T with an accepted frequency band of [illegible] is equal to the base of the system [illegible] = 2FT. If in this process m = $e^{H'T}$, then F = [illegible], whence it is apparent that with an increase in T, the accepted frequency band also increases without limit.

Since a binary channel with additive white noise is symmetrical, it is possible to use expression (2.28) from which it follows that the rate of information transmission increases with a decrease in the probability of error. Minimum error probability at the given P_s is provided by selecting opposed signals, for which

$$p = \frac{1}{2}[1 - \Phi(\sqrt{2}h)].$$

Substituting this value in expression (2.28) we get

$$I'(z',z) = \frac{1}{T}\Big\{1 + \frac{1}{2}[1 - \Phi(\sqrt{2}h)]\times$$
$$\times\log\frac{1}{2}[1 - \Phi(\sqrt{2}h)] + \frac{1}{2}[1 + \Phi(\sqrt{2}h)]\times \qquad (3.97)$$
$$\times\log\frac{1}{2}[1 + \Phi(\sqrt{2}h)].$$

It is easy to satisfy ourselves by analysis of this expression [12] that as duration of signal element T decreases, the information transmission rate monotonically rises, despite the decreased value of h. Therefore, the carrying capacity of a channel with the indicated restrictions must be considered the limit of expression (3.97) when T, and hence also h, tends toward zero. This limit may be easily found by using the property that with a small value of x

$$\Phi(x) \approx \sqrt{\frac{2}{\pi}}\,x,$$
$$\ln(1 + x) \approx x - \frac{x^2}{2}.$$

Converting expression (3.97) to natural units we find

$$C_{i} = \lim_{T\to 0}\frac{1}{T}\Big\{\ln 2 + \frac{1}{2}\ln\frac{1}{2}[1 + \Phi(\sqrt{2}h)]\,+$$
$$+[1 - \Phi(\sqrt{2}h)] + \frac{1}{2}\Phi(\sqrt{2}h)[1 + \Phi(\sqrt{2}h) -$$
$$- 1 + \Phi(\sqrt{2}h)] - \frac{1}{4}\Phi^2(\sqrt{2}h)[1 + \Phi(\sqrt{2}h) +$$
$$+ 1 - \Phi(\sqrt{2}h)]\Big\} = \lim_{T\to 0}\frac{1}{2T}\Phi^2(\sqrt{2}h) = \lim_{T\to 0}\frac{1}{T}\frac{2h^2}{\pi}.$$

Taking expressions (3.85) and (3.46) into account we obtain

$$C = \frac{2}{\pi}\frac{P_s}{\nu^2} = \frac{2}{\pi}C'_{r}. \qquad (3.98)$$

Thus, such a strong limitation imposed on the size of the code base decreases the carrying capacity by only π/2 times in comparison with the case in which there are no limitations placed on the method of coding.

The carrying capacity of a binary channel with a given power of signal P_s and a spectral density of additive white noise ν^2 when the signals are not opposed but orthogonal is also of interest. In this case (see (3.54))

$$p = \frac{1}{2}\left[1 - \Phi(h)\right]$$

Reasoning in the same way as above, we obtain

$$C_{0\,\mathrm{orth}} = \frac{1}{\pi}\frac{P_s}{\nu^2} = \frac{1}{2}C_0. \qquad (3.99)$$

It follows that the transition from opposing signals to orthogonal signals cuts the carrying capacity in half.

Notes

1. (See Section 3.2) Many works on communication theory (including a number by C. E. Shannon) are based on the conception of signal and noise as processes with their amplitude or power spectra lumped entirely in a limited frequency band F. This makes it possible to use Kotel'nikov's well-known readout theory (e.g., see [10]) which allows a continuous-time problem to be reduced to a discrete-time problem.

Very serious objections have been advanced in opposition to this concept. First, limited-spectrum signals are not in principle realizable since they must be infinite in duration. Second, every process with a limited spectrum is singular or determinate. This means that the value of this process may be restored at any moment in time over any finite segment of it [4]. Hence it follows that all the information included in a limited-spectrum signal is contained in any arbitrarily small segment of it.

The condition of nonsingularity of a process with a power spectral density of $G(\omega)$ amounts to convergency of the integral

$$\int_0^\infty \frac{\ln G(\omega)}{1+\omega^2}\,d\omega$$

This condition is broken if in any finite frequency segment $G(\omega) = 0$, and, in particular, if $G(\omega)$ differs from zero only in a finite band.

Let us note that this condition is similar to the Paley-Wiener [3] stipulation of physical realizability of linear systems which says that if $S(j\omega)$ is the frequency characteristic of the realized circuit, integral

$$\int_{-\infty}^{\infty} \frac{\ln|S(j\omega)|}{1+\omega^2}\,d\omega$$

converges. From this it is easy to conclude that if an indeterminate process is fed to the input of this circuit, then the process at its output will also be indeterminate.[1]

For this reason it may be stated that communication theory may draw meaningful conclusions only from the examination of indeterminate processes which have an unlimited spectral extent, to which Kotel'nikov's theorem does not apply. The observation must be made that in one of his works [1], which is devoted to the theory of potential interference resistance, Kotel'nikov does not make use of his own theorem.

What has been stated in no way contradicts the possibility that an indeterminate signal may exist which is representable in interval (0,T) by a Fourier series with a finite number of coefficients different from zero, i.e., by a finite trigonometric polynomial. Such a signal segment, being limited in time, has an infinite spectral extent. Thus, for example, the segment of signal $z(t) = a\cos\omega_0 t$ $(0 \le t \le T)$, with only one Fourier coefficient different from zero, has some amplitude spectral density unlimited by any band

$$S(j\omega) = a\left[\frac{\sin(\omega-\omega_0)T}{\omega-\omega_0}+\frac{\sin(\omega+\omega_0)T}{\omega+\omega_0}-j\frac{1-\cos(\omega-\omega_0)T}{\omega-\omega_0}-j\frac{1-\cos(\omega+\omega_0)T}{\omega+\omega_0}\right].$$

The extent of a signal power spectrum composed of a sequence of elements, each of which is represented by a finite trigonometric polynomial, is also infinite. Nevertheless, a signal element may be regarded as a segment of a periodic process of period T when this periodic process, being determinate, may have a spectrum concentrated in a finite band.

Based on physical considerations it is clear that any signals and interference in actual communication systems are nonsingular. Nevertheless, in the solution of various problems resort is often had to mathematical idealization, replacing a nonsingular process with a singular one close to it, specifically by a process with a limited spectrum. In so doing, despite a very good approximation of a spectrum (in the sense of absolute or mean-square error) it is possible to obtain paradoxical results [13,14]. Thus, a signal as weak as desired may be detected with a probability of unity against a background of singular noise. Furthermore, even with white noise it is possible with as small a probability of error as desired to detect the presence or absence of a weak signal against a background of noise, observing it for a given t[illegible] if the received signal together with the noise is passed through an ideal pi-shaped filter. The process at the filter output will have a spectrum limited in band and may be extrapolated as far as desired. Therefore, an observation over time T may be equivalent to an observation over a longer time. It is possible to select such an extrapolation interval so as to obtain a sufficiently great dummy energy of extrapolated signal which provides for a given probability of correct detection at a given spectral noise density.

---

[1] It may therefore in particular be stated that if $G(\omega)$ is the power spectrum of interdeterminate noise a filter of characteristic $S(j\omega)$ satisfying condition (3.71) is physically realizable.

In fact, a filter with a pi-shaped frequency characteristic is not physically realizable. This ideal filter characteristic may, of course, be approached, but it is well known that the closer the characteristic of the real filter is to the ideal, the greater the signal delay that such a filter has. In order to observe the signal at filter output for time T the signal must keep entering the filter for a considerably longer time. Thus, what is observed at filter output is the result of protracted signal action and therefore contains information about a sufficiently large signal segment. Great fidelity of signal detection is therefore attained here because of the utilization of a large real, and not dummy signal, and this resolves the paradox.*

To avoid erroneous conclusions we will nowhere posit a limited spectrum.

2. (See Sections 2.2 and 3.3) Expansion of a random process into the series represented by formulas (3.2) and others must be understood in the sense of convergence in the mean square. This means, for example, that for the second of formulas (3.2)

$$\lim_{K\to\infty}\overline{\left[n(t)-\sum_{k=0}^{K}(\alpha_k\cos k\omega_0 t+\beta_k\sin k\omega_0 t)\right]^2}=0.$$

Here α_k and β_k are some random numbers. In the same sense we may speak also of expanding a random process with a limited spectrum into a Kotel'nikov series [15].

Since n(t) is a normal random process, it follows from the convergence in the mean square that convergence is almost certain [17], i.e., any realization n(t) with a probability of unity may be expanded into a Fourier series where the coefficients of the expansion will coincide with the corresponding realization of the aggregate of random variables α_k, β_k. This same thing also refers to expansion of z(t) and of the other normal random processes encountered in this book.

The integration of random processes, for example, in formula (3.34) is also understood in the sense of convergence in the mean square, and, for example, the integral

---

*This example may be compared with the optimum decision system using matched filters (Figure 3.3). There, readout of the output voltages is practically instantaneous, but it contains the necessary information about the signal which was received during the considerably longer time T.

$$\int_0^T n(t)\,g(t)\,dt$$

(where $g(t)$ is a regular function) represents a random variable I such that with arbitrary subdividing of interval $(0,T)$ by the finite set of points $t_1 < t_2 < \dots < t_k < t_{m+1}$

$$\lim_{m\to\infty}\left[\sum_{k=1}^{m} n(t_k)\,g(t_k)\,(t_{k+1}-t_k) - I\right]^2 = 0,$$

where in passing to the limit all the intervals $t_{k+1} - t_k$ tend toward zero. Every realization of the integrable normal process $n(t)$ is almost certainly integrable and its integral equals the corresponding realization of variable I.

Let us observe that all results pertaining to interference resistance could be more rigorously derived without having recourse to series expansions of the random processes, but this would lead to more complex deductions and would require mathematical apparatus less familar to engineers.

3. (See Section 3.3) The optimal reception systems (decision systems), matched filters, in particular, found in this chapter are in certain works, e.g., [9], deduced based on a statistical criterion as systems which allow the derivation of the greatest ratio of instantaneous signal strength (at a certain readout moment) to mean-square interference value. This approach is entirely justified in cases where interference is normal (gaussian) noise and the additional requirement of linearity is made on the receiving system. In fact, the normal noise probability distribution is kept during passage through any linear system. From this it is easy to deduce that of all linear circuits the one with the smallest error probability is the one in which there is the greatest signal-to-mean-square-interference ratio.

Proceeding merely from the stipulation of maximizing the signal-to-noise ratio, however, it is impossible to prove that a linear system always provides the optimum. Actually, with certain types of interference (e.g., impulse) greatest reception fidelity is found in a nonlinear system. Therefore, in deducing the decision principle from the ideal observer criterion and from the structure of the receiving system in conformity to these principles we nowhere limit ourselves to examining linear systems alone, but look for the optimum with respect to all possible operations to which the received signal is subjected. The fact that some of these operations may be performed in a linear system and coincide with operations maximizing the signal-to-noise ratio results from the peculiarities of gaussian noise and may fail to occur under other conditions.

4. (See Sections 3.3 and 3.4) To effect an optimal coherent decision system it is essential, generally speaking, to know exactly and to be able to reproduce all realizations of signals $z(t)$. Under actual conditions the parameters of a channel are never known with absolute precision and therefore

a decision system contains a certain amount of inaccuracy. Actually, as a consequence of this the probability of error proves to be greater than that calculated for a completely known signal.

In designing communication systems it is important to evaluate tolerances in determining the parameters of received signals and ensure observance of them. These tolerances cannot be stated in general form since they depend greatly on the form of the signal. We will limit ourselves to certain remarks based on the assumption that a signal is a narrowband one, i.e., may be written in the form (3.48):

$$z(t) = E(t)\cos[\omega_{\text{ср}}t + \varphi + \Phi(t)]$$

The shape of the envelope of E(t) (with accuracy to a constant multiplier and start of reading of time) and also of instantaneous phase $\Phi(t)$ (with accuracy to start of reading of time) is selected during the design process and are reproduced with any degree of accuracy. Thus, the problem amounts to determining a constant multiplier for E(t), the start of reading of time, the initial phase φ, and the average frequency ω_{av}.

If we use a system with an active delay, knowledge of the constant multiplier is not at all required for designing an optimal decision system. As far as the remaining parameters of the signal are concerned, the allowable inaccuracy in determining them varies for different systems.

For example, we will consider the effect of inaccuracy in determining the initial phase. For this purpose we will assume that the decision system is designed for receiving the signal of (3.48) and in actuality the signal which arrives is

$$\begin{aligned} E(t)\cos[\omega_{\text{ср}}t + \varphi + \Phi(t) + \Delta\varphi] = \cos\Delta\varphi\, E(t)\cos[\omega_{\text{ср}}t + {} \\ + \varphi + \Phi(t)] - \sin\Delta\varphi\, E(t)\sin[\omega_{\text{ср}}t + \varphi + \Phi(t)] \end{aligned} \qquad (3.100)$$

Roughly speaking, this means that instead of signal z(t) there arrives signal $z(t)\cos\Delta\varphi$ and the second term of (3.100) can be regarded as additional interference. The effect of this additional interference, which in the first approximation is orthogonal to the useful signal, depends on what other useful signals are used in the given system. If one realization of the signal is $E(t)\sin[\omega_{av}t + \varphi + \Phi(t)]$, this interference goes to certain branches of the decision system and, being added with the noise component, greatly raises the probability of error. Even with complete absence of noise, an error occurs if $\Delta\varphi > \pi/4$.

In another case when a binary system with opposing signals is under consideration, the second term of (3.100) generally has no effect on the decision system and the inaccuracy in phase $\Delta\varphi$ can be compensated for by an increase in signal power (or the magnitude of h^2) by $1/\cos^2\Delta\varphi$ times. If a "loss" in power on the order of 10% is considered permissible, for $\Delta\varphi$ the tolerance is equal to 18°.

Inaccuracy in reproduction of the average frequency ω_{av} in the first approximation leads to an inaccuracy in the initial phase inasmuch as an inaccuracy in phase "runs after" a certain time T $\Delta\varphi = \Delta\omega_{av}T$.

Demands on accuracy in determining the instant of signal arrival or instant of reading in a decision system can be established similarly. The accuracy required for coherent reception in maintaining the average frequency with the present state of the art can be provided only by automatic adjustment of it based on the received signal itself. Inasmuch as the signal is received together with interference, even under these conditions the accuracy of establishing the frequency and phase of the signal and also the instant of reading is limited and this leads to an increase in the probability of error. Therefore, a preference is often expressed for refraining entirely from determining the initial phase and use is made of noncoherent methods of reception to which the following chapter is devoted. As will be shown there, in noncoherent reception the tolerances in precision with which the average frequency of signal and instant of reading are set are greatly extended.

5. (See Section 3.6) The method set forth for finding the optimal decision system under conditions of normal noise with a nonuniform spectrum belongs to V. A. Kotel'nikov [1]. However, in the discussion presented it is silently assumed that the time of processing of a received signal is not limited since otherwise complete "whitening" of the noise would be impossible.

If the additional requirement is made that processing of the signal be performed over a time interval (0,T), the statement of the problem changes. The main difficulty in seeking an optimal decision system according to the method described in Section 3.5 is that the Fourier coefficients for "colored" noise are mutually correlated. To overcome this difficulty the signals and interference are expanded into an orthonormalized system of functions which are the eigenfunctions of the integral equation

$$\int_0^T R(t,s)\varphi_k(s)\,ds = \lambda_k\varphi_k(t), \qquad (3.101)$$

where R(t,s) is a function of noise correlation.

With such an expansion the coefficients of a series for noise prove to be independent random variables with dispersions of λ_i [17]. We will point out that as a consequence of the positive definiteness of the correlation function, all eigenvalues of λ_i are not negative.

It has been demonstrated [17,18] that the decision rule which is optimal according to the criterion of maximal likelihood is that $s_i(t)$ is considered a received signal if for all signals the inequality $r \neq i$ is met:

$$\int_0^T V_{ir}(t)\left[x(t) - \frac{s_i(t) + s_r(t)}{2}\right]dt > 0 \qquad (3.102)$$

where $V_{ir}(t)$ is the solution of the integral equation

$$\int_0^T R(t,s)V_{ir}(s)\,ds = z_i(t) - z_r(t). \qquad (3.103)$$

It is easy to see that for white noise when $R(t,s) = \delta(t - s)$ the solution of this equation is trivial: $V_{ir}(t) = z_i(t) - z_r(t)$ and the optimal decision rule coincides with (3.27).

In the general case rule (3.102) can be realized in the circuit of Figure 3.12a if filter Φ_1 has a transfer function $g(t_1,t_2)$ which is a solution to the equation

$$\int_0^T R(t_1,s)g(s,t_2)\,ds = \delta(t_1 - t_2),$$

and the decision circuit PC_1 is optimal for signals passing through Φ_1 against a background of white noise. If the noise is stationary, i.e., $R(t,s) = R(|t - s|)$, then filter Φ_1 has constant parameters since $g(t_1,t_2) = g(t_2 - t_1)$. With a T that increases without limit function $g(t_2 - t_1)$ approaches the transfer function of a "whitening" filter.

The probability of error for a binary system is determined by the expression

$$P = \frac{1}{2}[1 - \Phi(\sqrt{a})],$$

where

$$a = \int_0^T V_{12}(t)[z_1(t) - z_2(t)]\,dt.$$

It can be demonstrated [8] that in those cases when among the eigenvalues of λ_i of equation (3.101) there is a least value λ_1, $z_1(t) = -z_2(t) = c\varphi_1(t)$, where $\varphi_1(t)$ is the eigenfunction of equation (3.101) corresponding to the least eigenvalue, are the optimal signals in the binary system and coefficient c is determined by the allowable power of the signal. In this case $a = 4c^2/\lambda_1$.

If among the eigenvalues of λ_i there is one equal to zero, i.e., if the function $\varphi_0(t)$ is such that $\int_0^T R(t,s)\varphi_0(s)\,ds = 0$, then there is a singular event in which a signal proportional to $\varphi_0(s)$ can be detected with a zero probability of error since $a = \infty$. This takes place when the noise spectrum is equal to zero over a finite interval of frequencies as indicated in Remark 1. A potentially singular event occurs if among the λ_i there are those as small as desired. In this case it is possible to select a shape of signal with which the probability of error will be less than any given value however small. In actual channels when the spectral density of the noise at any frequency exceeds a certain positive magnitude, i.e., when the noise contains a "white" component, there is no singularity.

6. Often devices which are parts of a channel amount to a circuit with a transfer function $L(j\omega)$ noticeably changing the shape of the signal. In this case we understand by $z(t)$ not the signal at the input of the channel but a distorted signal at its output. All discussion presented in this chapter holds if the elements of the distorted signal do not cease in time, otherwise the problem of reception becomes complicated. This case will be considered more in detail in Section 7.2.

7. (See Section 3.7) Formula (3.84) was obtained by Shannon [2] on the assumption that a channel represents an ideal filter passing signals and interference in a strictly limited band of frequencies of width F. It is often explained that this formula is an approximate one, giving a more accurate value for the carrying capacity of the channel the more pi-shaped is the frequency characteristic. For a channel with an actual frequency characteristic, formula (3.84) should determine the carrying capacity if the passband of F is properly determined.

Here, however, there occur certain difficulties in selecting a "proper determination" for the passband which result in indeterminacy of the carrying capacity which is being calculated. For example, if the channel frequency characteristic is close to a gaussian curve[1], then, adopting as the value of F the width of this characteristic at the 0.707 or 0.1 level, we obtain different values for C which vary by a factor of 1.6. If, of course, the channel has a frequency characteristic which is more rectangular, the carrying capacity value computed depends less on the level at which the passband is read; nevertheless, a certain ambiguity in determining the carrying capacity still remains.

As shown in [3] formula (3.84) gives an exact carrying capacity value for the case where the signals have a certain correlation interval τ_k, if frequency band F is understood to mean $1/\tau_k$. Under certain conditions this definition of the frequency band coincides with the channel's effective "noise" passband.

Formula (3.84), derived in Section 3.7, expresses the exact channel carrying capacity in which F represents the conditional frequency band occupied by the system described in expression (7.4). It may seem surprising that this formula coincides completely with Shannon's formula which is derived under absolutely different premises. This result is, however, entirely justifiable. It is easy to show that a substantial fraction of the signal power spectrum lies in frequency band F which coincides with the "conditional frequency band," where this fraction is larger, the larger is signal base 2FT. Without pausing to prove this we refer to Figure 3.15 which shows signal power spectra

---

[1] A multistage resonant amplifier is known to have this type of characteristic.

$$z(t) = \sum_{k=k_1}^{k_2} (a_k \cos k\omega_0 t + b_k \sin k\omega_0 t)$$

with FT values of 10 and 20. Here a_k and b_k are random independent identically distributed normal variables. This figure graphically shows that as FT increases a larger and larger portion of the signal power is found to be lumped in a frequency band of width F. This permits formulation of the theorem of the carrying capacity of a channel with additive white noise as follows.

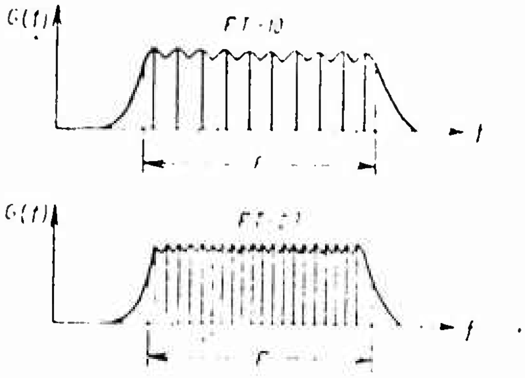


Figure 3.15. Signal Power Spectrum with Different Values of Base 2FT.

Given a frequency band of width F and an arbitrary quantity α ($0 < \alpha < 1$). There then exists a value of signal duration T_0 (dependent on F and α) such that when $T > T_0$ an ensemble of signals of length T and power P_s may be constructed where a signal power of not less than αP_s is lumped in frequency band F, and these signals may be used to transmit information with an arbitrary small error probability and at a rate arbitrarily close to

$$C = F \ln\left(1 + \frac{P_s}{P_n}\right) \text{ natural units/sec},$$

where P_n is the power of the additive white noise in band F.

From this formulation it is evident that Shannon's formula remains valid no matter how signal spectrum band width is determined. Stricter requirements for signal power concentration in frequency band F result only in a need to choose a larger base, i.e., in the given case, in employment of signal elements of greater duration, or, in other words, in greater enlargement of the source alphabet during coding.

In case the noise is not white, but has normal distribution of the probabilities of the instantaneous values, channel carrying capacity at a given signal power may be determined in the same way as was used in deriving formula (3.84). Thus, σ_i^2 must be substituted instead of σ_0^2 in expression (3.90). σ_i^2 is the dispersion of coefficients a_i and b_i when expanding the noise in a general interval into a Fourier series. Further, it is easy to find that the maximum quantity of transmittable information is provided by a signal power distribution where $\sigma_{ai}^2 = \sigma_{bi}^2 = \lambda - \sigma_i^2$, with λ determined by condition $\frac{1}{T}\sum_{k_1}^{k_2}(\lambda - \sigma_k^2) = P_s$. In such a choice the carrying capacity is

$$C = \frac{1}{T}\sum_{k_1}^{k_2}\ln\left(1+\frac{\sigma_{sk}^2}{\sigma_k^2}\right). \qquad (3.101)$$

When [illegible] tends toward infinity we obtain an expression for the carrying capacity of a channel with "limited" passband $F = f_2 - f_1$ in the form given by Shannon

$$C = \int_{f_1}^{f_2}\ln\left[1+\frac{G_s(f)}{G_n(f)}\right]df \text{ natural units/sec}, \qquad (3.102)$$

where $G_n(f)$ is the interference power spectral density and $G_s(f)$ is the signal power spectral density, so chosen that $G_s(f) = \max[K - G_n(f), 0]$ $(f_1 < f < f_2)$, and constant K is determined from condition

$$\int_{f_1}^{f_2}\max[K - G_n(f), 0]\,df = P.$$

With uniform spectral interference density $G_n(f) = \sigma_0^2$, formula (3.102) changes into expression (3.84).

BIBLIOGRAPHY

1. [illegible]

2. [illegible] "[illegible] Differential Theory of [illegible]" [illegible] Press, 19[illegible].

3. Zhelezn[illegible], N. A., [illegible] [Principles of Signal Theory and Tasks of Further Development on the Basis of New Stochastic Models], Radiotekhnika, [illegible] 11, 19[illegible].

4. Davenport, V. B. and V. L. Rut, [illegible] [Introduction to the Theory of Random Signals and Noise], Foreign Literature Press, 1960.

5. Privalov, I. I., [illegible] [Fourier Series], OFTI, Moscow, 1934.

6. Beletskiy, A. F., [illegible] [Fundamentals in the Theory of Linear Electrical Circuits], "Svyaz'" Publishing House, 1967.

7. Gutkin, L. S., [illegible] [Several Relationships in Optimal Circuits for Signal Detection], Radiotekhnika, Nos. 2 and 4, 1960.

8. Nesteruk, V. F., [illegible] [On the Theory of Signal Reception with Correlated Interference], Radiotekhnika, No. 6, 1962.

9. Teplov, N. L., [illegible] [Maximal Resistance to Interference of Systems with Discrete Signals], No. 4, 1960.

10. Fano, R., [illegible] [Information Transmission. Statistical Theory of Communication], "Mir" Press, 1965.

11. Kharkevich, A. A., [illegible] [Combatting Interference], Fizmatgiz, 1963.

12. Varshaver, B. Ya., [illegible] [On the Theory of Carrying Capacity in Binary Transmission], Radiotekhnika, No. 1, 1959.

13. [illegible], L. [illegible] and [illegible], "A Paradox Concerning Rate of Information," Inf. and Contr., No. 1, 1958.

14. [illegible], R. C. and M. F. Mathews, "Further Results on the Detectability of Known Signals in Gaussian Noise," BSTJ, No. 2, 1963.

15. Middleton, D., [illegible] [Introduction to Statistical Communication Theory], "Sovetskoye radio" Press, Vol. I, 1961; Vol. II, 1962.

16. Vaynshteyn, L. A. and V. D. Zubakov, [illegible] [Discrimination of Signals Against a Background of Random Interference], "Sovetskoye radio" Press, 1960.

17. Loev, M., [illegible] [Probability Theory], Foreign Literature Press, 1962.

18. Khelstrom, K., *Statisticheskaya teoriya obnaruzheniya signalov* [Statistical Theory of Signal Detection], Foreign Literature Press, 1963.

19. Gutkin, L. S., *Teoriya optimal'nykh metodov priyema pri fluktuatsionnykh pomekhakh* [Theory of Optimal Reception Methods in the Presence of Fluctuation Interference], Gosenergoizdat, Moscow and Leningrad, 1961.

20. Levin, B. R., *Teoreticheskiye osnovy statisticheskoy radiotekhniki* [Theoretical Fundamentals in Statistical Radio Technology], Vol. I, "Sovetskoye radio" Press, 1966.

CHAPTER IV

CHANNEL WITH RANDOMLY CHANGING SIGNAL PHASE AND ADDITIVE FLUCTUATION INTERFERENCE

4.1. General Characterization of Channel with Randomly Changing Initial Signal Phase

It was assumed in the preceding chapter that the set of transmitted signals $z_i(t)$ was accurately known on reception. It was furthermore assumed that the set of incoming signals $\mu z_i(t - t_p)$ was likewise known since the transmission factor μ and time t_p taken by the signal in passing through the channel were considered constant.

Under actual conditions some parameters of the incoming signals are unknown on reception and, at best, only the probability distributions of these parameters are known. At times these unknown parameters may be directly determined to some degree of accuracy by analyzing the received signal, and knowledge of them may be used in receiving the following signal elements. This often proves to be impossible because the unknown parameters do not remain constant during transmission, but fluctuate rather rapidly; and knowledge of the previous values of these parameters is practically useless for reception of the next part of the signal. Even in cases where the unknown signal parameters change very slowly the opportunity of determining them by analysis of the incoming signal is not always exploited. The fact is that the increased reception fidelity achieved by estimating the parameters does not always pay for the complication of the receiving unit which is required to perform this analysis. In many cases it is economically more profitable to produce the same reliability increase by raising transmitted signal power.

This chapter studies the case where the unknown parameter is the initial phase of the harmonic components of the signal. Phase indefiniteness may have various causes. This indeterminacy is provoked rather often in modern communication equipment by the conditions under which the signal is formed in the transmitting unit. Here it is not infrequent that every signal element is transmitted with an absolutely arbitrary initial phase.

Another reason for incoming signal phase indeterminacy is the fluctuation in signal propagation time t_p in the channel. Here the case is examined where t_p changes within such small limits Δt_p that the signal envelope changes in time Δt_p may be completely disregarded. Nevertheless, the phase of the high-frequency filler of the incoming signal changes within such significant limits that all phase shift values from $-\pi$ to $+\pi$ may be considered equiprobable. For this the condition must be fulfilled that

$$\frac{1}{f_{av}} \ll \Delta t_p \ll \frac{1}{F}, \qquad (4.1)$$

where f_{av} is the average frequency of the signal spectrum and F is the conditional frequency band occupied by the signal.

It is apparent that condition (4.1) may be fulfilled only for relatively narrow band signals in which $F \ll f_{av}$, but it is just these signals which are ordinarily used in radio communications and in long-range wire communication.[1]

We will show that under condition (4.1) propagation time fluctuations may be reduced to phase fluctuations. Let the signal be transmitted

$$s(t) = \sum_{k=k_1}^{k_2} (a_k \cos k\omega_0 t + b_k \sin k\omega_0 t) = \sum_{k=k_1}^{k_2} c_k \cos(k\omega_0 t + \varphi_k), \qquad (4.2)$$

where

$$c_k = \sqrt{a_k^2 + b_k^2}; \quad \varphi_k = \operatorname{arctg}\frac{a_k}{b_k}.$$

The incoming signal in union with the interference is

$$\begin{aligned} x(t) &= \mu s(t - t_p) + n(t) = \mu \sum_{k=k_1}^{k_2} c_k \cos[k\omega_0(t - t_p) + \varphi_k] + \\ &+ n(t) = \mu \sum_{k=k_1}^{k_2} c_k \cos[k\omega_0(t - \bar{t}_p) + \varphi_k + \psi_k] + n(t), \end{aligned} \qquad (4.3)$$

where μ is a constant transmission factor; $\bar{t}_p$, average propagation time; $n(t)$, additive interference; and

$$\psi_k = k\omega_0(\bar{t}_p - t_p) = 2\pi \frac{k}{T} \Delta t_p.$$

From condition (4.1) when $F = (k_2 - k_1 + 1)/T$ it follows that the different values of ψ_k lie between $2\pi(k_1/T)\Delta t_p$ and $2\pi(k_2/T)\Delta t_p$, and that the difference between them does not exceed $2\pi\Delta f\, t_p \ll 2\pi$. This permits the values of φ_k for all k to be considered approximately identical and equal to $2\pi f_{av}\Delta t_p$.

---

[1] In narrow-band signals the phase indeterminacy of the harmonics involves indeterminacy of the initial phase of the "high-frequency" filler.

The fluctuations in propagation time t_p may be caused by changes in the medium in which the radio signals are being propagated (e.g., changes in the height of the reflecting region in ionospheric communication, changes in cable and amplifier temperature in wire communication, etc.), and also given by changes in the mutual arrangement of transmitter and receiver.

The conditions for receiving signals depend to a considerable degree on the rapidity with which the fluctuations occur.

The following cases may be distinguished:

1) very rapid fluctuations where signal phase changes substantially over a single signal element;

2) rapid fluctuations where the initial phases of adjacent signal elements may be considered uncorrelated, but which do not perceptibly change within one element of the signal phase (phase fluctuations caused by signal-shaping conditions within the transmitter also usually belong to this case);

3) slow fluctuations where the initial phases of adjacent elements are almost identical, but the phase changes considerably over several elements;

4) very slow fluctuations where signal phase changes little over a substantial number of signal elements.

This classification is, of course, arbitrary and there are various intermediate cases. It is useful, however, as a certain abstraction to facilitate theoretical analysis.

The first case is usually accompanied by just as rapid fluctuations in the transfer coefficient (by signal fading) and will be considered in Chapter VIII. The fourth case differs almost not at all from the case of a completely known signal as discussed in chapter III, inasmuch as when there are very slow fluctuations in phase it is possible by analysis of preceding signal elements to determine with sufficient accuracy the expected phase relationships in the subsequent elements and to effect coherent reception.

For the second case complete absence of information about the initial phase of the received signal element is characteristic. As will be shown later, this does not hinder reception of the information contained in the element if only it is not contained in the value itself of the initial phase. Discrimination of a signal in the complete absence of (or with the complete failure to use) information about the initial phase of each element will be called absolutely incoherent reception.

The third case occupies an intermediate position between the second and fourth. As in the fourth case coherent reception is in principle possible but for evaluation of the initial phase of a signal element only a small number of preceding elements may be used and this leads to significant error and to an increase in the probability of error. In the third case, as in the fourth, understandably, it is possible to employ absolutely incoherent reception by refraining from the use of any information about the initial phase of the expected signal element. However, use is made here of relatively incoherent reception in which the initial phase of a certain sequence of elements is unknown, but possible phase relations among adjacent signal elements are known.

4.2. Conjugate Signals, Envelope, Instantaneous Phase, and Instantaneous Frequency. Orthogonality in the Intensified Sense.

In our study of incoherent reception we need such concepts as signal envelope, its instantaneous phase, and its instantaneous frequency. These concepts are rather widely used in engineering practice but they are not always unambiguously understood. In this paragraph we give definitions which will be used in this and subsequent chapters. Although these definitions are not the most general, they are convenient in the mathematical model of signal and interference used here and are adequate for solving the problems presented.

Let an element of signal $z_r(t)$, which is given in the interval $0 \leq t \leq T$ be represented in this interval by the series (3.2):

$$z_r(t) = \sum_{k=0}^{\infty}(a_{rk}\cos k\omega_0 t + b_{rk}\sin k\omega_0 t),$$

where $\omega_0 = 2\pi/T$.

Let's assume that all harmonic components of this signal are shifted in phase by a certain magnitude ψ. As a result, we obtain the signal

$$\begin{aligned} z_{r,\psi}(t) &= \sum_{k=0}^{\infty}[a_{rk}\cos(k\omega_0 t + \psi) + b_{rk}\sin(k\omega_0 t + \psi)] \\ &= \cos\psi\sum_{k=0}^{\infty}(a_{rk}\cos k\omega_0 t + b_{rk}\sin k\omega_0 t) + \sin\psi\sum_{k=0}^{\infty}(b_{rk}\cos k\omega_0 t - \\ &\quad - a_{rk}\sin k\omega_0 t) = z_r(t)\cos\psi - \tilde{z}_r(t)\sin\psi, \end{aligned} \tag{4.4}$$

where the series

$$\tilde{z}_r(t) = \sum_{k=0}^{\infty}(-b_{rk}\cos k\omega_0 t + a_{rk}\sin k\omega_0 t) = z_{r,-\pi/2}(t). \tag{4.5}$$

is called conjugate[1] with the series $z_1(t)$. It is obtained from $z_1(t)$ by changing the phases of its components by $-\pi/2$.

Expression (4.4) can be written in complex form:

$$z_{1}(t) = \mathrm{Re}\{[z_1(t) + j\tilde{z}_1(t)]e^{j0}\} = \mathrm{Re}[Z_1(t)e^{j0}] \qquad (4.6)$$

We will call the complex function

$$Z_1(t) = \begin{cases} z_1(t) + j\tilde{z}_1(t) & \text{when } 0 \leq t < T, \\ 0 & \text{when } t < 0 \text{ and } t \geq T \end{cases}$$

a finite analytical signal. We will write the analytical signal in exponential form:

$$Z_1(t) = \begin{cases} E_1(t)e^{j\varphi_1(t)} & \text{when } 0 \leq t < T, \\ 0 & \text{when } t < 0 \text{ and } t \geq T \end{cases}$$

Here

$$E_1(t) = |Z_1(t)| = \sqrt{z_1^2(t) + \tilde{z}_1^2(t)} \qquad (4.9)$$

is the signal envelope; and

$$\varphi_1(t) = \arg Z_1(t) = \operatorname{arctg}\frac{\tilde{z}_1(t)}{z_1(t)} \qquad (4.10)$$

is the instantaneous phase of the signal.

We call the derivative with respect to time of the instantaneous phase the instantaneous angular frequency:

$$\omega_{1i}(t) = \frac{d\varphi_1(t)}{dt} = \frac{\frac{d\tilde{z}_1}{dt}z_1(t) - \frac{dz_1}{dt}\tilde{z}_1(t)}{z_1^2(t) + \tilde{z}_1^2(t)}. \qquad (4.11)$$

It is easy to see that

---

[1] We will not discuss conditions of convergency of series (4.5). In all practical applications we will consider signals with a finite base for which series (3.2) is a trigonometric polynomial. In these series (4.5) degenerates into a polynomial and always has a finite value. Incidentally, many of the following results hold even if series (4.5) diverges at some points.

[illegible]

[illegible]

We will [illegible] especially convenient [illegible] for narrow-band signals[1] [illegible] in this case the [illegible] are slowly changing functions of time [illegible] with the high-frequency filling of the signal cos [illegible] arbitrarily chosen angular frequency within the limits of that band of frequencies in which most of the power of a narrow band signal is lumped, then function [illegible] cos [illegible] are also called slowly changing. In this case the following notation is often used instead of (4.12):

$$\left.\begin{array}{l} z_r(t) = [illegible] \\ \tilde{z}_r(t) = E_r(t) \sin[illegible] \end{array}\right\} \qquad (4.14)$$

We call the operation of transforming function $z(t)$ into its envelope $E(t)$ or into instantaneous frequency $\omega(t)$ ideal amplitude or frequency detection respectively. For a signal element given in interval T these operations are physically realizable if a delay for a time greater than T is permissible. Indeed, knowing function $z(t)$ over this interval, it is possible to determine its Fourier coefficients (see Figure 3.1) to construct a conjugate function $\tilde{z}(t)$, and then to reproduce (for example, using a computer) $E(t)$ and $\omega(t)$ from formulas (4.9) and (4.11)). An actual "linear" amplitude detector discriminates the envelope $E(t)$ of a signal $z(t)$ delivered to it (or a certain monotonic function of E) on condition that its load is inertialess for the envelope and has complete inertia for the high-frequency filling [1]. Obviously, these conditions are contradictory and can be met only approximately, the accuracy being greater the less is the ratio between the effective width of the signal spectrum and its average frequency. A similar assertion holds also for ordinary frequency detectors.

In what follows we will discuss only signals with a finite base, i.e., we will consider the upper limit of the summation in (3.2) and (4.5) as large as desired but a finite number K.

Conjugate signals $z_r(t)$ and $\tilde{z}_r(t)$ are orthogonal in the interval (0, T), i.e.,

---

[1] A signal is called relatively narrow-band if the effective width of its spectrum is much less than the average frequency.

$$\int_0^T z_i(t)\tilde{z}_i(t)\,dt = 0 \qquad (4.15)$$

we can see this easily by substituting (5.2) and (4.5) in this integral and performing term-by-term integration:

$$\int_0^T z_i(t)\tilde{z}_i(t)\,dt = \int_0^T \sum_{k=1}^{K}(a_{ik}\cos k\omega_0 t + b_{ik}\sin k\omega_0 t)\times$$

$$\times\sum_{k=1}^{K}(-b_{ik}\cos k\omega_0 t + a_{ik}\sin k\omega_0 t)\,dt = \frac{T}{2}\sum[-a_{ik}b_{ik} + a_{ik}b_{ik}] = 0.$$

If two signals $z_r(t)$ and $z_i(t)$ are mutually orthogonal, then signals $\tilde{z}_r(t)$ and $\tilde{z}_i(t)$ which are conjugate with them are also orthogonal with one another. To prove this it is sufficient, after representing the signals by appropriate trigonometric polynomials, to cross-multiply them and perform integration, as a result of which we obtain

$$\int_0^T z_r(t)z_i(t)\,dt = \int_0^T \tilde{z}_r(t)\tilde{z}_i(t)\,dt = \frac{T}{2}\sum_{k=1}^{K}(a_{rk}a_{ik} + b_{rk}b_{ik}). \qquad (4.16)$$

However, from the orthogonality of signals $z_r(t)$ and $z_i(t)$ it generally does not follow that signals $z_r(t)$ and $\tilde{z}_i(t)$ (or $\tilde{z}_r(t)$ and $z_i(t)$) will also be mutually orthogonal. Indeed,

$$\int_0^T z_r(t)\tilde{z}_i(t)\,dt = -\int_0^T \tilde{z}_r(t)z_i(t)\,dt = \frac{T}{2}\sum_{k=1}^{K}(a_{rk}b_{ik} - b_{rk}a_{ik}). \qquad (4.17)$$

and if the right side of (4.16) is equal to zero, the right side of (4.17) may be not equal to zero. If still the following conditions are met simultaneously

$$\left.\begin{array}{l}\int_0^T z_r(t)z_i(t)\,dt = 0,\\ \int_0^T z_r(t)\tilde{z}_i(t)\,dt = 0,\end{array}\right\} \qquad (4.18)$$

then signals $z_r(t)$ and $z_i(t)$ are called orthogonal in an intensified sense.

Signals (5.5[illegible]), (5.5[illegible]) and (5.5[illegible]) are examples of pairs of signals which are orthogonal in an intensified sense. Signals (5.5[illegible]) and (5.57) are orthogonal but not in an intensified sense. We can easily see this by replacing either of a pair of signals with a conjugate signal and computing its scalar product with the second signal.

We will note that the condition of orthogonality in an intensified sense can be written using analytical signals:

$$\int_0^T Z_r(t) Z^*_l(t)\,dt = 0, \qquad (4.18a)$$

where $z^*_l(t) = z_l(t) - j\tilde{z}_l(t)$ is a function complex conjugate with $Z_l(t)$.

A system of m signals is called orthogonal in an intensified sense if condition (4.18) holds for any pair of signals.

4.3. Decision System in the Case of Absolutely Incoherent Reception

Based on the criterion of maximal likelihood, we will find the optimal decision principle in the case of reception of a single signal element

$$z'(t) = \mu z_{r,\varphi}(t) + n(t) = \mu z_r(t)\cos\varphi + \mu\tilde{z}_r(t)\sin\varphi + n(t), \qquad (4.19)$$

$$r = 1, 2, \ldots, m, \quad 0 < t < T,$$

and assuming the initial phase of φ to be a random variable uniformly distributed over the interval from zero to 2π.

$$w(\varphi) = \frac{1}{2\pi}, \quad 0 \leq \varphi \leq 2\pi$$

For this purpose it is necessary to find the conditional densities of probability distribution $w(z'|z_r)$ $(r = 1, \ldots, m)$ and to determine the greatest of them corresponding to the most likely of the possible transmitted signals. These densities are

$$w(z'|z_r) = \int_0^{2\pi} w(\varphi)\, w(z'|z_r, \varphi)\, d\varphi, \qquad (4.20)$$

where $w(z'|z_r, \varphi)$ is the density of the probability distribution of signal $z'(t)$ on condition that signal $z_r(t)$ was transmitted and the shift in place in φ assumed the value φ.

In light of (3.2) and (3.3) we may express the signal received in the form

$$\begin{aligned} z'(t) = \sum_{k=1}^{\sigma} [(\mu a_{rk}\cos\varphi + \mu b_{rk}\sin\varphi + \alpha_k)\cos k\omega_1 t + \\ + (\mu b_{rk}\cos\varphi - \mu a_{rk}\sin\varphi + \beta_k)\sin k\omega_1 t] = \\ = \sum_{k=1}^{\sigma} (A_k\cos k\omega_1 t + B_k \sin k\omega_1 t), \end{aligned} \qquad (4.21)$$

whence

$$A_k = \mu(a_{rk}\cos\varphi + b_{rk}\sin\varphi) + \alpha_k,$$
$$B_k = \mu(b_{rk}\cos\varphi - a_{rk}\sin\varphi) + \beta_k.$$

With known r and φ the probability of reception of signal $z'(t)$ is nothing other than the probability that the magnitudes α_k and β_k, which describe actual interference, will take the values

$$\left.\begin{aligned} \alpha_k &= A_k - \mu(a_k \cos\varphi + b_k \sin\varphi) \\ \beta_k &= B_k - \mu(b_k \cos\varphi - a_k \sin\varphi) \end{aligned}\right\} \qquad (4.21)$$

If the interference is normal white noise then all α_k and β_k are mutually independent random variables with a normal probability distribution. We will first assume that a receiver analyzes signal components with frequencies less than $K\omega_0$ where K is a number as large as desired but finite. Then the joint probability density for α_k and β_k is expressed as:

$$w(\alpha_1, \alpha_2, \ldots, \alpha_K, \beta_1, \beta_2, \ldots, \beta_K) = \frac{1}{(2\pi\sigma_0^2)^K} \exp \times \left[-\frac{1}{2\sigma_0^2}\sum_{k=1}^{K}(\alpha_k^2 + \beta_k^2)\right]. \qquad (4.22)$$

Here $\sigma_0^2 = \overline{\alpha_k^2} = \overline{\beta_k^2}$ and, as was shown in Chapter III

$$\sigma_0^2 = \frac{\sigma^2}{T},$$

where σ^2 is the spectral density of white noise.

Based on (4.21) and (4.22) it is easy to determine the conditional probability density of reception of signal $z'(t)$ with transmission of symbol y_r and when $\varphi < \psi < \varphi + d\varphi$:

$$w(z' \mid y_r, \varphi)\, d\varphi = \frac{1}{(2\pi\sigma_0^2)^K}\frac{d\varphi}{2\pi} \exp\left\{-\frac{1}{2\sigma_0^2}\sum_{k=1}^{K}\left[A_k - \mu(a_k\cos\varphi + b_k\sin\varphi)\right]^2 + \left[B_k - \mu(b_k\cos\varphi - a_k\sin\varphi)\right]^2\right\}. \qquad (4.23)$$

Substituting the expression obtained in (4.20) we find the conditional distribution density of signal z' with transmission of symbol y_r to be:

$$w(z' \mid y_r) = \frac{1}{2\pi(2\pi\sigma_0^2)^K} \exp\left[-\frac{\sum(A_k^2 + B_k^2)}{2\sigma_0^2} - \frac{\mu^2\sum(a_k^2 + b_k^2)}{2\sigma_0^2}\right] \times \int_0^{2\pi} \exp\left\{\frac{\mu}{\sigma_0^2}\left[\sum(A_k a_k + B_k b_k)\cos\varphi + \sum(A_k b_k - B_k a_k)\sin\varphi\right]\right\} d\varphi. \qquad (4.24)$$

in which summation everywhere extends from $k = 1$ to $k = K$.

Making use of (3.26) and introducing the additional magnitudes

$$\left.\begin{aligned} Y_r &= \mu\sum_{k=1}^{K}(A_k b_{rk} - B_k a_{rk}),\\ V_r &= |X_r - jY_r| = \sqrt{X_r^2 + Y_r^2},\\ \xi_r &= \operatorname{arctg}\frac{Y_r}{X_r}, \end{aligned}\right\} \tag{4.25}$$

we will transform the integrand in (4.24) in the following way:

$$\exp\left\{\frac{\mu}{\sigma_0^2}\left[\sum_{k=1}^{K}(A_k a_{rk} + B_k b_{rk})\cos\varphi + \sum_{k=1}^{K}(A_k b_{rk} - B_k a_{rk})\sin\varphi\right]\right\} = \exp\left\{\frac{1}{\sigma_0^2}\operatorname{Re}\left[(X_r - jY_r)e^{j\varphi}\right]\right\} =$$

$$= \exp\left\{\frac{1}{\sigma_0^2}\operatorname{Re}\left[V_r e^{j(\varphi - \xi_r)}\right]\right\} = \exp\left\{\frac{1}{\sigma_0^2}V_r\cos(\varphi - \xi_r)\right\}.$$

Thus, after setting $\varphi - \xi_r = \psi$, we obtain

$$w(z'|z_r) = \frac{1}{2\pi\sigma_0^2}\exp\left(-\frac{P_x + P_r}{\sigma_0^2}\right)\times\int_0^{2\pi}\exp\left(\frac{V_r}{\sigma_0^2}\cos\psi\right)d\psi.$$

The latter integral is expressed by a modified Bessel function of the zero order [2]:

$$I_0(x) = \frac{1}{2\pi}\int_0^{2\pi}e^{x\cos t}\,dt = \sum_{n=0}^{\infty}\frac{\left(\frac{x}{2}\right)^{2n}}{(n!)^2}. \tag{4.26}$$

Finally, we have

$$w(z'|z_r) = \frac{1}{\pi\sigma_0^2}I_0\left(\frac{V_r}{\sigma_0^2}\right)\exp\left(-\frac{P_x + P_r}{\sigma_0^2}\right). \tag{4.27}$$

In accordance with the criterion of maximum likelihood the decision that signal $z_r(t)$, corresponding to symbol y_r, was transmitted must be received when with all $s \neq r$, $w(z'|z_r) > w(z'|z_s)$ or, according to (4.27), when

$$I_0\left(\frac{V_r}{\sigma_0^2}\right)\exp\left(-\frac{P_r}{\sigma_0^2}\right) > I_0\left(\frac{V_s}{\sigma_0^2}\right)\exp\left(-\frac{P_s}{\sigma_0^2}\right). \tag{4.28}$$

This decision principle can be represented in a more convenient form by taking the logarithms of both sides of the inequality:

$$\ln I_0\left(\frac{V_r}{\sigma_0^2}\right) - \frac{P_r}{\sigma_0^2} > \ln I_0\left(\frac{V_s}{\sigma_0^2}\right) - \frac{P_s}{\sigma_0^2}. \tag{4.29}$$

Parameters X_r, Y_r, and V_r can be written in integral form:

$$X_r = \frac{2k}{T}\int_0^T z'(t)\,x_r(t)\,dt$$

$$Y_r = \frac{2k}{T}\int_0^T z'(t)\,y_r(t)\,dt \qquad (4.29)$$

$$V_r = \frac{2k}{T}\sqrt{\left[\int_0^T z'(t)\,x_r(t)\,dt\right]^2 + \left[\int_0^T z'(t)\,y_r(t)\,dt\right]^2} = \frac{k}{T}\left|\int_0^T Z'(t)\,Z_r^*(t)\,dt\right|,$$

where $y_r(t)$ is a signal conjugate with $x_r(t)$; $Z'(t)$ and $Z_r(t)$ are analytical signals corresponding to $z'(t)$ and $x_r(t)$, and the "asterisk" indicates a complex conjugate function.

It is easy to see that the expressions (4.29) hold by substituting in them (3.2), (3.5), and (3.7).

We will examine systems which realize the algorithm obtained.

Quadrature System

Decision rule (4.28a) is optimal for absolutely incoherent reception. Based on this fact it is possible to construct a reception system (Figure 4.1) which is suitable for any given signal system. This system contains m generators which produce the shape of expected signals with an accuracy to the phase of the high-frequency filling and m pairs of multipliers. A received signal (together with interference) and the voltage of one of the local generators, either directly or with a phase change of 90°, arrive at each of the multipliers. The output of each of the multipliers is integrated in the same way as described in Chapter III, as a result of which voltages are obtained which are numerically equal to kX_r and kY_r (where k is an arbitrary scale of signal shape reproduction).

These voltages arrive at nonlinear devices with a quadrature characteristic after which the voltages corresponding to the same r subscripts are added in pairs. The resulting summated voltages are equal numerically to $k^2V_r^2$. They arrive at nonlinear devices having the characteristic

$$u_{out} = \ln I_0\left([illegible] \right)$$

and then at subtractors in which the voltage delivered to the input is decreased by a magnitude numerically equal to [illegible]. The voltages at the output of each channel are equal to the right side, or, when r = [illegible], the left side of inequality (4.28). They are compared with one another and the greatest of them determines that one of the possible symbols which must be selected by the decision system. The system shown in Figure 4.1 is called a quadrature system.

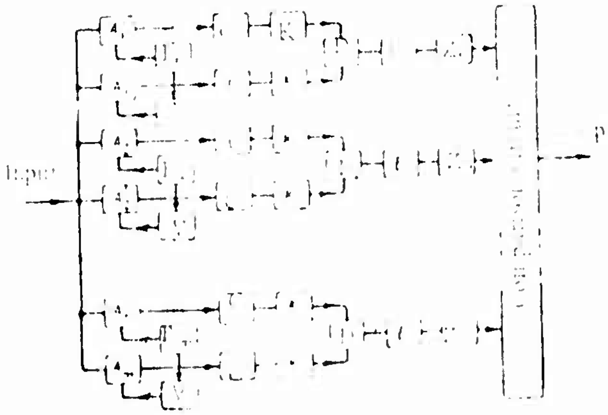


Figure 4.1. Decision System for a Signal with an Indeterminate Phase: A, Multiplier; B, Local signal generator; V, 90° phase inverter; G, Integrator; K, Device with a quadrature characteristic; D, Summator; E, Nonlinear device; Zh, Computer.

The decision principle obtained (4.28a) can be greatly simplified for systems with an active delay when the powers of all variants of the signal are the same. In light of the fact that with all $x > 0$ the function $\log I_0(x)$ is monotonically increasing, the decision principle can be formulated as follows: an ideal receiver or system with an active interval and with an indeterminate phase of signal must register the symbol x_i if for all $r \neq i$

$$V_r < V_i \qquad (4.30)$$

For all systems with an active interval the quadrature system is greatly simplified (Figure 4.2). In this case nonlinear devices E and subtracting devices Zh are not needed and voltages numerically equal to V_r are delivered directly to the system for comparison.

System with Matched Filters

Besides the quadrature reception systems there are other possible ones which permit reception in accord with principle (4.28) or (4.30). These systems may be based on matched filters [3,4].

As was shown in Chapter III, filter matched to signal $z_r(t)$ has the impulse response

$$g_r(t) = a s_r(t_0 - t)$$

or, with accuracy to a constant factor

$$g_r(t) = \sum_{k} [A_{rk} \cos k\omega (t_0 - t) + B_{rk} \sin k\omega (t_0 - t)] \quad \text{when } t_0 - T < t < t_0, \qquad g_r(t) = 0 \tag{4.31}$$

at other values of t ($t_0 \geq T$ is any delay, but common to all the filters of a given delay system).

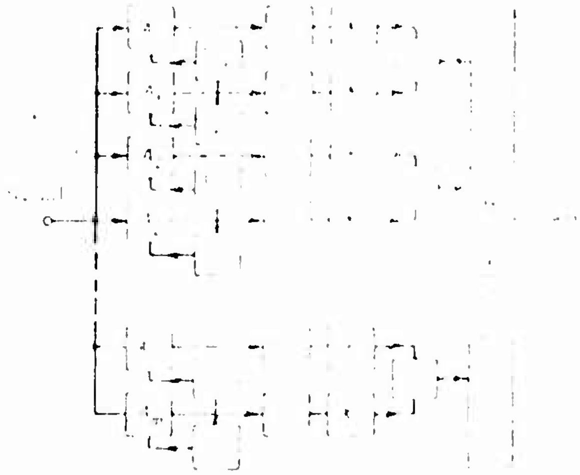


Figure 4.[illegible]. Decision System for Signals with an Active Interval and Indeterminate Initial Phase: A, Multiplier; B, Local signal generator; [illegible], 90° phase inverter; [illegible], Integrator; [illegible], Device with a Quadrature Characteristic; [illegible], Summator

Let the receiver contain matched filters for each of the [illegible] of expected signal. The received signal z'(t) goes to the inputs of all the filters. Voltage $e_r(t)$ at the output of the r-th filter at some moment t between t_0 and $t_0 + T$ is determined by Duhamel's integral

$$e_r(t) = \int_0^\infty z'(t - x) g_r(x)\, dx = \int_{[illegible]}^{[illegible]} z'(x) g_r(t - x)\, dx \tag{4.32}$$

Substituting expression z'(t) in the form of a series (3.2) and the value of $g_r(t - x)$ from (4.31), and assuming for simplicity that [illegible] = 1, we find

$$e_r(t) = \int_{t - t_0}^{t - t_0 + T} \sum_{k=1} (A_k \cos k\omega x + B_k \sin k\omega x) [illegible]$$

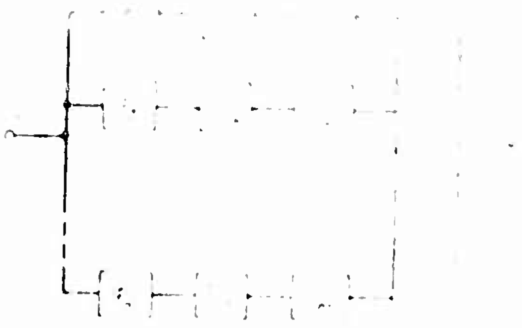


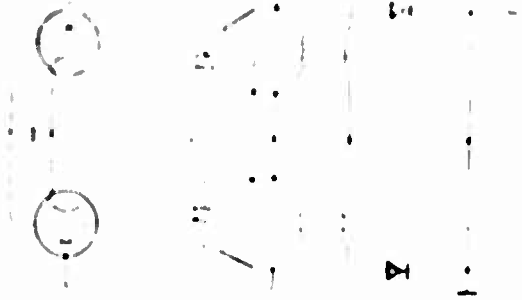
Figure 4.3. Decision System with Matched Filters for Signals with [illegible] Initial Phase: Φ_r, Matched Filter for r-th signal; D, Rectifier with a Characteristic of $y = \log I_0(x)$; [illegible], Subtracter.

For systems with an active internal there is no need for [illegible] devices and the result of rectification of the voltage envelopes at the filter outputs at instant $t = t_0$ can be delivered directly to a comparator. The rectification characteristic in this case can be [illegible] as it is the same for all the filters in the system in accordance with the decision principle expressed by (4.5[illegible]).

Both the systems presented provide for optimal incoherent reception inasmuch as they are based on the optimal decision principle [illegible] of the maximal likelihood criterion). We will note that in the case of a system which is orthogonal in an intensified sense each signal at the instant of reading creates a voltage only at the output of one of the [illegible] in a quadrature reception circuit (Figure 4.1[illegible]). At the other outputs [illegible] are created only by interference. This is a consequence of condition (4.2[illegible]) and the identity [illegible]. In fact, in the [illegible] of [illegible] a signal [illegible] orthogonal in an intensified sense to that signal with which a given filter is matched arrives at the input of a filter, then the envelope of the output voltage at the instant of reading is equal to [illegible]. Orthogonality in an intensified sense is sometimes called envelope orthogonality.

Instead of [illegible] matched filters in the circuit of Figure 4.5 can be replaced [illegible] the interval [illegible] pulse [illegible] and when [illegible] arbitrary [illegible] [illegible] ory loop [illegible]

If a signal arrives at a filter [illegible] orthogonal to the signal with which the filter is matched, but not in an intensified sense, then the voltage at the output of the filter at the instant of reading [illegible]



[illegible]

[illegible] tuned to these frequencies. As shown in the figure, the currents of the res- [illegible] and [illegible] across the resistor [illegible] called a [illegible] without particular justification.

[illegible] would be [illegible] resonant frequency. If the oscil- lations are damped [illegible] entirely equivalent to a [illegible] with signal [illegible] shows [illegible] the output [illegible] in figure [illegible] with [illegible] and with a frequency difference [illegible]

In the first case [illegible] linearly and in the second case it changes [illegible] are multiples of [illegible] the difference between the frequencies of signals [illegible] and [illegible] multiple of [illegible], then at the instant of reading each signal creates [illegible] at the output of its rectifier while at the other rectifier outputs [illegible] are created only by interference. As can be seen, the condition coincides with the condition

---

*In such circuits use is often made of electromechanical (piezoelectric or magnetostrictive) instead of the usual electrical oscillatory loops, sometimes with positive feedback. Instead of short-circuiting, the oscillations are damped by delivering a negative feedback.

of orthogonality in the intensified sense which, particularly, an EK system satisfies (3.56).

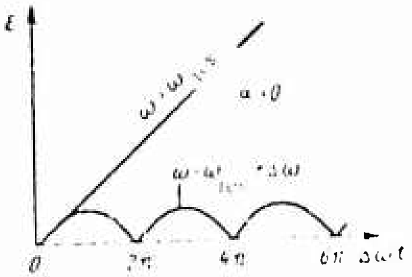


Figure 4.5. Voltage Envelope at Output of Ideal Loop.

An actual loop has losses and its pulse response is

[illegible]

where [illegible] res Q; and Q is the [illegible] Q factor. When [illegible] res [illegible], this pulse response differs little from the ideal. Still the shape of the envelope in this case will differ from that shown in Figure 4.5 and, in particular, the envelope when [illegible] 0 does not become zero. Figure 4.6 shows the course of envelope when [illegible] 0.1 [illegible]. If we consider [illegible] 2π·1, [illegible] res [illegible] corresponds to this value of [illegible].

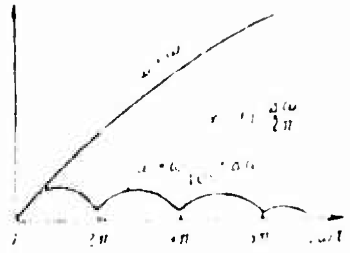


Figure 4.6. Voltage Envelope at Output of Actual Loop.

By achieving a high Q-factor in a loop, it is possible to approach the ideal case shown in Figure 4.5. We will note that damping oscillations permits reception as shown in Figure 4.4 with as high Q-factors in the loops as desired. Without damping of oscillations this would be impossible since the oscillations caused by preceding elements of a received signal would be preserved in a loop with a high Q-factor.

[illegible]

where P_1 is the power of signal s_1 [illegible] inasmuch as they have the same coefficients in a Fourier expansion, and where we use X^0 and Y^0 to designate the integrals $\frac{2}{T}\int$ [illegible] and $\frac{2}{T}\int_0^T$ [illegible]. We will note that for a system with a [illegible] interval the power of all signals are the same and in what follows we will designate that [illegible]

$$[illegible]$$

$$[illegible]$$

[illegible]

$$[illegible]$$

[illegible]

$$[illegible]$$

[illegible]

$$[illegible]$$

$$[illegible]$$

Based on [illegible]

The results obtained permit us to write the joint density of the probability distribution 2m of the random variables V_r, θ_r:

$$[illegible] \qquad (4.43)$$

We will shift from variables V_r, θ_r to V_r, φ_r using the formula [illegible], and then we will set [illegible]

$$[illegible] \qquad (4.44)$$

(the product V_r enters this expression as a Jacobian transform).

The density obtained does not depend on [illegible]. In order to find the joint density of m magnitudes of V_r, we must integrate (4.44) m times with respect to φ_r, as a result of which we obtain

$$[illegible] \qquad (4.45)$$

In accordance with the principle of (4.50), the decision system will select symbol y_i which was actually transmitted if the magnitude of V_i is greater than any other magnitude of V_r. Therefore, the probability of inequality (4.50) when all V_r have the density of (4.45). For computing the probability q of correct reception it is necessary to integrate (4.45) for all V_r in that range where V_i [illegible] V_r (r = 1,...,m; r ≠ i). This problem is rather simple since all the variables of integration separate as follows:

$$q = \int\cdots\int w(V_1, \ldots, V_m)\,dV_1 \ldots dV_m$$

[illegible] (4.46)

where we use the substitution [illegible] and the designation [illegible] $= 2P_s/Q$ introduced in Chapter III.

After expanding [illegible] by Newton's binomial formula we reduce integral (4.46) to the sum of the tabular integrals

[illegible] (4.47)

From this the probability of error is

[illegible] (4.48)

The result derived indicates that error probability in an active-interval system orthogonal in the intensified sense is, just as when the signal phase is unknown, uniquely determined by the ratio of the signal element power to spectral noise density h^2. With a given signal power and spectral density of fluctuation interference, neither the frequency band occupied by the signal nor any other signal parameters affect error probability, provided that the signals satisfy condition (4.18).

We also easily ascertain that in an active-interval system orthogonal in the intensified sense the discrete transform of the channel is symmetrical, i.e., the probabilities of all types of error are identical.

In the case when m = 2, it follows from (4.48) that

Figure 4.7 shows the relation (curve a). The same figure (curve b) shows the dependence of the error probability on h² for optimum coherent reception plotted from formula (4.54) for a binary orthogonal system. It ensues from this figure that the lack of a priori information about the expected signal phase increases relatively little the probability of error and may be compensated for by a slight increase in signal power.

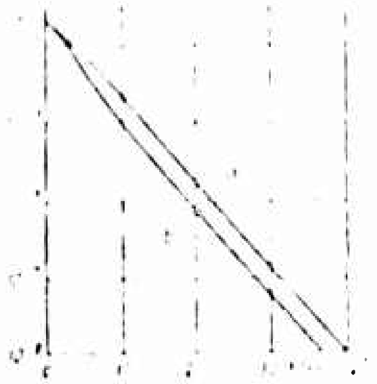


Figure 4.7. Probability of Error (a) for Coherent and (b) Incoherent Reception (Active-Interval Orthogonal System).

Let some error probability p, characterizing the required reception reliability, be attainable when $h^2 = h_1^2$ in the case of coherent reception and when $h^2 = h_2^2$ in the case of incoherent reception. Then ratio h_2^2/h_1^2 represents the power loss caused by lack of a priori information about initial signal phase. Figure 4.8 shows the dependence of this loss in decibels on tolerable error probability. When adequately high requirements are made on reception reliability the power loss does not exceed 1 db. This may explain the fact that in practice incoherent reception is employed comparatively more often than coherent, even in cases where received signal phase does not fluctuate or fluctuates very slowly, because the power gain afforded by coherent reception does not make up to the system for its loss [illegible] phase alignment in the receiving unit.

Analysis of formula (4.48) can satisfy us that with a fixed value of h² the probability of error increases with increase in code base m, but from this we should not draw the premature conclusion that the resistance of communication to noise decreases as the code base increases. As was shown in Chapter II, the equivalent error probability should be taken into account to evaluate the reliability of information transmission. With nonredundancy coding the equivalent error probability is $p_e = p/\log_2 m$ (2.65). Furthermore, at a given transmission rate the signal element duration, and hence its power, is proportional to log m. Therefore the different systems should be compared when the they have the same values of parameter $h^2 \log m$, which is an invariant if information transmission rate and signal power are given.

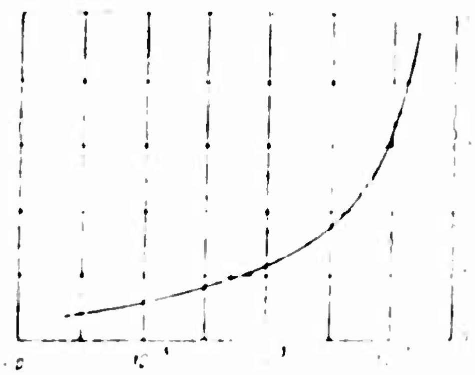


Figure 4.8. Dependence of Power Loss on Permissible Error Probability in Converting from Coherent to Incoherent Reception.

Figure 4.9 shows how [illegible] depends on parameter $h^2 \log_2 m$ with various code bases m in [illegible] systems orthogonal in the intensified sense. It may be deduced from the figure that the increase in communication reliability is [illegible] with a large base, but it should be taken into account that increasing the code base almost always involves increased instrumentation complexity. It moreover often results in [illegible] widening the frequency band [illegible] by the signal, and this is undesirable in many cases.

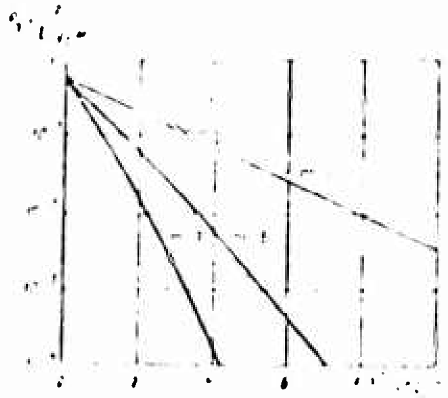


Figure 4.9. Comparison of Resistance to Interference in Orthogonal Systems with Different Code Bases.

System with Passive Interval

When the powers of signals are not the same, even when orthogonality is retained, a general expression for error probability cannot be obtained and it must be computed separately for each specific system. By way of example we will consider a simple binary system with amplitude keying (AK) with signals

$$[illegible]$$

where [illegible] is a random initial phase.

Here the signal power is [illegible] and P_2 = [illegible]. Assuming the a priori probabilities of the signals to be the same, we obtain the average power of a signal

$$[illegible]$$

We will determine the magnitudes of λ_i from formula [illegible]

$$[illegible]$$

According to [illegible] it must be decided that symbol x_1 was transmitted if

$$[illegible]$$

An error will occur only when inequality [illegible] is not fulfilled in transmitting "send" symbol x_1 and when in transmitting "interval" symbol x_2 it is fulfilled.

We will first assume that [illegible] 1. Under this condition there is great probability that

$$[illegible]$$

When $x \ll 1$, $\ln I_0(x) \approx x$ and inequality (4.50) may be approximately replaced by a simpler inequality

---

*A similar problem of optimal incoherent detection of a sinusoidal signal in white noise has been studied in detail in radar theory. However, in view of the important difference in the cost of error (false alarm or missing a signal) use is usually made in radar of the Neiman-Pearson criterion. Here, in accordance with peculiarities of a system for transmitting discrete messages, use is made of the maximal likelihood criterion which coincides in the case of the same a priori probabilities of a signal with the ideal observer criterion.

$$\sqrt{X_k^2 + Y_k^2} > \frac{\mu a}{2}. \qquad (4.50a)$$

which means that when there is a strong signal the ideal receiver must register symbol y_1 in the case where the amplitude of the component of the received signal with a frequency of $k\omega_0$ is greater than the operating threshold which is half the amplitude of the expected signal.

Let us find the error probability during transmission of an interval, i.e., the probability that inequality (4.50) will be fulfilled during the interval. In this case

$$r = \sqrt{X_k^2 + Y_k^2} \quad [illegible]$$

represents a random variable with the Rayleigh distribution

$$w(r) = \frac{r}{\sigma_0^2}\exp\left(-\frac{r^2}{2\sigma_0^2}\right) \text{ (where } r > 0\text{)}.$$

The probability $p\_$ of error during the interval is the probability that r exceeds $\mu a/2$

$$p = \int_{\mu a/2}^{\infty} \frac{r}{\sigma_0^2}\exp\left(-\frac{r^2}{2\sigma_0^2}\right) dr = \exp\left(-\frac{\mu^2 a^2}{8\sigma_0^2}\right) = e^{-\frac{h^2}{2}}, \qquad (4.51)$$

where $h^2 = \frac{\mu^2 a^2}{4\sigma_0^2} = \frac{\mu a^2 T}{4\nu_0}$ is the ratio of average signal element power to spectral noise density.

Now let us find the probability of error during sending, i.e., the probability that inequality (4.50) will not be fulfilled when

$$\sqrt{X_k^2 + B_k^2} \quad [illegible]$$

Variable

$$r = \sqrt{X_k^2 + Y_k^2}$$

submits to the generalized Rayleigh distribution [6]

$$w_1(r) = \frac{r}{\sigma_0^2}\exp\left(-\frac{r^2 + \mu^2 a^2}{2\sigma_0^2}\right) I_0\left(\frac{r\mu a}{\sigma_0^2}\right) \quad (r > 0)$$

The probability that an error will occur during sending, i.e., that r will adopt a value less than $\mu a/2$, is

$$p_+ = \int_0^{\frac{\mu a}{2}} \frac{r}{\sigma_0^2}\exp\left(-\frac{r^2 + \mu^2 a^2}{2\sigma_0^2}\right) I_0\left(\frac{r\mu a}{\sigma_0^2}\right) dr \qquad (4.52)$$

$$= 1 - \int_h^{\infty} x\exp\left(-\frac{x^2 + 4h^2}{2}\right) I_0(2hx)\, dx$$

$$= 1 - Q(2h, h).$$

where $Q(x,y)$ is a special tabulated function, and

$$Q(x,y) = \int_{y}^{\infty} t \exp\left(-\frac{t^2+x^2}{2}\right) I_0(xt)\,dt \qquad (4.53)$$

Integrating by parts, we may represent the Q-function in the form of a series

$$Q(x,y) = e^{-\frac{x^2+y^2}{2}} \sum_{n=0}^{\infty} \left(\frac{x}{y}\right)^n I_n(xy) \qquad (4.53a)$$

The probability of error during send is somewhat less than the probability of error during an interval. The complete probability of error is

$$p = \frac{1}{2}(p_- + p_+) \qquad (4.54)$$

Therefore an amplitude keying system with optimal incoherent reception is asymmetrical. The operating threshold could also be chosen such that the error probabilities during sending and interval were identical, but in this case reception will not proceed in accordance with optimum principle (4.50), hence the total error probability will rise.

We will now drop the condition that $a/\sigma_0 \gg 1$.

We will designate by the function which is inverse to $\log I_0$, i.e., $y = (x)$, if $x = \ln I_0(y)$. Figure 4.10 displays the graph of function $y = (x)$. Inequality (4.50) may then be written:

$$\sqrt{z_0 + E_i} > \frac{\Gamma\left(\frac{\mu^2 a^2}{2\sigma}\right)}{\mu a}\, z_0 \quad \text{or} \quad \sqrt{z_0 + E_i} > z_0 \frac{\Gamma(\ldots)}{\ldots}$$

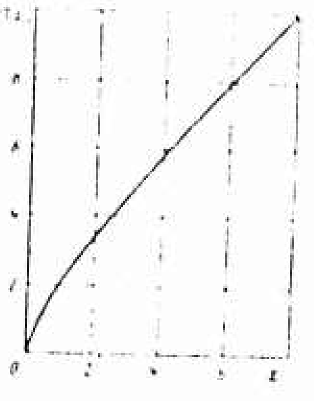


Figure 4.10. Graph of Function y = .(x).

This inequality differs from (4.50a) in that the expression on the right side depends not only on [illegible]a, but also on σ_0.

Therefore with a weak signal the optimum operating threshold is determined not only by the amplitude of the incoming signal, but also by the interference level. Let us denote the optimum operating threshold by

$z_0 = \sigma_0 \frac{1(2h^2)}{2h}$. Variable $\frac{z_0}{\sigma_0} = \frac{1(2h^2)}{2h}$ we will then call the optimum relative operating

threshold. Figure 4.11 shows the relationship between h and this threshold. With larger values of h it tends toward h = a/2σ_0, which agrees with (4.50a).

We will determine the error probabilities by substituting z_0 for a/2 as the integration limits in expression (4.51) and (4.52):

$$P' = \exp\left(-\frac{z^2}{2\sigma_0^2}\right) = \exp\left[-\frac{[illegible]}{[illegible]}\right], \qquad (4.53)$$

$$P'' = 1 - Q\left([illegible]\right) = 1 - Q\left[[illegible]\right].$$

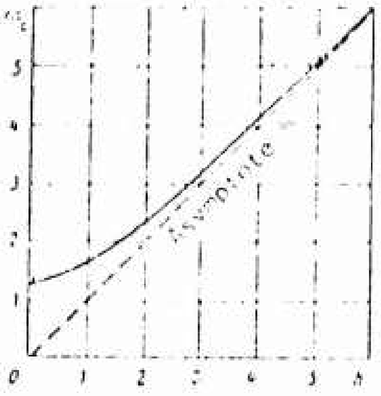


Figure 4.11. Dependence of Optimum Relative Operating Threshold on h.

Figure 4.12 represents the total error probability figured by substituting (4.53) in (4.54). We will note that in the area of error probabilities which are not very small (p > 10[illegible]) this curve approximates formula (4.49) satisfactorily, i.e., an amplitude modulation system differs very little in resistance to interference from binary orthogonal systems with an active interval. However, it should be kept in mind that here the comparison is made with the same average power (the same h). In this case the peak power (sending power) in an amplitude modulation system is twice as great as in a frequency modulation system.

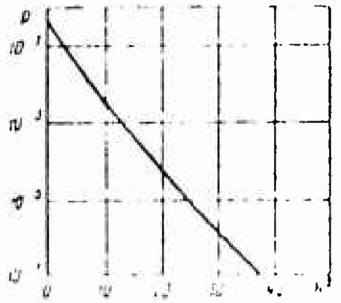


Figure 4.12. Dependence of Error Probability on h in an Amplitude Keying System.

Nonorthogonal Systems with an Active Interval

Calculation of error probability in the case when the conditions of (4.18) are not met can be performed in the same way as in the preceding examples. However, great difficulty arises here in computing the joint density of V_r since X_r, Y_r, $\tilde{X}_r$ and $\tilde{Y}_r$ are not independent.

We will first consider a binary system with an active delay and assume that signal $z_1(t)$ was transmitted and the initial phase had a value of φ. To compute the error probability with optimal incoherent reception it is necessary to find the probability distribution density of the variables X_1, Y_1, X_2, and Y_2 and integrate it over the area for which

$$X_1^2+Y_1^2<X_2^2+Y_2^2.$$

Obviously, in this case the distribution of X and Y is normal. We will compute their moments:

$$\begin{aligned}\overline{X}_1&=\frac{2a}{T}\int_0^T \overline{z'(t)z_1(t)}\,dt\\&=\frac{2a}{T}\left\{\int_0^T z_1^2(t)\cos\varphi\,dt+\int_0^T z_1(t)\tilde{z}_1(t)\sin\varphi\,dt\right\}+\\&+\int_0^T \overline{n(t)z_1(t)}\,dt=2P_c\cos\varphi;\end{aligned}\tag{4.56}$$

and similarly

$$\overline{Y}_1=2P_c\sin\varphi,$$

$$\overline{X}_2=\frac{2a}{T}\int_0^T \overline{z'(t)z_2(t)}\,dt$$

$$= \frac{2\mu}{T}\left[\int_0^T \mu z_1(t)\, z_2(t)\cos\beta\, dt + \int_0^T \mu \tilde{z}_1(t)\, z_2(t)\sin\beta\, dt + \int_0^T n(t)\, z_2(t)\, dt\right] = 2P_s(\rho_1\cos\beta + \rho_2\sin\beta), \tag{4.57}$$

$$\bar{Y}_2 = 2P_s(\rho_2\cos\beta - \rho_1\sin\beta),$$

where we introduce the designation

$$\left.\begin{aligned} \rho_1 &= \frac{\mu^2}{TP_s}\int_0^T z_1(t)\, z_2(t)\, dt = \frac{\mu^2}{TP_s}\int_0^T \tilde{z}_1(t)\, \tilde{z}_2(t)\, dt, \\ \rho_2 &= \frac{\mu^2}{TP_s}\int_0^T \tilde{z}_1(t)\, z_2(t)\, dt \\ &= -\frac{\mu^2}{TP_s}\int_0^T z_1(t)\, \tilde{z}_2(t)\, dt. \end{aligned}\right\} \tag{4.57a}$$

We will note that a Bunyakovskiy-Shwartz inequality yields the following expression

$$0 \le \rho = \sqrt{\rho_1^2 + \rho_2^2} \le 1 \tag{4.57b}$$

The variable ρ characterizes the degree of deviation from orthogonality inasmuch as when $\rho = 0$ the signals are orthogonal in an intensified sense and when $\rho = 1$ they coincide with an accuracy to the initial phase.

As in (4.41) we obtain for the second moments

$$\overline{X_1'^2} = \overline{(X_1 - \bar{X}_1)^2} = \frac{4\mu^2}{T^2}\overline{\left[\int_0^T n(t)\, z_1(t)\, dt\right]^2} =$$

$$= \mu^2\sigma_0^2\sum_{k=1}^{K}(a_{1k}^2 + b_{1k}^2) = 2\sigma_0^2 P_s = \frac{2}{T}\nu^2 P_s.$$

Obviously, for X_2, Y_1, and Y_2 the value of dispersion is given by expression (4.41). It is further easy to see that

$$\overline{X_1'Y_1'} = \overline{X_2'Y_2'} = 0,$$

$$\overline{X_1'X_2'} = \frac{4\mu^2}{T^2}\overline{\int_0^T n(t)\, z_1(t)\, dt\int_0^T n(t)\, z_2(t)\, dt} =$$

$$= \mu^2\sum_{k=1}^{K}(a_{1k}a_{2k}\overline{\alpha_k^2} + b_{1k}b_{2k}\overline{\beta_k^2}) =$$

$$= \mu^2\sigma_0^2\sum_{k=1}^{K}(a_{1k}a_{2k} + b_{1k}b_{2k}) = 2\frac{\mu^2}{T}\sigma_0^2\int_0^T z_1(t)\, z_2(t)\, dt =$$

$$= 2\sigma_0^2 P_s\rho_1 = \frac{2}{T}\nu^2 P_s\rho_1.$$

where, as everywhere, $h^2 = P_s T / N_0$.

After integrating this density with respect to φ_1 and φ_2 within limits from 0 to 2π we find the two-dimensional density of V_1 and V_2:

$$[illegible] \qquad (4.60)$$

To find the error probability it is only necessary to integrate this two-dimensional density in the range where $V_1 < V_2$. Expanding the Bessel function into a series after simple but rather cumbersome transformations, we obtain the following result [ ]:

$$[illegible] \qquad (4.61)$$

where I_n [illegible] of the nth order.

[illegible], with signals [illegible] the series approach [illegible]

[illegible] was pre[illegible]

[illegible] terms [illegible]

$$[illegible]$$

[illegible]

$$p = Q\left(\sqrt{\frac{h^2}{2}\left(1 - \sqrt{1-\rho^2}\right)},\ \sqrt{\frac{h^2}{2}\left(1 + \sqrt{1-\rho^2}\right)}\right) - \frac{1}{2} e^{-\frac{h^2}{2}} I_0\left(\frac{\rho h^2}{2}\right). \qquad (4.61b)$$

Figure 4.13 shows the dependence of error probability on h^2 for various values of ρ. As is apparent from this figure, the most resistant to interference is the orthogonal system ($\rho = 0$). With an increase in ρ from 0 to 0.4, the probability of error increases relatively little and the deviation from orthogonality may be compensated for by a slight increase in the power of the signal. This is even more clearly demonstrated in Figure 4.14 which shows the dependence of the required value of h^2 on ρ with a given probability of error. When ρ approaches unity, the signals become indistinguishable (with incoherent reception and no increase in power will compensate for the drop in fidelity.

We will note that variable ρ has a simple physical meaning. The reader can easily see that it is equal to the ratio between the envelope in a filter matched with $z_2(t)$ and the envelope in a filter matched with $z_1(t)$ at the instant of reading if signal $z_1(t)$ is delivered to them without interference.

For systems with a code base of $m > 2$ and nonorthogonal signals it is not possible to obtain with reasonable simplicity general expressions for the probability of error in the case of optimal incoherent reception. However, for some particular cases it is possible to obtain exact solutions for at least evaluations by using more or less artificial procedures. Here, as in the case of coherent reception, it is sometimes possible to reduce the problem to a more simple one by using the isomorphism of systems. However, in the case of incoherent reception equality of Kotelnikov distances is not sufficient for isomorphism of systems. It is further necessary that this equality be retained with changes in the initial phases of signals. The possibility of renumbering the signals so as to meet the following equalities is a sufficient condition for two systems to be isomorphic:

$$\left(\int_0^T z_i^{(1)}(t) z_k^{(1)}(t)\,dt\right)^2 + \left(\int_0^T z_i^{(1)}(t) \tilde{z}_k^{(1)}(t)\,dt\right)^2 =$$

$$\left(\int_0^T z_i^{(2)}(t) z_k^{(2)}(t)\,dt\right)^2 + \left(\int_0^T z_i^{(2)}(t) \tilde{z}_k^{(2)}(t)\,dt\right)^2, \qquad (4.62)$$

$$i, k = 1, 2, \ldots, m,$$

where the superscripts indicate the system.

In fact, in fulfilling condition (4.62) all magnitudes of X_i and Y_i in both systems during reception of a certain signal have the same joint distribution of probabilities and it unambiguously determines the probability of error.

By way of example we will find an estimate of the probability of error for a system with an active interval when $m = 4$, the signals of which satisfy the condition

$$\left(\frac{1}{T}\int_0^T s_i(t)\,s_k(t)\,dt\right) \quad [illegible]$$ [illegible]

$$\begin{cases} P & \text{when } [illegible] \\ 0 & \text{when } [illegible] \\ [illegible] & \text{when } [illegible] \end{cases}$$

In other words, for each of the four signals there [illegible] and there is no orthogonality for the remaining two [illegible] following satisfies this condition:

$$s_1(t) = a[\cos [illegible] + \cos [illegible]]$$
$$s_2(t) = a[\cos [illegible] + \cos [illegible]] \qquad [illegible]$$
$$s_3(t) = a[\cos [illegible] + \cos [illegible]]$$
$$s_4(t) = a[\cos [illegible] + \cos [illegible]]$$
$$0 < t \leq T,$$

where $\omega_0 = 2\pi/T$; k_1, k_2, k_3, k_4 are any whole numbers not the same; [illegible] is a random initial phase; and $[illegible]a^2 = P_s$. Despite the seeming artificiality of this example, it is useful inasmuch as it serves to demonstrate the principal methods of obtaining estimates for error probability when it is not possible to calculate an exact value. Furthermore, this example will be used later for analysis of one system which is widely used in practice.

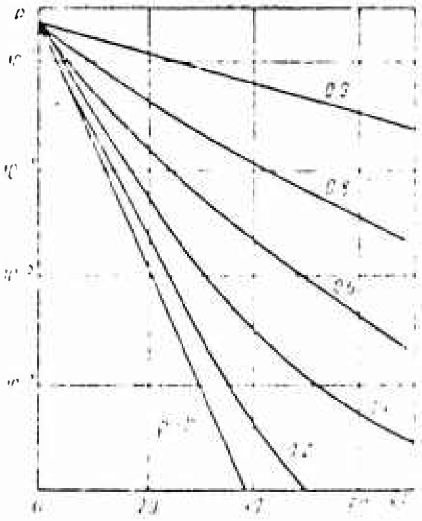


Figure 4.13. Probability of Error for a Binary System with Nonorthogonal Signals.

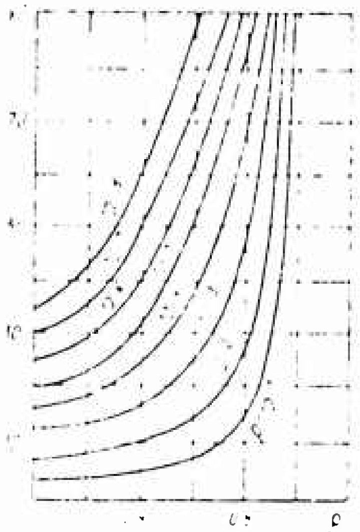


Figure 4.14. Dependence of Required Value of h² or [illegible] with Given Probabilities of Error.

An optimal system registers symbol y_1 if at the same time $V_1 > V_2$, $V_1 > V_3$, and $V_1 > V_4$, i.e.,

[illegible]

[illegible]

The [illegible] probability of error [illegible] decision system [illegible] with [illegible] under these conditions [illegible] probability of error. If we replace the principle of [illegible] the probability of error does not decrease but [illegible]. Therefore, after calculating the probability of error with a certain decision principle differing from (4.61), we obtain an estimate from above for the probability of error in an optimal decision system.

We will change the decision principle so as to simplify calculations of the probability of error and, specifically, we will assume that the decision system regulates symbol y_1 (corresponding to z_1) if the following pair of inequalities is met

$$\left(\int_0^T z\xi_1 dt\right)^2 + \left(\int_0^T z\xi_1' dt\right)^2 > \left(\int_0^T z\xi_3 dt\right)^2 + \left(\int_0^T z\xi_3' dt\right)^2 \qquad [illegible]$$

$$\left(\int_0^T z\xi_2 dt\right)^2 + \left(\int_0^T z\xi_2' dt\right)^2 > \left(\int_0^T z\xi_4 dt\right)^2 + \left(\int_0^T z\xi_4' dt\right)^2 \qquad [illegible]$$

If both these inequalities are not met, symbol y_3 is registered. Symbol y_2 is registered when inequality (4.62a) is met and inequality (4.62b) is not met. Otherwise symbol y_4 is registered. Thus, the postulated principle amounts to the following: the hypothesis that "signal" $z_1(t)$ was transmitted is compared in likelihood with the hypothesis that $z_3(t)$ was transmitted and regardless of the result the hypotheses that $z_2(t)$ or $z_4(t)$ was transmitted are compared with one another. The selection of these particular hypotheses is made based on the optimal principle of incoherent reception.

We will note that "signals" z_1, z_2, z_3, and z_4 are orthogonal in pairs in an intensified sense. Therefore, the probability of error in selection of the first pair of "hypotheses" (z_1 and z_3) does not depend on how the selection of the second pair (z_2 and z_4) was made and both probabilities are determined by formula (4.49). It is only necessary to bear in mind that the power of signal ξ_i is half the power of full signal z_i and, therefore, retains the designation h^2 for the ratio between the power of signal z_i and the

[illegible] power of the [illegible]
[illegible]

Thus, [illegible]

[illegible] principle [illegible]

[illegible]

[illegible]

and making [illegible] the signal [illegible]

$$P_{coh} = 1 - \frac{1}{4}[1 + \Phi(\sqrt{2}h)]^2$$

and inasmuch as p_{coh} is the evaluation from below for the probability of error in optimal incoherent reception, then, finally, combining it with (4.66), we find

$$1 - \frac{1}{4}[1 + \Phi(\sqrt{2}h)]^2 \leq p \leq e^{[illegible]} - \frac{1}{4}e^{[illegible]} \qquad (4.67)$$

The evaluations obtained are shown by the solid lines in Figure 4.15. These curves are sufficiently close to one another so that with practical calculations they permit evaluating the probability of error with satisfactory accuracy. If, as is often encountered in practice, it is necessary to determine for a given probability the required value of h, the average of the two evaluations gives a degree of accuracy in the range p [illegible] 10 [illegible] not greater than [illegible]. The broken line in this figure shows the probability of error of an orthogonal system when m [illegible] 4 computed with formula (4.48)

$$p [illegible]$$

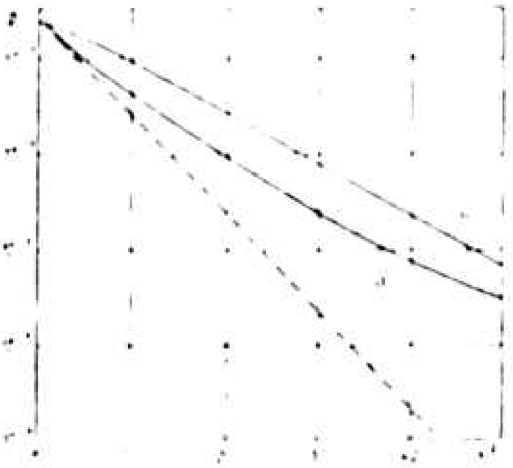


Figure 4.15. Evaluations of Error Probability [illegible] System with KAM From Below and From Above [illegible] Comparison [illegible] Orthogonal Systems [illegible] shown also.

From a comparison of these curves it is apparent that the deviation from orthogonality, as in binary systems, leads to an increase in the probability of error which can be compensated for by a rather large increase in the power of the signal.

In conclusion, we will note that discrete representation of the channel in the system under discussion proves to be asymmetrical. If symbol y_1 is transmitted and an error has occurred, symbol y_2 or y_4 will occur with a greater probability than y_3. Symbol y_2 will become y_1 or y_3 with a greater probability than y_4, etc. We will return to this in Chapter IX.

Frequency Keying (FK)

An overwhelming majority of existing communication systems which use absolutely incoherent reception are based on frequency keying. From the results obtained above it follows that the greatest resistance to interference is provided by those systems which are orthogonal in the intensified sense. Two signals which represent segments of a sinusoid of duration T with arbitrary initial phases are orthogonal in the intensified sense if their frequencies are multiples of 1/T. In order to make sure of this we will calculate the value of [illegible] for the signals

[illegible]

[illegible]

According to [illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

Similarly,

[illegible]

[illegible]

[illegible]

$$[illegible]$$

Obviously, [illegible] = 0 when and only when [illegible] = 0, and [illegible] = 0. In the given case this is met for arbitrary values of [illegible] and [illegible] if

$$(\omega_1 + \omega_2)T = 2\pi n_1 \text{ and } (\omega_1 - \omega_2)T = 2\pi n_2$$

where n_1 and n_2 are whole numbers. In this case

$$\left.\begin{array}{l} \omega_1 = \frac{\pi k_1}{T} \\ \omega_2 = \frac{\pi k_2}{T} \end{array}\right\} \qquad (4.69)$$

where $k_1 = n_1 - n_2$ and $k_2 = n_1 + n_2$ are also whole numbers.

In actual FK practice condition (4.69) is most often not observed. Instead of achieving exact orthogonality of signals in the intensified sense, systems are limited to providing approximate orthogonality, understanding by this the condition [illegible] ≪ 1. As can be seen from Figure 4.11 a binary system with [illegible] on the order of 0.1 or even 0.2 is hardly different with respect to resistance to interference from an orthogonal system.

In modern systems of "narrow-band" FK approximate orthogonality is achieved by replacing condition (4.69) with the less rigid condition

$$[illegible]$$

Indeed, in this case

$$[illegible]$$

$$[illegible]$$

Therefore,

$$[illegible]$$

and, consequently,

$$[illegible]$$

If [illegible] ≥ 30, and this is always met in practice, then [illegible] 0.1 and the signals can be considered approximately orthogonal.

In older systems of "wide-band" FM approximate orthogonality is achieved by making the difference $\omega_2 - \omega_1$ sufficiently large:

$$[illegible] \qquad (4.69b)$$

Inasmuch as [illegible] the magnitude of [illegible] in all cases is limited by the following approximate inequality:

$$[illegible]$$

and if [illegible] 2, then also [illegible] 0.1.

In order to meet condition (4.69b) it is necessary to increase the accepted band of signal frequency. If condition (4.69a) is used where k = 1, the accepted band of frequencies is [illegible] while with condition (4.69b) if [illegible] need not exceed [illegible], the accepted band of frequencies must be greater than 4 [illegible]. However, wide-band FM systems have an advantage under conditions when it is impossible to provide for a very high degree of accuracy in signal frequency inasmuch as in this case small changes in the frequency of a signal lead only to a certain [illegible] the voltage delivered to the comparison circuit (Figures 4.1 [illegible] 4.3) from that channel in which the signal is located. In a narrow-band system [illegible] signals is [illegible] simultaneously and this leads to a significant increase in the probability of error.

For an approximate quantitative evaluation of the permissible drift in frequency in arbitrary [illegible] we will consider that case when the decision system is optimal for signals with nominal frequencies and the actual frequencies of signals deviate from the nominal within limits of [illegible]. We will stipulate that a drop in resistance to interference which can be compensated for by increasing the signal power by 10 [illegible] is permissible.

Let the decision system be optimal for reception of signals $s_1(t) = [illegible]$ and $s_2(t) = [illegible]$ and let signal [illegible] actually arrive. The envelope at the instant of reading at the output of the filter matched with signal [illegible] or the voltage in the appropriate branch of the correlation system, according to (4.5) and (4.6), is equal to

$$[illegible] \qquad (4.70)$$

if we ignore interference, then

$$[illegible]$$

Substituting this in (4.70), after simple transformations we obtain

$$[illegible] \qquad (4.71)$$

As should be expected, the greatest value of $I_1(t)$ occurs when [illegible] = 0. With a drift in the frequency $I_1(t)$ decreases and this can be compensated for by increasing the power of the signal (or a^2) by [illegible]

As far as the filter matched with signal $z_2(t)$ is concerned, the voltage created across it by incoming signals $z_1(t)$, at the instant of reading, is equal practically to zero if condition (4.69b) is met, inasmuch as with small values [illegible] signals $z_1(t)$ and $z_2(t)$ remain approximately orthogonal. Thus, for a wide-band system the permissible frequency drift which can be compensated for by increasing a^2 by 10 [illegible] is determined from the equation

$$[illegible] \; 1.1$$

Expanding sin([illegible]T/2) into a Taylor series and limiting ourselves to the [illegible] term, we obtain

$$[illegible]$$

$$[illegible]$$

The matter stands otherwise with narrow-band [illegible]. In this case a small drift in frequency causes a sharp [illegible] quality which is expressed by the fact that signal [illegible] at the instant of reading, a noticeable voltage proportional to [illegible] across the filter matched with signal [illegible]. [illegible]

[illegible]

$$[illegible]$$

[illegible] considering that [illegible] and [illegible]

$$[illegible]$$

Thus, with a drift in signal frequency in narrow-band FM the increase in power must compensate not only for the decrease in $I_1(l)$ but also for the disruption in orthogonality.

By using formulas (4.71) and (4.72) we may ascertain that with a probability of error on the order of $10^{[illegible]}$ and [illegible] ≈ 0.6 T (or Δf ≈ 0.1/T) the magnitude ρ ≈ 0.1 may be compensated for by an increase of approximately [illegible] signal power. At the same time, for compensation of the decrease in $I_1(l)$ it is necessary to increase the power of the signal by an additional [illegible]. Thus it can be considered that Δf ≈ 0.1/T is a permissible value of deviation of signal frequency from the nominal in narrow-band FM. This tolerance is [illegible] times smaller than in a wide-band FM system.

In those cases when it is not possible to provide precision of [illegible] frequency even within the limits which are permissible for wide-band FM, use is made either of a nonoptimal decision system of wide-band reception, which will be discussed in the following section, or of binary modulation (see Chapter IX).

4.5. Nonoptimal Methods of Incoherent Reception

Optimal systems of incoherent reception have come into use in communication practice only in recent years. At present various systems are widespread which differ from optimum systems and whose advantage is, in some instances, simplicity, and in others, less rigid requirements on frequency stability. The majority of these systems are designed for the FM binary system.

Despite the relative simplicity of these systems a rigorous theory of noise resistance for them is complex and has not been completely worked out. We will limit ourselves to an approximate analysis of certain nonoptimum reception methods and to comparing them with the optimum methods studied above.

Narrow-Band Envelope Reception

The narrow-band system differs from the optimum matched filter system (Figure 4.[illegible]) in that instead of filters matched to the signal it uses unmatched "separation" filters with relatively narrow passbands. Thus, for binary FM systems filters are generally used which have the impulse response

$$\left.\begin{aligned} g_1(t) &= G(t)\cos(\omega_{1,0}t + \varphi_1), \\ g_2(t) &= G(t)\cos(\omega_{2,0}t + \varphi_2), \end{aligned}\right\}$$

where G(t) is the envelope of the impulse response and is ordinarily the same for both filters, φ_1 and φ_2 are certain determinate phase shifts, and $\omega_{1,0}$ and $\omega_{2,0}$ are filter resonant frequencies coinciding (in principle) with the signal frequencies of $s_1(t)$ and $s_2(t)$.

Depending on the type of function G(t) the system provides different noise-resistances. If

$$\left.\begin{array}{ll} G_0(t) = \text{const} \neq 0 & \text{when } 0 < t < T, \\ G_0(t) = 0 & \text{when } t < 0 \text{ and } t > T, \end{array}\right\}$$

then these filters are obviously matched to the signal and the system coincides with the optimum system of Figure 4.5. Here the probability of error is expressed by formula (4.49). But filters like this are difficult to realize. Therefore simpler filters are used, for example, filter in the form of a single oscillatory loop, for which

$$G_0(t) = [illegible] \qquad (4.73)$$

or band filters which approximate the ideal pi-response (physically unrealizable) filter, for which

$$G(t) = [illegible], \qquad (4.73a)$$

here Δf is the effective (or "noise") passband of the filter which is defined by the equality

$$\Delta f = \frac{\int_0^\infty [illegible]}{[illegible]},$$

where [illegible] is the filter transfer function. For a pi-response filter it coincides with a passband in the usual sense.

When signal [illegible] is fed to a filter with resonant frequency [illegible], the oscillation amplitude at its output gradually rises. In the case of a matched filter the amplitude, as we have seen, rises in accordance with a linear law.

The amplitude of a single loop changes after the law [illegible], which for the ideal pi-response filter it follows the law [illegible].

Noise, however, acts on the filter all the time and therefore [illegible] filter output [illegible] from the assumption of steady state. Since the filters under consideration are linear the noise at the output of each of them [illegible] and has power

$$P_n = [illegible] \Delta f$$

[illegible] of readout moment [illegible] filter [illegible] signal [illegible] of amplitude [illegible] depends on the type of filter, the

---

[1] For simplicity we assume the transmission factor of the filter at resonance to be unity, which, of course, does not impair the generality.

amplitude of a at its input, and on Δf, and also noise of normal probability distribution and intensity dependent on Δf. The envelope of the total voltage, as is known [6], has a generalized Rayleigh probability distribution just as in the case of a matched filter. At the output of a matched filter, however, the ratio of signal power to noise power at the moment of readout is

$$\frac{a^2T}{2\nu} = \frac{W_sT}{\nu} = h^2,$$

while with nonoptical filters it depends on the ratio between effective passband Δf and signal duration T. By varying the value of Δf we can find the value of it at which the signal power to noise power ratio q^2 at readout moment is maximum.

With reception of a single pulse this passband for a single resonant loop is equal to Δf = 0.65/T and for a pi-response filter Δf = 1.37/T [8,9]. With the effective filter passband thus selected the maximal value of q^2 is [9]:

$$q^2 \approx 0.815\frac{a^2T}{2\nu} = 0.815h^2 \text{ for a resonant loop,}$$

$$q^2 \approx 0.825\frac{a^2T}{2\nu} = 0.825h^2 \text{ for a pi-response filter.}$$

If the voltage across a filter, not tuned to the frequency of the received signal, were determined only by fluctuation interference, decision making in this system would reduce to comparing the values of the two envelopes at the moment of readout, and one of them (in the filter without the signal) would have a Rayleigh distribution, while the other (in the filter with the signal) would have a generalized Rayleigh probability distribution. Repeating the same calculations as in conclusion (4.49) the error probability

$$p = \frac{1}{2}e^{-\frac{q^2}{2}} \approx \frac{1}{2}e^{-\frac{0.82h^2}{2}}.$$

could be found, i.e., it is the same as in the optimum system if signal power is diminished 18%.

There is, however, no basis for this conclusion. We must take into account, first, that with unmatched filters the received signal at moment of readout creates a voltage both in the filter tuned to its frequency and in the other one.* Second, at the moment of readout residual voltages from transient processes created by the preceding signal elements are kept at the outputs

---

[*] This phenomenon, despite the existing fallacy, does occur in ideal pi-response filters with nonoverlapping bands, since at readout moment they do not yet have the steady state in which a signal outside the passband does not traverse the filter. Only in unmatched filters and orthogonal signals is the voltage at readout moment at the output of the "empty" filter determined by interference alone.

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

As can be seen from Figure [illegible], [illegible] error less than 10 [illegible] takes a value of [illegible] leads to an increase of approximately double in the power of the interference. If the width of the filter passband is doubled the power of the interference is indeed doubled but the ratio U_{reqd} [illegible] drops to 0.07 and for all practical purposes such a widening of the band will have no effect on resistance to interference.

Obviously, the optimal value of a loop passband which provides for a minimal probability of error in the given system lies between these two limits

Exact calculations confirmed by experiment yield an [illegible] of residual oscillations [illegible] value of the passband [illegible] and for a pulse response filter [illegible]. In this case [illegible] of the ratio between the power of the signal and the [illegible]

[illegible] and the probability [illegible]

$$[illegible]$$

[illegible]

Narrow-Band Instantaneous Frequency Reception

A second system of receiving the signals [illegible] of a narrow-band filter, limiter, and discriminator [illegible] are used for a discriminator [illegible] limiter and filter to [illegible]

[illegible]

$$[illegible]$$

where [illegible] is the probability [illegible] quency [illegible] with transmission of signal [illegible] account of the characteristic of the filter [illegible]

Obviously the probability density [illegible] depends on the [illegible] powers [illegible] of the signal and the interference at the [illegible] on the characteristics of the filter, and [illegible] [illegible] = (1/2)([illegible]).

One possible circuit for a discriminator differs from [illegible] separation filters only in the existence of a limiter [illegible]

Figure 6.16. Instantaneous Frequency Reception of FM Signal

[illegible] [illegible] density [illegible] [illegible] the probabilities of error [illegible] [illegible] [1] for the particular [illegible] [illegible] [illegible] and [illegible] 1.5 (where [illegible] [illegible] [illegible] performed [illegible]

[illegible] general [illegible] [1] [illegible] [illegible]

[illegible]

with [illegible] the amplitude [illegible] frequency [illegible] in this case is complicated by the fact that q and [illegible] will depend on what signal element preceded the one under consideration.

In all actual systems filters are used in which a steady state is established by the moment of reading and the signal is practically not weakened.

Formula (4.76) acquires an exceptionally simple form if [illegible]. Considering that $Q(0,y) = e^{-y(2,2)}$ and $I_0(0)$ [illegible] 1, in this case

$$p \; [illegible] \; \frac{1}{2}\exp(\; [illegible] \; q^2) \qquad (4.[illegible])$$

The authors of work [12] show that [illegible] apprently is an optimal value for deviation in instantaneous-frequency FK reception. For the case of a [illegible] response filter, when

$$\omega_{[illegible]} \; [illegible]$$

This is confirmed, at least for large q, in work [13].

In comparing formula (4.[illegible]a) with (4.71), it should be borne in mind that the effective filter passband in a circuit with a discriminator which passes signals $z_1(t)$ and $z_2(t)$ must be at least twice the effective passband of the separating filter in a circuit for narrow-band envelope reception. This requirement flows from a comparison of the processes for establishing amplitude and instantaneous frequency (see, e.g., [14]). Therefore, for one and the same noise spectral density at filter output, the magnitude of q^2 in an instantaneous-frequency reception system will be exactly half that in a circuit for envelope reception. It follows that the probability of error in both these circuits will be approximately the same.

Wide-Band Reception with Post-Detection Integration

The above-studied methods of narrow band detection are simpler than the optimum methods, but the requirements for frequency stability in narrow-band reception are about as rigid as in the quadrature system. These requirements can to a considerable degree be lessened if instead of narrow-band separation filters wide-band filters are used whose passband exceed the possible signal frequency changes under the effect of destabilizing factors. Nevertheless, the nominal degree of frequency shift 2[illegible], of course, be of the same order as the effective filter passband 2[illegible]f.

If [illegible]f [illegible] the natural oscillations in the filter damp so fast that the residual voltages formed by the preceding signal elements may be

---

[illegible] This condition does not contradict the fact that the effective filter passband 2[illegible]f must be great in comparison with [illegible] inasmuch as for all filters actually used in practice 2[illegible]f is much greater than the average quadratic band of [illegible].

completely disregarded. But expansion of the filter passband increases the power of the noise passed through this filter. Since signal voltage at the output of the wide band filter is rather rapidly established, the ratio of signal power to noise power at the output is

$$q^2 = \frac{P_c}{P_n} = \frac{h^2}{2\Delta fT} \qquad (1.77)$$

We will also consider that the filter frequency characteristics for all practical purposes do not overlap. This permits us to say that the noise at filter outputs is not correlated.

After registering the received message by comparison of the instantaneous values of the envelopes at the outputs of the filters (one of which has the generalized and the rest the ordinary Rayleigh probability distribution), the same expression for error probability may be derived as in the optimum system, but with the difference that the variable h is replaced by q:

$$P_e = \frac{1}{2}e^{-\frac{q^2}{2}} = \frac{1}{2}e^{-\frac{h^2}{4\Delta fT}}, \qquad (1.78)$$

i.e., this receiving method is equivalent to signal power loss by a factor of 2ΔfT as compared to optimum reception.

The noise-resistance of wide-band reception may, however, be substantially increased if a decision is made not on the basis of the instantaneous values of the envelope, but their whole course throughout duration T of a signal element is taken into account. Let us observe that in narrow-band reception our taking into account the values of the envelope at different moments in time throughout a single element cannot increase noise-resistance because all these values are highly intercorrelated and therefore contain no additional information. In the wide-band filter the correlation interval [6]

$$\tau_c = \int_0^\infty R(\tau)d\tau = \frac{1}{2\Delta f} \qquad (1.79)$$

where R(τ) is the envelope of the correlation coefficient of the noise passed through the filter, is significantly less than T. Therefore there appears a possibility of increasing noise-resistance by taking into account the whole course of the envelope.

Let us assume that a decision is made from taking into account the voltage envelopes at the filter outputs at moments in time divisible by 1/2Δf. Let us suppose as a first approximation that the noise values separated by interval τ_c are mutually uncorrelated.[1] Let us denote these values for the first filter by $x_1, x_2, \ldots, x_n$ and for the second, by $y_1, y_2, \ldots, y_n$, where $n = 2\Delta fT$. Let us find the optimum decision principle (based on the ideal observer criterion) which is realizable with respect to these values.

---

[1] If there were an ideal pi-response filter these values at its output would be strictly uncorrelated among themselves (and hence also independent for gaussian noise).

[illegible]

[illegible] distributed [illegible] probability distribution [illegible]

[illegible] the general [illegible]

[illegible]

[illegible]

[illegible]

[illegible]

By taking the logarithm of the inequality we [illegible] convenient form

[illegible]

[illegible]

[illegible]

$$[illegible]$$

[illegible]

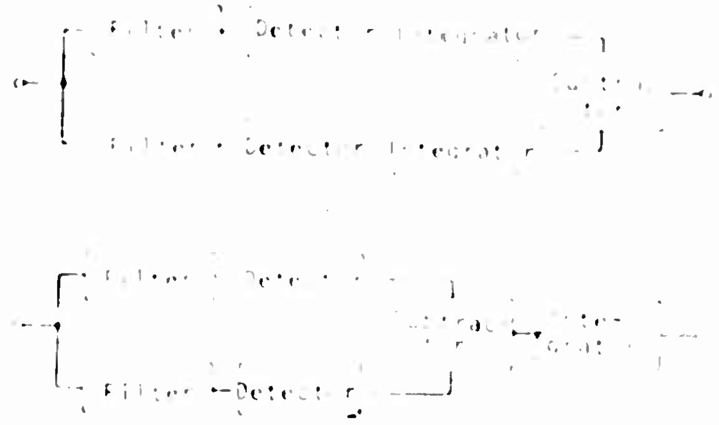

Filter
Detector
Integrator
Filter
Detector
Integrator
Filter
Detector
Subtracter
Integrator
Filter
Detector

Figure 4.17. Block Diagram of Receiver with [illegible] Detector and Integrator

Instead of separately integrating the voltages at the output of the detectors and subtracting them, we may first subtract them and then integrate the difference of these voltages (Figure 4.17) [illegible] of the sign of the integral derivative. Both schemes are absolutely equivalent. In practice the second is employed because it is simpler, but the first is more convenient for purposes of analysis.

The rectification characteristic is here expressed by function $\ln I_0$ (see note after formula (3.56)). When signals are small this characteristic is well approximated by a quadratic relationship; when the signals are large, by a linear one.

[illegible]

where $I_f(t)$ is the fluctuation portion of the detector current.

The value of integral $\int_0^T$ [illegible] is a random variable with [illegible] expectation of zero because that [illegible] of this random variable [illegible]

[illegible] (4.89)

[illegible] changing the order of [illegible] statistical [illegible], we derive

[illegible] (4.90)

[illegible] the fluctuating portion of the [illegible]

[illegible] the fourth equation [illegible] equation is accomplished on the [illegible] coefficient.

[illegible] correlation fades so rapidly that [illegible] only in the initial portion of interval [illegible]

[illegible] (4.90a)

[illegible] correlation interval at first detector output.

[illegible] fulfilled, then the probability distribution of $\int_0^T I(t)\,dt$ [illegible] close to normal. In fact, this integral may be regarded [illegible] of integrals:

$$\int_0^T I(t)\,dt = \int_0^{T_1} I(t)\,dt + \int_{T_1}^{T_2} I(t)\,dt + \dots + \int_{T_k}^{T} I(t)\,dt$$

$$0 < T_1 < T_2 < \dots < T_k < T$$

If the region of integration in each of these integrals is larger than correlation interval $\tau_c \approx 1/\Delta f$, they may then be considered independent

[illegible]

---

[1] [illegible]

[2] Let us remind the reader that first integrator is the name here given to the integrator in whose channel there is a signal present at a given time. Therefore the whole reasoning remains valid during transmission of either of the two symbols if the integrators are appropriately numbered.

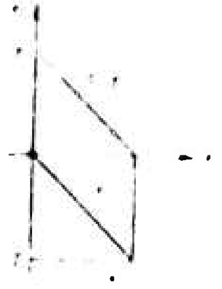


Figure 4.18. Integration Region of Expression (4.90).

Substituting these values in expression (4.91) we [illegible] oscillators loop

[illegible]

[illegible]

[illegible]

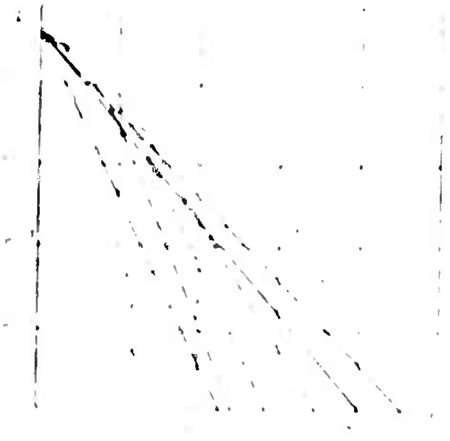
[illegible]

[illegible]

[illegible]

[illegible]

[illegible]



[illegible]

[illegible]

[illegible]

[illegible]

Here for a single loop

[illegible]

while for a pi-response filter

[illegible]

The relationship derived is shown in Figure 4.19 (curves 3, 4, [illegible]). When h [illegible] the curves differ little from the corresponding curves for the quadratic detector, and when h > [illegible] the noise resistance of the system with linear detection is substantially higher than with quadratic and approaches the noise resistance of an optimum system of incoherent reception. When q [illegible] 1 we may consider approximately that [illegible] and expression (4.9[illegible]) will be simplified:

For a single key:

$$P = \frac{1}{2}\{1 - [illegible]\} [illegible]$$

[illegible] filter with [illegible]

$$P = \frac{1}{2}\{1 - \Phi[illegible]\} [illegible]$$

In practice a low-frequency filter which is not tuned to be a square-wave pulse is often used instead of the integrator at detector output. Here voltage at filter output will be proportional, not to integral (4.8[illegible]), but to Duhamel's integral

$$u(t) = \int_0^t [illegible] \qquad (4.96)$$

where g(t) is the low-frequency filter pulse response.

This integral represents a random variable whose numerical characteristics may be computed if g(t) is known. If g(t) differs little from expression (4.8[illegible]), then the noise-resistance of this reception method will also approach the noise-resistance of reception with post-detection integration. But the ordinarily used, relatively simple low frequency filters have an impulse response which differs from zero at any value of t. This has as its result that filter output voltage at the moment of readout depends both on the received signal element and on the preceding elements in a way similar to the way this occurs at a high-frequency filter output in the narrow band envelope reception system.

This phenomenon substantially increases error probability and to combat it relatively broad passband filters, in which response g(t) is adequately damped by time t = 1, should be used. Resultantly, variable g(1 - τ) in integral (4.96) proves to be considerably less than unity in a significant portion of its integration range; this leads to "incomplete integration" of the noise, i.e., to reducing the ratio of the constant component to the fluctuation component of the voltages at the output of the filter which appears behind function symbol Φ (in expression 4.91).

Numerous computations [9] show that for the different low-frequency filter characteristics the best compromise between the conditions of obtaining insufficiently small residual voltagees from preceding signal elements and the best noise average is made when the effective low-frequency filter

[illegible]

Integration of the [illegible] instantaneous frequency [illegible] bution of the [illegible] separation of the [illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible] extent the phase of the incoming signal changes slowly [illegible] of the preceding [illegible] results of the [illegible] estimate of the [illegible] phase of the expected signal waveform [illegible] The accuracy of this [illegible] depends, on the one hand, on the rate at which the channel is fluctuating, and, on the other, on the ratio of signal power to spectral noise density; a large [illegible] noise provides exact measurement of the phase of the incoming [illegible].

[illegible]

It [illegible] that channel [illegible] the expected signal [illegible] predicted [illegible] from the [illegible] except [illegible] binary [illegible] that [illegible] fluctuating [illegible] significant.

[illegible] systems [illegible] signal [illegible].

In a channel with slowly fluctuating phase, however, signal [illegible] more efficient than orthogonal [illegible] the alternative [illegible] phase [illegible] with [illegible] signals [illegible] Chapter III showed that [illegible] binary coherent [illegible] was

$$P = \tfrac{1}{2} [\text{illegible}] \qquad (3.38)$$

while for the orthogonal system under the same conditions

$$P = \tfrac{1}{2} [\text{illegible}] \qquad (3.39)$$

Therefore, a binary system with phase modulation can give an equivalent power gain of 3 db as compared with an orthogonal binary system.

Unfortunately, for a long time practical utilization of phase modulation in real channels was prevented by the phenomenon of spontaneous phase slips in the reference voltage. The essence of the matter is that the phase of the reference voltage going to the phase detector (Figure 5.9) must be set by averaging the phase of the preceding signal elements. But the phases of these elements may, in conformity to the message being transmitted, adopt two values differing by 180°:

[illegible]

[illegible]

The reference voltage, however, must have an entirely determinate phase, coinciding, for example, with the phase of signal $u_1(t)$, and must follow after the slow phase changes in the channel without reacting to instantaneous phase jumps during modulation. One of the simplest methods of obtaining this reference voltage, proposed by A. A. Pistol'kors [17], is to lead the received oscillations to a frequency doubler. Doubling the frequency (and phase) gives a voltage of frequency $2\omega_0$ and of unchanged phase (since an initial phase of 2π does not differ from an initial phase of zero). Then the voltage of doubled frequency goes to a divider in which the frequency is divided by two. In concept this should give a voltage of frequency ω_0 and unchanged phase which is used as the reference voltage. Every frequency divider which uses a division of two has, however, two equilibrium states; and therefore the reference voltage thus derived may have one of two phase values (0 or π). Various effects which cannot be accounted for (mainly transient processes in the circuits of the nonlinear signal during modulation, as well as power supply voltage changes) may at certain times cause the phase of the reference voltage to skip from one equilibrium value to the other. This skipping causes the so-called "reverse operation" in which the "0" symbols are received as "1" symbols, and vice versa.

Many other systems for extracting the reference voltage from the received signal have been proposed (see, for example, [18, 19]), but they all lead to the same result because there are two equilibrium states corresponding to the two phases of the reference voltage. This is obviously unavoidable since the signals $u_1(t)$ and $u_2(t)$ have absolutely equal rights in the channel. The same disadvantage (so-called reverse operation) is thus to some degree inherent in all these systems.

This defect was eliminated in radical fashion when the system of relative phase modulation [20] was proposed instead of the "classical" phase modulation. In this system the information which is being transmitted is not embedded in the phase value itself of the given signal element, but in the phase difference between this element and the preceding one. In the RPT (relative phase telegraphy) binary system, this phase difference may, for example, take on the values of 0° and 180°. To transmit symbol v_1 a signal element is emitted whose phase coincides with the phase of the preceding element, and to transmit symbol v_2 the element's phase is opposite to that of the preceding element.

The first element (at the beginning of the communication session) carries no information, but serves only for reading the phase difference in the following element.

As was shown in Chapter II, this system may also be regarded as an ordinary system with phase modulation, but with a special recoding of the transmitted signals which is designed to correct transitions to negative.

Signals may be received in RPT in a channel with slowly changing phase by using the coherent method after the scheme in Figure 5.9, but with recoding of the received symbols. This recoding is performed by addition modulo 2 of each received symbol to the preceding. In other words, symbol y_1 is registered in the case where the voltage polarity at system output coincides with the polarity occurring during reception of the preceding element, while symbol y_2 is registered where these polarities are opposed. This method of coherent signal reception by RPT is therefore also called the polarity comparison method [20].

We easily satisfy ourselves that in case the reference voltage phase jumps to the opposite side, only one signal element will be erroneously received. This advantage of the RPT system, however, involves a drawback which was also mentioned in Chapter II. When interference changes voltage polarity at integrator output (Figure 5.9) both the symbol corresponding to this element and the following symbol will be incorrectly registered. If, however, incorrect polarity occurs during reception of n elements in a row, then the first symbol of this sequence and the first symbol following the sequence of elements of distorted polarity will be erroneously recorded.

Therefore probability of incorrect registration of a symbol in the RPT system is not the same as the probability that distorted polarity will appear at phase detector output or, which is the same thing, as the probability p_{PT} in the system of "classical" phase telegraphy (PT) defined by formula (4.98).

The value of p_{RPT} is easy to determine from the following considerations. Erroneous registration of some symbol may happen only as the result of two incompatible events:

a) The polarity of a given element is incorrectly received and that of the preceding element has been correctly received; or

b) the polarity of a given element is correctly received and that of the preceding, incorrectly.

Each of these events has probability $p_{PT}(1 - p_{PT})$. Hence,

$$p_{RPT} = 2p_{PT}(1 - p_{PT}) = \frac{1}{2}[1 - \Phi(\sqrt{2h})][1 + \Phi(\sqrt{2h})] = \frac{1}{2}[1 - \Phi^2(\sqrt{2h})] \qquad (4.99)$$

This value of p_{RPI} should be supplemented by the probability of reference voltage phase skipover which also causes a symbol to be misrecorded in the RPI system. But this probability is hard to estimate; it depends on the system chosen for forming the reference voltage and on the quality of equipment adjustment.

Another correction which must be introduced in (4.99) is [illegible] the small deviations in phase of the reference voltage caused by averaging over a finite period of time. Inasmuch as we are dealing here with slow fluctuations in phase in a channel this correction is not great. Several quantitative evaluations of it will be given in Chapter V. Considering that in the described method of reception use is made of knowledge of the initial phase which is not quite exact, some authors call it quasi-coherent. A block diagram for implementing this method is shown in Figure 4.20.

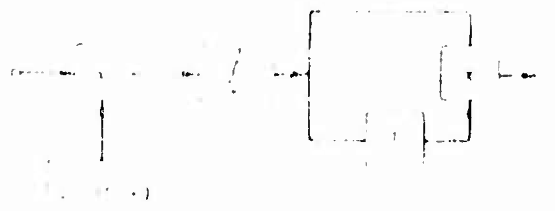


Figure 4.20. Coherent Reception of RPI Signals.

Incoherent Reception of RPI Signal

In relatively incoherent reception use is not made of even the approximate knowledge of the initial phase but the fact is used that the phase relationship between two adjacent signal elements is for all practical purposes not disrupted in the case of slow fluctuations. There are many different methods of incoherent RPI reception which in one way or another implement the optimal incoherent decision principle.

To explain this principle we will [illegible] than in other systems [illegible] the [illegible] transmitted symbol is determined not by a signal element itself [illegible] but by a sequence of two elements [illegible] symbol "1" corresponds to [illegible]

$$[illegible]$$ [illegible]

[illegible]

$$[illegible]$$ [illegible]

[illegible] fore, the [illegible] which pr[illegible] [illegible] on the [illegible]

Expression (4.28) [illegible] dance with which [illegible] determined from [illegible] which signals (4.16) are determined. [illegible] vector with [illegible] and can be written [illegible] time period "T":

[illegible]

[illegible]

[illegible]

Figure 4.21. Quadrature reception of RPI signals.

Figure 4.21 shows a quadrature [illegible] for receiving binary [illegible] signals which is in essence no different from a general active internal system (Figure [illegible]). Local reference generators generate voltages which are proportional to $z_1(t)$ and $z_2(t)$, i.e., a continuous sinusoid $z_1(t)$ and a series of [illegible] for which the period [illegible] the phase changes by 180°. Initial phase [illegible]. Integration over the summing interval is within limits from [illegible] with Figure [illegible]. As can be seen [illegible] the first [illegible] voltages are formed which [illegible] are compared with one another [illegible] in accordance with the voltage [illegible] reading.

[illegible] in the following form:

$$[illegible]$$

Removing the parentheses we can easily reduce (4.1[illegible]) to the following form:

$$A\,V_0\,T_0\,[illegible] \qquad (4.1[illegible])$$

where

$$[illegible]$$

$$[illegible]$$

The magnitudes V_a, V_b, [illegible] may be obtained [illegible] in-terval of duration T as shown in Figure 4.2[illegible]. [illegible] V_1 and V_2 are taken directly from the integrators [illegible] of the line with a delay of T. The output voltage of the [illegible] Figure 4.22 is proportional to the left side of inequality [illegible] decision is reached in accordance with the sign of the [illegible] of the system shown in Figure 4.2[illegible] were first suggested [illegible] and are used in the M[illegible] system [21]. The difference [illegible] the cosinusoidal voltage with a rectangular one [illegible] the cost of an insignificant reduction in the [illegible] use elements of discrete equipment [illegible] In the M[illegible] system analog memory devices are used [illegible]

Figure 4.2[illegible] shows another [illegible] It is based on the [illegible] circuit shown in Figure 4.[illegible] [illegible] and [illegible] signal [illegible] and [illegible]

$$[illegible]$$

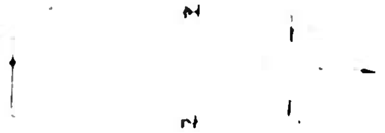
[illegible] filter [illegible]

$$[illegible]$$

[illegible] technique for synthesis [illegible] implement [illegible] filters [illegible]. Nevertheless, the system shown in Figure 4.2[illegible] is [illegible] in practice.

The circuit shown in Figure [illegible] system shown in Figure 4.2[illegible] [illegible] 4.[illegible]. Here [illegible] filter matched with a segment of [illegible] used. [illegible] at instant [illegible] signal [illegible] filter input. Then, [illegible] (4.[illegible]), we find that [illegible] will be equal to the following [illegible]

$$[illegible]$$



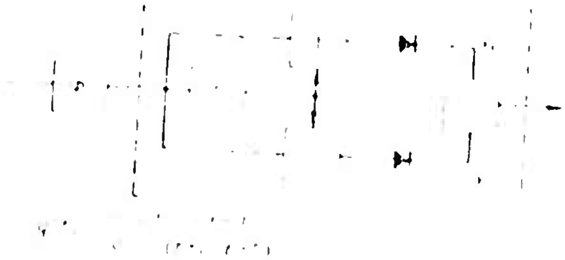


Figure [illegible]. [illegible]

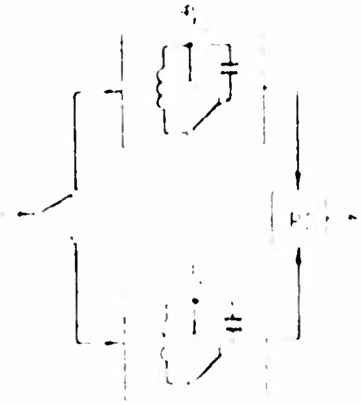
The [illegible] shown [illegible] circuit [illegible] differs from Figure [illegible] pass filter [illegible] optimally [illegible] the [illegible] transient processes, the [illegible] amounts to [illegible] with a generalized Rayleigh [illegible] with an ordinary Rayleigh [illegible]

where [illegible] the signal-to-noise [illegible]

When transient processes [illegible]

[illegible]

At this moment the signal is cut off from the first filter and delivered to the second filter which is brought to zero initial conditions. In the first filter the oscillations in this process are not damped and persist until moment of reading [illegible] in the form (4.104). At this time the first filter performs the same role as the delay line in the circuit shown in Figure 4.21. [illegible] instant [illegible] oscillations of the form (4.104) are set up in the second filter. [illegible] these oscillations arrive at a phase detector (PD), which is the same as that shown in Figure 4.2[illegible], or takes the form of a multiplier with an integrator. The sign of the voltage at the output of the phase detector will be positive if condition (4.10[illegible]) is met and this makes it possible to reach an optimal decision.



[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible] of Errors in Binary [illegible]

[illegible]

order to stress that a discrete channel with relative phase keying is a channel with a memory in which errors have a tendency to group in twos. This must be taken into consideration in coding.

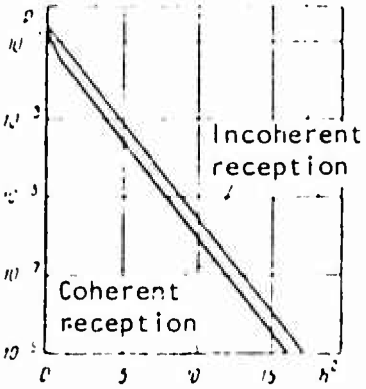


Figure 4.26. Probability of Error for a Binary RPT System.

It should not be thought that all errors in a received sequence of symbols are joined in pairs. In coherent reception single (isolated) errors may appear when interference destroys two or several elements of a signal in a row. In this case two isolated errors appear at the beginning and end of a group of elements with incorrectly determined polarity. Thus, the probability that in a PT system there will be not less than two incorrectly received symbols between two correctly received symbols. In light of the fact that in RPT interference in the form of white noise creates independent errors, we have

$$p_{is} = 2(1 - p_{PT})p_{PT}\ \text{[illegible]}$$
$$\approx 2(1 - p_{PT})^2 p_{PT}\ \text{[illegible]} \approx 2(1 - p_{PT})p_{PT}$$

or, in light of (4.99),

$$\frac{p_{is}}{p_{RPT}} = p_{PT} \qquad (4.107)$$

Thus, in a channel of good quality when $p_{PT} \ll 1$, isolated errors in coherent RPT reception constitute an insignificant part of all errors. To them must be added those isolated errors which occur during spontaneous skips in phase of the reference voltage.

Altogether different relationships are found when reception is incoherent. In this case there is also a tendency for errors to group themselves in pairs and this is caused by the fact that the intervals of time which are used for reaching two sequential decisions partially overlap. However, isolated errors still constitute a significant proportion of all errors.

The evaluations obtained in [24] show that when $p_{RPT} \approx 4\cdot10^{-3}$ from 51.2 to 75.6% of all errors are isolated and when $p_{RPT} \approx 5.6\cdot10^{-8}$ from 57.2% to 78.6%. We will note that in a channel with independent errors practically all errors are isolated when the average fidelity is the same.

As a consequence of error grouping in RPT, there is no point in direct application of codes which correct single errors. Fidelity can be increased here by using Abramson codes which were mentioned in Section 2.8. These codes permit correcting single as well as double adjacent errors. It is also possible to use codes which are intended for the purpose of correcting independent errors in introducing decorrelation. In the given case this is done by joining odd and even (in order of occurrence) symbols in a combination of a correcting code. When using a recurrent code decorrelation is accomplished if the step of the code is equal to two or more.

RPT Systems When m > 2

Along with binary RPT systems, rather wide use is made of RPT systems whose code base is $m > 2$ (most frequently $m = 4$ and $m = 8$). Usually such systems provide for multiplexing of a channel, i.e., simultaneous transmission of messages from several sources and several (most frequently two or three) binary channels are considered to be an aggregate. We will talk about them from these positions in Chapter IX, determining the probability of incorrect reception of a binary symbol in each of the aggregate messages. However, in the past few years increasingly greater importance has been acquired by use of symbols with a larger code base for suitable transmission of coded data from one source. In this respect the probability of correct or incorrect reception of the n-th symbol is of interest.

By way of example we will consider the case where $m = 4$. Let there be transmitted symbols 0, 1, 2, and 3 and let information about them be embedded in the difference between phases $\Delta\varphi$ between adjacent sinusoidal signal elements. For example, $\Delta\varphi = 0$ correspond to symbol "0", $\Delta\varphi = \pi/2$ to symbol "1", $\Delta\varphi = \pi$ to symbol "2", and $\Delta\varphi = 3\pi/2$ to symbol "3". Thus, each signal element has the form

$$z(t) = a\cos\left(\omega t + \varphi + k\frac{\pi}{2}\right),$$

where φ is the initial phase and coefficient k assumes the values of 0, 1, 2, and 3, depending on the transmitted symbol and the phase of the preceding element.

If the initial phase φ fluctuates so slowly that it can be considered known, quasi-coherent reception, such as in a PT system with $m = 4$ with subsequent recoding, is possible. The rules for this are obvious. The probability of correct reception for such a biorthogonal PT system was computed in Chapter III (3.70a). Hence, the error probability is

$$P_{P,1} = 1 - \frac{1}{4}[1 + \Phi(h)]^2 =$$

$$= 1 - \frac{1}{4}\{2 - [1 - \Phi(h)]\}^2 = 1 - \frac{1}{4}\{4 - 4[1 - \Phi(h)] + [1 - \Phi(h)]^2\} = \qquad (4.108)$$

$$= [1 - \Phi(h)] - \frac{1}{4}[1 - \Phi(h)]^2 \qquad (4.108)$$

To compute the probability of error in a RPT system it is necessary to consider the doubling of them. However, this is not as simple as it was in reaching the conclusion of (4.99) inasmuch as two adjacent errors in a PT system will lead sometimes to two and sometimes to three errors after recoding to RPT. Therefore, we will limit ourselves to obtaining an evaluation from above, bearing in mind that an error in recoding never leads to more than two errors after recoding. Consequently,

$$P_{RPT,1} \leq 2[1 - \Phi(h)] - \frac{1}{2}[1 - \Phi(h)]^2 \qquad (4.109)$$

When h ≫ 1 and when the occurrence of adjacent errors prior to recoding is exceedingly small, this evaluation is a good approximation. Ignoring under these conditions the square of the small value 1 - Φ(h), we obtain

$$P_{RPT,1} \approx 2[1 - \Phi(h)] \quad (h \gg 1), \qquad (4.109a)$$

i.e., the probability of error is four times greater than in a binary system with orthogonal signals in the case of coherent reception.

In case of incoherent reception the initial phase φ is considered to be unknown. The transmitted signal is determined by a segment of the signal joining two elements, namely:

The following signal corresponds to symbol "0"

$$s_1(t) = a\cos(\omega t + \varphi), \quad -T \leq t \leq T,$$

to symbol "1"

$$s_2(t) = \begin{cases} a\cos(\omega t + \varphi), & -T < t < 0, \\ -a\cos(\omega t + \varphi), & 0 < t < T, \end{cases} \qquad (4.110)$$

to symbol "2"

$$s_3(t) = \begin{cases} a\cos(\omega t + \varphi), & -T < t < 0, \\ a\cos(\omega t + \varphi), & 0 < t < T, \end{cases}$$

to symbol "3"

$$s_4(t) = \begin{cases} a\cos(\omega t + \varphi), & -T < t < 0, \\ a\sin(\omega t + \varphi), & 0 < t < T \end{cases}$$

An optimum incoherent decision circuit for such a system can be made in various ways. Specifically, general-purpose circuits are possible, i.e., a quadrature one and one with matched filters. They differ from Figures 4.21 and 4.23 only in the doubling of the number of branches and the replacement of a final subtractor with a circuit for comparing the four magnitudes. Other variation will be considered in Chapter IX.

For an evaluation of error probability with optimal incoherent reception we make use of the fact that the system in (4.110) is isomorphic with system (4.63a) if in the latter the signals are delivered not in the interval (0, T) but in the interval (-T, T). It can easily be seen that for the system in (4.110) the conditions of (4.53) are met when the limits of integration are changed in accordance with the doubled power of the signal. Therefore, the evaluations in (4.67) hold if h^2 is replaced by $2h^2$, which gives

$$[illegible] \qquad (4.111)$$

As already noted, systems with different code bases should be compared for equivalent error probability and with the same value of the parameter $h^2/\log_2 m$. In the given case when m = 4 and sufficiently large values of h the equivalent probability of error is approximately $1/2\, p_{\text{RPI}\, m=4}$.

It is not possible to show graphically the dependence of the equivalent probability of error on $h^2/\log_2 m$ for m = 2 and m = 4 for coherent RPI reception as was done in Figure 4.9 for orthogonal systems, inasmuch as the curves for m = 4 practically coincide with those for m = 2. Thus, with the use of RPI with a given rate of transmission the increase in m does not increase the fidelity of reception in distinction from orthogonal systems. However, an increase in the code base in a RPI system permits increasing the rate of message transmission without broadening the frequency band, while with retention of orthogonality the increase in m involves broadening the frequency band even if the rate of transmission remains the same.

4.7 Carrying Capacity of a Channel with an Indeterminate Phase

Fluctuations in the phase of a signal reduce the carrying capacity of a channel inasmuch as two signals which differ only in the initial phase are indistinguishable, even in the absence of additive interference.

We will examine the case when the initial phase of a received signal is not known and may with equal probability take any value from 0 to 2π but does not change for the duration of transmission of the entire message. We will here proceed from expression [illegible] for the carrying capacity of a channel with completely known signals. For such a channel let there be a certain signal system $s_i(t)$ of length T and average power P_s, which we will present as

$$[illegible]$$

By the means of these signals information may be transmitted in the ideal channel at a certain rate I'. The given signal system may be regarded as two independent systems:

$$[illegible]$$

and

$$[illegible]$$

which are being transmitted simultaneously. The average power of signals $z_i^{(1)}(t)$ and $z_i^{(2)}(t)$ is P_s. It is evident that half of the information transmitted is carried by the former signals and half by the latter.

In a channel with known and invariable initial phase these two signal systems are easily separated by phase selection. In a channel of indeterminate phase this selection is, generally speaking, impossible.

In fact, if, for example, $A_k(t) = B_j(t)$, then signals $z_k^{(1)}(t)$ and $z_j^{(2)}(t)$ will be indistinguishable in incoherent reception. If, however, transmission is restricted only to either one of these signal sets they may be distinguished from each other even in incoherent reception.

This assertation may be supported as follows. We will regard signals $a_i(t)$ as points in a B-dimensional space (where B = 2FT is the signal base). Then signals $z_i^{(1)}(t)$ and $z_i^{(2)}(t)$ will be the projections of $z_i(t)$ on two mutually orthogonal B/2-dimensional spaces S_1 and S_2. By shifting all the phases of the components of $z_i^{(1)}(t)$ by π/2 we may match these signals with subspace S_2. If the phases of all the components of either signal is shifted by angle φ, which changes from 0 to 2π, the corresponding point will describe a circle lying in a plane perpendicular to subspace S_1. The points lying on this circle are indistinguishable in incoherent reception.

But different nonintersecting circles correspond to the various signals $z_i^{(1)}(t)$. Therefore all signals $z_i^{(1)}(t)$, if they are distinguishable in coherent reception, remain distinguishable even in incoherent reception. The same is true of the system of signals $z_i^{(2)}(t)$.

Thus, an ideal channel with exactly known parameters of signals of power $2P_s$ may be represented as the superposition of two channels of indeterminate phase, each of which has signals of power P_s and transmits half of all the information. Hence, carrying capacity C_{ip} of a channel of indeterminate phase must be half the carrying capacity of an ideal channel which has a signal of twice the power, or by expression (3.84)

---

*Strictly speaking, the circles corresponding to the two signals $z_j^{(1)}(t)$ and $z_k^{(1)}(t)$ may coincide in the case where $z_j^{(1)}(t) = -z_k^{(1)}(t)$ or $A_j(t) = -A_k(t)$. These opposite signals are indistinguishable in incoherent reception and one signal of each pair of opposed signals must be excluded from system $z_i^{(1)}(t)$, which is suitable for a channel with indeterminate phase. The number of distinguishable signals may, moreover, be reduced by not more than half. This curtailment, however, introduces only a slight correction to the rate at which information is transmitted, and when passing to a limit (T → ∞), this correction tends toward zero.

$$C_{ip} = \frac{1}{2} F \ln\left(1 + \frac{2P_s}{P_n}\right) \text{ natural units/sec.} \tag{4.112}$$

When $P_s \gg P_n$

$$C_{ip} \approx \frac{1}{2} F \ln\left(\frac{2P_s}{P_n}\right) = \frac{1}{2} F \left(\ln \frac{P_s}{P_n} + \ln 2\right)$$

$$\approx \frac{1}{2} F \ln \frac{P_s}{P_n} = \frac{1}{2} C_i ,$$

where C_i is the carrying capacity of the ideal channel; and when $P_s \ll P_n$

$$C_{ip} \approx \frac{1}{2} F \frac{2P_s}{P_n} = F \frac{P_s}{P_n} \approx C_i$$

Therefore, when the signals are weak the carrying capacity of a channel of indeterminate phase hardly differs from that of an ideal channel. As signal power increases, however, the difference between these carrying capacities widens and, at the extreme, phase indeterminacy reduces carrying capacity by half. This is not an unexpected result. With weak signals the small differences in initial phase of the two signals are masked by noise, and therefore the ideal channel with completely known incoming signal phase has no essential advantage (in the sense of signal indistinguishability) over a channel of indeterminate phase. Increased signal power uncovers the possibility of better distinguishing signal phases in the ideal channel, thus resulting in significant difference in carrying capacity.

If signal power P_s and spectral noise density σ^2 are prescribed and frequency band F is not restricted, the maximum carrying capacity when $F \to \infty$ may be determined from expression (4.112), taking into account that $P_n = \sigma^2 F$:

$$C_\infty = \lim_{F \to \infty} C_{ip} = \frac{P_s}{\sigma^2} , \tag{4.113}$$

which coincides with the earlier derived expression (5.85) for an ideal channel.

The question as to whether it is possible to show a regular method for the selection of signals which provide for the attainment of the carrying capacity in (4.112) has not been resolved for the general case. However, for a channel with an unlimited passband such a method does exist. We will show that with a given signal power and interference spectral density a system of signals orthogonal in an intensified sense provides for the carrying capacity of (4.113). This means that for a system consisting of $m = e^{HT}$ signals are orthogonal in an intensified sense, the probability of correct reception with a sufficiently large T exceeds $1 - \varepsilon$, where ε is a positive number as small as desired if

$$H < C_\infty = \frac{P_s}{\sigma^2} . \tag{4.114}$$

We will proceed from expression (4.46) for the probability of correct reception q in the case of an optimal incoherent decision principle:

$$q = e \int [illegible]$$

where

$$m = e^{[illegible]}.$$

Using the integral representation of a modified Bessel function, we write it in the following form

$$q = [illegible]$$

$$\times \exp\Big[-\frac{1}{2} [illegible] \Big] [illegible]$$

Designating $\eta\cos\varphi = x$, $\eta\sin\varphi = y$, and then $x - \sqrt{2}\,h = z$, we obtain

$$q = \frac{1}{2\pi} \int \int [illegible]$$

$$\times \exp\Big[-\frac{1}{2} [illegible] \Big] [illegible]$$

$$= \frac{1}{2\pi} \int \int e^{[illegible]} \Big\{1 - \tag{4.115}$$

$$- \exp\Big[-\frac{1}{2} [illegible] \Big]\Big\}^{m-1} [illegible]$$

We will define a number $a_1 > 0$ with a given [illegible] such that

$$\frac{1}{\sqrt{2\pi}} \int [illegible] = \frac{1}{2} [illegible] \tag{4.116}$$

Inasmuch as the integrand in (4.115) is not negative and $e^{-y^2/2} \le 1$, we have

$$q \geq \frac{1}{2\pi} \int dz \int e^{[illegible]} \Big\{1 - \exp\Big(-\frac{1}{2} [illegible]\Big)\Big\}^{m-1}$$

$$dy = \frac{1}{2\pi} \int dz \int e^{[illegible]} \Big\{1 - \exp\Big[-\frac{1}{2} [illegible]\Big]\Big\}^{m-1} dy \tag{4.117}$$

We select $T = T_0 =$ [illegible] $/2P_s$. Then $\sqrt{2}\,h$ [illegible] and the expression included in the braces in (4.117) will be an increasing function of z with z [illegible]. Therefore, with an exchange of z for $-a$, the right side of (4.117) does not increase and

$$q \geq \frac{1}{2\pi} \int e^{[illegible]} dy \int e^{[illegible]} \Big\{1 - \exp\Big[-\frac{1}{2} [illegible]\Big]$$

$$[illegible] \Big\}\Big\}^{m-1} dz = \Big\{1 - \exp\Big[-\frac{1}{2} [illegible]\Big]\Big\}^{m-1}$$

$$\times \frac{1}{2\pi} \int e^{[illegible]} dy \int e^{[illegible]} dz$$

or, in light of (4.116),

$$q \geq \left\{1 - \exp\left[-\frac{1}{2}(\sqrt{2}\,h - a)^2\right]\right\}^{m-1} \cdot \times \left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{a} e^{-z^2/2}\,dz - \frac{\varepsilon}{2}\right] > \left\{1 - \exp\left[-\frac{1}{2}(\sqrt{2}\,h - a)^2\right]\right\}^{m-1} - \frac{\varepsilon}{2}. \tag{4.118}$$

If condition (4.114) is met, it is possible to find $\delta > 0$ such that

$$h^2 > T(H' + \delta)$$

or

$$h > \sqrt{T}\,\sqrt{H' + \delta}. \tag{4.119}$$

Having been given a sufficiently small δ satisfying condition (4.119), it is possible to find $\eta > 0$ such that

$$\frac{h}{\sqrt{T}} > \sqrt{H' + \delta} + \eta \tag{4.120}$$

Then, considering that $\sqrt{2}\,h > a$,

$$1 - \exp\left[-\frac{1}{2}(\sqrt{2}\,h - a)^2\right] \geq 1 - \exp\left[-\frac{1}{2}\left(\sqrt{2T(H' + \delta)} + \sqrt{2T}\,\eta - a\right)^2\right].$$

When $T \geq T_1 = a^2/2\eta^2$

$$1 - \exp\left[-\frac{1}{2}(\sqrt{2}\,h - a)^2\right] \geq 1 - \exp[-T(H' + \delta)] \tag{4.121}$$

From (4.118) and (4.121)

$$\begin{aligned} q &\geq \{1 - \exp[-T(H' + \delta)]\}^{m-1} - \frac{\varepsilon}{2} \geq \\ &\geq 1 - (m-1)\exp[-T(H' + \delta)] - \frac{\varepsilon}{2} > \\ &> 1 - m\exp[-T(H' + \delta)] - \frac{\varepsilon}{2}. \end{aligned}$$

Considering that $m = e^{H'T}$, we obtain

$$q > 1 - e^{H'T}\left[e^{-H'T - T\delta}\right] - \frac{\varepsilon}{2} = 1 - e^{-\delta T} - \frac{\varepsilon}{2}.$$

Setting $T_2 = (1/\delta)\ln 2/\varepsilon$, we finally obtain the result that with $T \geq \max(T_0, T_1, T_2)$

$$q > 1 - \varepsilon.$$

which is what we wanted to prove.

We will remind the reader that the initial formula (4.46) was obtained under the assumption that during time T the initial phase of the signal remains practically unchanged. Inasmuch as in the proof presented it was assumed that T could be as large as desired, it holds only for a channel in which the phase of the signal does not fluctuate but remains unknown when the decision system is designed, so that only incoherent reception is possible.

For the case when the initial phase of the signal fluctuates rather rapidly, computation of the carrying capacity of the channel entails great difficulty. For the purpose of obtaining an evaluation of this carrying capacity from below it is possible to resort to the following reasoning. We will select a sufficiently small interval of time T_1 during which the phase practically does not fluctuate and we will transmit a message with the help of a sequence of binary signals of duration $T = T_1$. We will perform coding in a discrete channel by joining rather long sequences of information symbols and by guaranteeing a given (as small as desired) probability of error of decoding with a transmission rate which is as close as desired to the carrying capacity of a discrete channel (2.28). In natural units this carrying capacity can be written in the following form

$$C = \frac{1}{T}[\ln 2 + p\ln p + (1-p)\ln(1-p)] \text{ natural units/sec.} \quad (4.122)$$

where p is the probability of error in a discrete channel depending on $h = P_s T/\nu$.

The maximal carrying capacity of a discrete symmetrical binary channel will occur when the probability of error is minimal. The latter is provided for in incoherent reception by the selection of an RPT system for which $p = (1/2)e^{-h^2}$. Expressing T by h^2 and substituting in (4.122) the value of the probability of error we find[1]

$$C = \frac{P_s}{\nu h^2}\left[\ln 2 - \frac{1}{2}e^{-h^2}(\ln 2 + h^2) + \left(1 - \frac{1}{2}e^{-h^2}\right)\ln\left(1 - \frac{1}{2}e^{-h^2}\right)\right] \quad (4.123)$$

---

[1] Strictly speaking, formula (4.122) for RPT is not correct. It holds for a channel without memory while with RPT there is a tendency toward paired grouping of errors. However, this formula can be used as an evaluation from below inasmuch as a channel with RPT can be converted into a channel with independent errors by considering separately symbols with odd and even numbers and joining them separately into combinations of a correcting code. If the dependence of errors is considered, the carrying capacity proves to be somewhat greater than (4.122).

By changing the magnitude [illegible] so that [illegible] remains less than T_p, we would be able to change [illegible].

In distinction from the case of a channel with a constant phase where the rate of information transmission [illegible] according to (3.87) increases monotonically with a decrease in [illegible], here we have an optimal value [illegible] with which (4.125) achieves a maximum. An analysis of expression (4.125) [2.3] shows that this maximum occurs when h = 1.551. This corresponds to the optimal value

$$T_{opt} \; [illegible].$$

If the power of a signal is sufficiently great, so that T_{opt} [illegible] T_p, we will select a duration of signal equal to T_{opt}. Substituting [illegible] in (4.125), we find

$$[illegible] \qquad [illegible]$$

Thus, the carrying capacity of a channel with fluctuations in phase and with an unlimited passband [illegible] 2.5 [illegible] of the carrying capacity of a channel with constant parameters if the signal power and rate of fluctuation are such that during time

$$[illegible]$$

the phase remains practically unchanged. The case [illegible] requires separate consideration.

Notes

1. (see Section 4.1) the definition of an analytical signal (4.7) [illegible] here departs somewhat from that generally accepted. Usually the imaginary part of an analytical signal which is used is not function $\tilde{x}_r$ expressed in interval (0, T) by a conjugate series (4.5) and equal to zero outside this interval, but function [illegible] conjugate with [illegible] according to Hilbert which is defined as the principal value of the integral

$$[illegible] \qquad [illegible]$$

This function does not [illegible] outside the interval (0, T).

The definition of an analytical signal used here is convenient for analysis of resistance to interference in the case of incoherent reception inasmuch as series (4.5) will appear in a natural way in reaching the conclusion for an optimal decision principle, e.g., that in (4.29).

We will note that if [illegible] and [illegible] are understood to be not finite functions different from zero only in interval (0, T) but periodic functions expressed along the entire time axis by series [illegible] and [illegible], then [illegible] coincides with the Gilbert transform of $[illegible]_r(t)$, i.e.,

[illegible] (4.126)

The question of convergence of the series in [illegible] does not arise in this work, as we everywhere assume that only a finite number of [illegible] differ from zero.

2. (See Section 4.5.) [illegible] assumption that the initial phase of the received signal is distributed evenly over an interval from 0 to [illegible] is natural, but still it has no basis other than an experiment.

If the distribution of [illegible] is not known, to deduce the decision principle it is possible to use the generalized criterion of maximum likelihood [illegible], i.e., to consider that signal [illegible] was transmitted, if for all r ≠ [illegible]

[illegible] (4.127)

To find a maximum with respect to [illegible], we differentiate (4.25) and setting the derivative equal to zero, we determine the value of [illegible] for which the likelihood function attains a maximum. After simple transformations, we find that the maximum is reached when

[illegible] (4.128)

Substituting this value of [illegible] in (4.25), we find

[illegible]

whence the decision principle governing the transmission of signal [illegible](t) adopts the following form:

[illegible]

for all r ≠ [illegible]. Taking the logarithms of this inequality, we find

[illegible] (4.129)

For systems with an active interval this coincides with the principle of (4.30)

[illegible]

obtained with a uniform distribution of [illegible].

3. (See Section 4.[illegible].) A more simply realizable system [27] may be used instead of the nonlinear devices K (Figures 4.1 and 4.2) to square the integration results when effectuating a quadrature system. For this purpose voltages from

the outputs of the two integrators and proportional to X_i and Y_i go to the multipliers (Figure 4.27) which are fed the respective voltages [illegible] cos $\omega_{aux}t$ and sin $\omega_{aux}t$ from an auxiliary frequency ω_{aux} [illegible] of arbitrary amplitude.

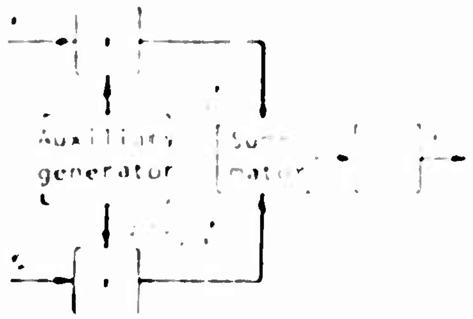


Figure 4.27. Squaring Unit for Quadrature Reception.

After summation of the output voltages of these multipliers a voltage is obtained which is proportional to $X_i \cos \omega_{aux}t + Y_i \sin \omega_{aux}t$ and whose amplitude is proportional to $\sqrt{X_i^2 + Y_i^2}$. This amplitude is discriminated by detector 3 with a characteristic of log I_0 and goes directly to the comparator circuit (in an active interval system) or through threshold unit 2 (Figures 4.1 and 4.2). It is easily seen that the results of this system do not differ from those of Figures 4.1 and 4.2.

4. (See Section 4.3). A keyed filter in the form of an oscillatory circuit in which the oscillations are periodically (with period T) extinguished represents a linear system with variable parameters. If at moment t = 0 a harmonic of some frequency ω with unit amplitude is fed to the input of this system, then at moment t = T the amplitude at the output will reach some value F(ω). The function $F(\omega - \omega_{res})$, where ω_{res} is the resonant frequency, is conventionally called the dynamic amplitude-frequency characteristic of a keyed filter. This characteristic may be calculated by means of Duhamel's integral if it is taken into account that the impulse response of the keyed filter is $e^{-\alpha t}\cos \omega_{res}t$, where α is circuit damping. The voltage at the output of a keyed filter[1] at moment T is

$$u(T) = \int_0^T [illegible] \qquad (4.130)$$

[illegible]

---

[1] In deducing expression (4.130) we have disregarded terms on the order of $\alpha/2\omega_{res}$, as is permissible with small misalignments.

where ω [illegible] res[illegible]

Hence

$$[illegible] \qquad (3.13)$$

and

$$[illegible]$$

In the ideal circuit [illegible] = 0 the expression converts into the following

$$[illegible]$$

where

$$[illegible]$$

For estimating [illegible] dependence [illegible] of the frequency characteristic of the [illegible] filter [illegible]. There is another expression [illegible] of the [illegible] which [illegible] frequency [illegible]

[illegible]

$$[illegible] \qquad (3.1[illegible])$$

and [illegible] coincides with the [illegible] of the [illegible] frequency characteristic of the circuit with the [illegible] of tuning [illegible] where [illegible] Figure 3.28 represents these [illegible] characteristics.

[illegible] Section 3.5. Besides the described systems realizing the optimum decision principle for incoherent reception, other systems can also be proposed which are equivalent with respect to noise resistance and have certain advantages with regard to their technical realization. One of these is the system of synchronous heterodyning, partially used in the communication system known as "Rake." This system is outwardly reminiscent of the coherent reception system (Figure 3.5) and is designed for the reception of complex signals with a large base. Figure 3.29 represents the functional diagram of synchronous heterodyning for the reception of binary signals. Circuits may be constructed on the same principle for systems with any code base.

Let

[illegible]

[illegible]

where coefficients k adopt values from k_1 to k_2. [illegible] local oscillators (heterodynes). [illegible] which differ from the signals in that they are shifted [illegible] frequency [illegible]

[illegible]

[illegible]

We will assume that [illegible] multipliers [illegible] factors. Utilizing the previously [illegible] coefficients, let us find [illegible]

[illegible]

[illegible]

This condition is not obligatory for the circuit in question, but [illegible] is to complicate analysis and somewhat decrease resistance to interference because of the generation of "image" frequencies.

The voltages e_1 and e_2 are fed [illegible] frequencies $f_{res\,1}$ and $f_{res\,2}$ [illegible], the [illegible] deducing formula (4.[illegible]) we easily [illegible] that [illegible] $t = T$ the voltage on the first [illegible] filter [illegible] of frequency $f_{res\,1}$, the amplitude [illegible] expression (4.[illegible]) with a different [illegible]

[illegible]

[illegible] proportional to the value [illegible] plitude in the second keyed filter [illegible] fore, it is sufficient to [illegible] compare their value [illegible]

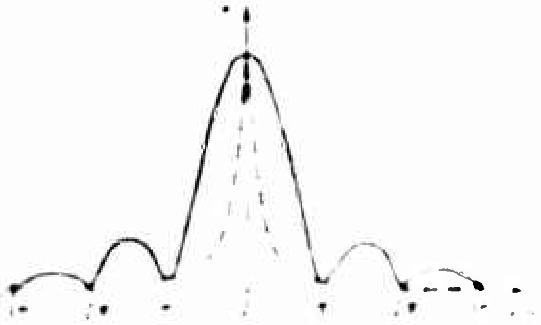


Figure 4.28 Frequency characteristic [illegible] keyed filter [illegible] [illegible] state [illegible]

Figure 4.29 [illegible] diagram [illegible] heterodyning

Such a system is convenient when [illegible] Matched (or keyed filters) [illegible] (e.g., pi-response filters). This permits [illegible] precision at the cost [illegible] Section 4.[illegible]

6. (See Sections 4.4 and 4.5). We will try to determine in general form an active-interval binary system the loss occurring due to the exchange in the circuit shown in Figure 4.5 of unmatched filters for the matched ones. Let $z_1(t)$ and $z_2(t)$ be the signals of the system and let the filters have a pulse response of $g_1(t)$ and $g_2(t)$. We will first consider the case when signal $z_1(t)$ is transmitted and draw attention to the fact that the probability of error in this case is determined entirely by this signal and the characteristics of the filters and does not depend on the second signal $z_2(t)$. This follows from the fact that the voltages arriving at the comparator do not depend on $z_2(t)$. Therefore, the probability of error in transmitting signal $z_1(t)$ does not change if instead of signal $z_2(t)$ in the system under consideration we use signal $z_2'(t) = bg_2(T - t)$ which is matched with filter $g_2(t)$ where the constant b is so selected that the powers of signals $z_2'(t)$ and $z_1(t)$ are the same.

We will replace temporarily the filter with the pulse response of $g_1(t)$ by a filter $g_1'(t)$ which is matched with signal $z_1(t)$. We obtain in this way a new system with signals $z_1(t)$ and $z_2'(t)$ and matched filters $g_1'(t)$ and $g_2(t)$. The probability of error in this active interval system is determined by formula (4.61) where the magnitude h represents the ratio between the power of the signal $z_1(t)$ and the interference spectral density and the magnitude ρ is calculated from formulas (4.57) for signals $z_1(t)$ and $z_2'(t)$. We will remind the reader that the voltage envelopes in the matched filters at the instant of reading are random values which have, generally speaking, a generalized Rayleigh distribution and the ratio between their regular components is equal to ρ.

We will now return to the initial filter $g_1(t)$. In this case the voltage in the second filter does not change and the voltage envelope in filter $g_1(t)$ (in the absence of interference) is somewhat less than in matched filter $g_1'(t)$. We will designate the ratio between the values of the envelope in the matched g_1' and unmatched g_1 filters at the instant of reading by the letter m. If the power of the signal $z_1(t)$ is increased by m times, we obtain in filter g_1 the same value of envelope which previously was in filter g_1'. But in this process the voltage in filter g_2 increases by m times, i.e., the magnitude of ρ is as if also increased by m times. If the magnitude of m is close to unity, in the first approximation the probability of error in transmitting signal $z_1(t)$ with the power increased by m times is determined by formula (4.61a), if in it we replace ρ by mρ. We will now return to the initial power of the signal. In this process the magnitude h is decreased m times.

and, consequently, the probability of error in transmitting signal $z_1(t)$ in an actual circuit with filters g_1 and g_2 is equal to

$$P_e = Q\left(\sqrt{\frac{h^2}{2m}\left(1-\sqrt{1-m^2\rho^2}\right)},\ \sqrt{\frac{h^2}{2m}\left(1+\sqrt{1-m^2\rho^2}\right)}\right) - \frac{1}{2}e^{-\frac{h^2}{2m}} I_0\left(\frac{\rho h^2}{2}\right), \tag{4.134}$$

where ρ is determined for signals $z_1(t)$ and $z_2(t)$ and m is the ratio between the values of the envelope at instant of reading in filters g_1 and g_2 when signal $z_1(t)$ is delivered to them:

$$m = \frac{\int_0^T z_1^2\,dt}{\sqrt{\left[\int_0^T z_1(t)\,z_1^*(t)\,dt\right]^2 + \left[\int_0^T \tilde{z}_1(t)\,z_1^*(t)\,dt\right]^2}},$$

$z_1^*(t)$ is a signal matched with filter g_1 and having the same power as signal $z_1(t)$.

Similarly, we may determine the probability of error when transmitting signal $z_2(t)$.

7. (See Section 4.5) The results obtained for reception with post-detection integration, particularly formulas (4.93) and (4.95), are true only to the degree of accuracy to which the probability distribution of the fluctuations at the output of the integrator may be considered normal. In fact, with a finite value of ΔfT this probability distribution differs from normal. Asymmetry in fluctuation probability distribution (especially noticeable at the output of an integrator which contains only noise) increases the probability of error. With large values of ΔfT the absolute value of this correction is very slight, but if error probability is low, as occurs when $h \gg 1$, the relative error of these formulas may become very substantial. Therefore, the larger h is, the larger must also be the value of ΔfT at which these expressions give a good appreciation.

If we do not take this circumstance into account we may arrive at the paradoxical conclusion that it is impossible for a nonoptimum system to have an error probability which is beneath the limit determined by the theory of potential resistance to interference. In fact, assuming, for example, that the value of ΔfT in expression (4.95a) is fixed and selecting a value for h which satisfies the condition

$$h > \frac{1}{0.21}\sqrt{2\Delta fT} \tag{4.135}$$

we get

$$P = \frac{1}{2}\left[1-\Phi\left(0.21\,h^2/\sqrt{2\Delta fT}\right)\right] < \frac{1}{2}\left[1-\Phi(h)\right] \tag{4.136}$$

This conclu[illegible], in fact, [illegible] relative error in using the normal [illegible] distribution [illegible] that formulas (4.93) and [illegible].

Furthermore, with [illegible] the output of the [illegible] detector following [illegible] and they begin to arrive [illegible] performed [illegible] increase the [illegible] this pro[illegible] integrat[illegible] signal [illegible]

Thus, the [illegible] reception [illegible] less than in the case of [illegible]

[illegible] detector [illegible] printer, etc. [illegible] the probability [illegible] filter [illegible] the stability [illegible] power gain. [illegible] mulas close [illegible] a certain gain [illegible] (compare Figure 4.1[illegible]).

[illegible] which is similar to [illegible] of signals in binary FPI was [illegible] indication as to how it was [illegible] (4.102) was deduced [illegible] a rather [illegible] bution of phase difference [illegible] a constant signal and normal noise [illegible] [51] in a somewhat different way but also on the basis of the results given in [30].

The approach suggested here, which takes into consideration the fact that a decision about the transmitted FPI signal is reached on the basis of an analysis of the arriving signal in interval (-T,T) made it possible to obtain formula (4.102) as a direct generalization of formula [illegible].

BIBLIOGRAPHY

1. [illegible] [Transformation of Superhigh Frequencies and Detection], [illegible] Moscow, Leningrad, 1953.

2. [illegible] Publishing, [illegible], 1963.

[illegible]

[illegible] Radio [illegible]

[illegible]

[illegible] [Effect of Ratio between Receiver Passband and Frequency Band [illegible]], "Elektrosvyaz'", No. 11, 1965.

14. Yevtyanov, S. I., [illegible] [Transient Processes in Amplifying Receiving Circuits], Svyaz'izdat, 1948.

15. Bunimovich, V. I., [illegible] [Fluctuation Processes in Radio Receiving Devices], "Sovetskoye radio" Press, 1951.

16. Slepyan, D., [illegible] [Fluctuations in the Power of a Random Signal], In the Collection: [illegible] [Determining the Parameters of Random Processes], Gostekhizdat, Ukrainian SSR, 1962.

17. [illegible]ristol'kors, A. A., [illegible] ["Multiple Telegraphy with a Change in Phase], [illegible], No. 3, 195[illegible].

18. Fetin, Yu. [illegible] [Practical Elimination of "Reverse Operation" in an Amplitude Phase Detector During the Action of Impulse Interference], [illegible], No. [illegible], 195[illegible].

19. [illegible] Stas, [illegible] "[illegible] Teletype," [illegible], No. [illegible], 19[illegible].

20. Petrovich, N. [illegible] [New Methods of Differential Phase Telegraphy], Radiotekhnika, No. [illegible], 19[illegible].

21. [illegible], V. [illegible] [Phase Difference "Relative" [illegible] and Its [illegible] for Transmitting Discrete Information], "[illegible]" Press, [illegible].

22. Mosier, [illegible] and [illegible], "Kineplex, a Bandwidth Efficient Binary Transmission System," Commun. and Electr., No. 34.

23. Petrovich, N. T., [illegible] [Transmitting Discrete Information in Phase Modulation Channels], "Sovetskoye radio" Press, 196[illegible].

24. Fink, L. M. and V. F. Georgiyev, [illegible] [Error Distribution in Reception of Binary Signals in Relative Phase Telegraphy], "Elektrosvyaz'," No. 7, 1963.

25. Fink, L. M., [illegible] [Carrying Capacity of Symmetrical Channels Having Variable Parameters with an Unlimited Frequency Band], "Radiotekhnika," No. 7, 1960.

26. Davenport, V. B. and V. L. Rut, [illegible] [Introduction to the Theory of Random Signals and Noise], Foreign Literature Press, 1960.

27. Gutkin, L. S., [illegible] [Theory of Optimal Methods of Reception with Fluctuation Interference], Gosenergoizdat, Moscow, Leningrad, 1961.

28. Mel'nikov, V. S., [illegible] [Evaluation of Resistance to Interference in a Telegraph Receiver with Post Detection Integration of Signals], "Elektrosvyaz'", No. 5, 1965.

29. Lawton, J. G., "Theoretical Error Rates of Differentially Coherent Binary and 'Kineplex' Data Transmission Systems," PIRE, No. 2, 1959.

30. Tsvetnov, V. V., [illegible] [Statistical Properties of Signals and Interference in Two-Channel Phase Systems], "Radiotekhnika," No. 5, 1957.

31. Georgiyev, V. K. and N. T. Petrovich, [illegible] [Calculation of Error Probability in Reception of EPI Signals Accompanied by Fading], "Radiotekhnika," No. 10, 1964.

CHAPTER V

CHANNEL WITH SLOW GENERAL FADING
(SINGLE-TRANSMISSION RECEPTION)

5.1. Nature of Fading Phenomena and Their Classification

We will give the name of fading channel to a channel in which the amplitude of signal components arriving at the receiver have been subjected to fluctuations. Under real conditions phase fluctuations are always observed with the amplitude fluctuations of the signal components. We will therefore assume that when fading is present the phase of the incoming signal is also indeterminate to a certain degree.

Fading is a phenomenon which is characteristic of most radio channels. In the fading channel the signal is usually physically propagated via several routes. Because of the variations in the course of the beams coming to the receiver from the transmitter the signal in the receiving antenna is the sum of the separate oscillations with different phases and amplitudes. The interference of these oscillations under conditions where the course of the beams does not remain constant is the basic reason for fluctuations of the signal components, of its amplitude and its phase.* We will consider these differences in path lengths to be slight as compared to the duration of a signal element [illegible] and will not take their influence on the moments of element beginning and end into account. The situation involving large differences in path will be examined in Chapter VII.

Let us take a short look at the model of fading phenomena of different types. Let us assume that the beams arriving at the receiver are reflected (or scattered) from some part of the ionosphere or troposphere in such a way that the difference in course is commensurate with the wavelength (Figure 5.1). The phenomenon occurs because neither the ionosphere nor any other reflector is an ideal mirror, but, rather, may be imagined as a very rough surface, which, furthermore, is not constant over time.

Let signal

$$s(t) = \sum [illegible]$$

be transmitted.

---

*Magnetic splitting of the beams also plays a certain role in this.

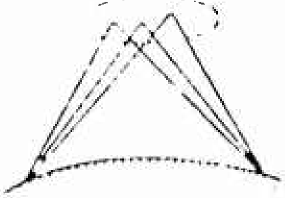


Figure 5.1. Multiple Beam Propagation of Signal.

At the input of the receiver arrive n beams, each with its own propagation period t_{pi} and its own transmission coefficient μ_i. Relatively narrow-band signals may be thought of as having identical t_{pi} and μ_i for all their components, i.e., independent of subscript k. Then the signal being received may be represented as

[illegible] (5.1)

where $\bar{t}_p$ is the average propagation time for all beams, $\varphi_{ik} = k\omega_0(\bar{t}_p - t_{pi})$ [illegible] $2\pi k\Delta t_i$ [illegible]; and n(t) is additive noise.

In the case in question the following inequality holds true:

[illegible] (5.2)

hence the values of φ_{ik} for a certain subscript i which lie in the region from $2\pi k_1\Delta t_i$ to $2\pi k_2\Delta t_i$ differ from each other by no more than $2\pi F\Delta t_i \ll 2\pi$. It may therefore be supposed that in a first approximation the values of φ_{ik} do not depend on the number of a component k, although for different values of subscript i, i.e., for different beams, they may be substantially different.

Therefore,

[illegible] (5.3)

where

$$\mu_c = \sum_{i=1}^{n} \mu_i \cos\psi_i; \quad \mu_s = \sum_{i=1}^{n} \mu_i \sin\psi_i;$$

$$\mu = \sqrt{\mu_c^2 + \mu_s^2}; \quad \theta = \arctan \mu_s/\mu_c; \quad t' = t - \bar{t}_p$$

(in the following we will drop the prime from t and assume $\bar{t}_p$ to be the moment of starting to read the time). The magnitude μ may be formally regarded as the length of a vector with components μ_c and μ_s.

The incoming signal thus differs from the transmitted by random transmission coefficient μ and a phase shift θ which is random, but approximately the same for all the frequency components. This fading is called general (or smooth) since the relations between the amplitudes and phases of the signal components do not change.

To analyze the conditions for transmitting information in a fading channel we must know the probability distribution of the random variables μ and θ. This may be found by assuming that the number n of incoming beams is so great as to permit the central limit theorem to be applied.[1]

Let us study two extreme cases where the differences in propagation time Δt_i attain values which substantially exceed the period of the average signal frequency $2\pi/\omega_{av}$ where $\Delta t_i \gg 2\pi/\omega_{av}$.[2]

In the first case ψ_i may be much larger than 2π (Figure 5.2). In this case the random variables $\cos\psi_i$ and $\sin\psi_i$ have a mathematical expectation of practically zero and identical dispersions of 0.5, while $\mu_i \cos\psi_i$ and $\mu_i \sin\psi_i$ are values of limited dispersion with mathematical expectations of zero. When n is large the sums of μ_c, μ_s may be considered normally distributed random variables with average values of zero and identical dispersion. Under these conditions μ has a Rayleigh distribution and its unidimensional probability density is[3]

---

[1] Here a good enough approximation is here obtained when n is no more than five, which is almost always the case in practice.

[2] The inequality $\Delta t_i \gg 2\pi/\omega_{av}$ does not at all contradict condition (5.1a) since signal frequency band F is practically always at least hundreds or thousands of times less than $\omega_{av}/2\pi$.

[3] The following is the more customary notation for a Rayleigh probability distribution density:

$$w(\mu) = \frac{\mu}{\sigma^2} \exp\left(-\frac{\mu^2}{2\sigma^2}\right)$$

It does not differ from that used in the text if $\mu_0^2 = 2\sigma^2$. The advantage of the notation in (5.3) is that the mean-square value of transfer coefficient μ_0 figures in clearly as a parameter.

$$\left.\begin{aligned} w(\mu) &= \frac{2\mu}{\mu_0}\exp\left(-\frac{\mu^2}{\mu_0}\right) \quad (\mu > 0), \\ w(\mu) &= 0 \quad (\mu < 0) \end{aligned}\right\} \tag{5.3}$$

where $\mu_0 = \overline{\mu^2}$ is the mean-square value of transmission factor μ.

Phase shift θ, as the arctangent of the ratio of two independent normal identically distributed random variables, has uniform probability density in the range from 0 to 2π. Rayleigh fading is the name we will give to this type of fading.

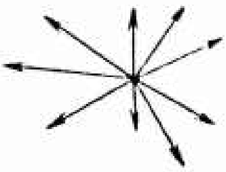


Figure 5.2. Vectorial Representation of Beams at Receiver Output When $\Delta t_{i_{max}} \gg 2\pi/\omega_{av}$.

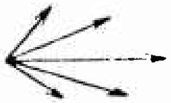


Figure 5.3. Vectorial Representation of Incoming Beams When $\Delta t_{i_{max}} \gg 2\pi/\omega_{av}$

In the second case the probability that ψ_i will reach 2π is very slight, i.e., the phases of the incoming beams (Figure 5.3) group themselves around an average value of zero. Assuming a symmetrical probability density for ψ_i we easily satisfy ourselves that the mathematical expectation of $\sin\psi_i$, as an odd function of ψ_i is also zero, while the mathematical expectation of $\cos\psi_i$ (an even function) differs from zero and is positive. Therefore the mathematical expectation of variable μ_s is zero and that of μ_c, which we will denote by μ_p, is more than zero (since variable $\mu_i > 0$). Propagation factor μ, as the length of a vector with normal components, one of which has an average value other than zero, obeys the generalized Rayleigh distribution. Its probability density is

$$\left.\begin{aligned} w(\mu) &= \frac{2\mu}{\mu_f}\exp\left(-\frac{\mu^2+\mu_p^2}{\mu_f}\right) I_0\left(2\frac{\mu\mu_p}{\mu_f}\right) \quad (\mu > 0), \\ w(\mu) &= 0 \quad (\mu < 0) \end{aligned}\right\} \tag{5.4}$$

Here $\mu_f^2 = \mu_0^2 - \mu_p^2$ is the average value of the square of the fluctuating portion of the propagation factor.

If we imagine μ_c as the sum of $\mu_p + \mu_{cf}$ the mathematical expectation of μ_{cf} is zero. Therefore μ may be considered the geometrical sum of the constant vector of μ_p, which is called the regular component of the transmission factor, and of the two normally distributed fluctuating vectors of average value zero, μ_{cf} and μ_s (Figure 5.4). The variable μ_f is the average square of the geometrical sum of μ_{cf} and μ_s.

Phase shift [illegible] in this case is nonuniformly distributed. Its probability density has its maximum at [illegible] 0, the value of which depends on the ratio between E_p and σ_0 (e.g., see [1]). To abbreviate we will call the fading characterized by the probability density of expression (5.4) quasi Rayleigh. Many authors come to the distribution of expression (5.4) under the supposition that in ionospheric radio communications mirror reflection which determines the regular portion of the transmission factor occurs along with the diffuse dissemination which generates the fluctuating portion of this factor. The reasoning adduced shows that such a model is unnecessary. In particular, in the medium wave range and in the lower sector of the shortwave range the condition $t_i > 2$ [illegible] is often fulfilled when waves are reflected from the ionosphere; and this determines the quasi Rayleigh nature of the fading without the hypothesis that there is a separate "regular" beam. In some cases, however, distribution (5.4) owes its origin to the direct passage of a beam, for example, along the surface of the earth, together with the arrival of diffuse reflected beams. It is apparent that distribution (5.3) is a particular case of expression (5.4) where $E_p = 0$.

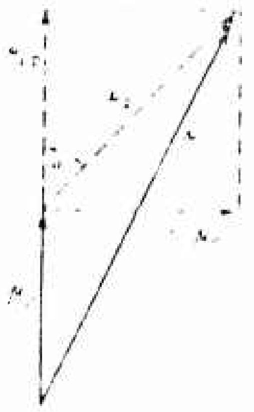


Figure 5.4. Resolution of Transmission Factor Into its Cophasal and Quadrature Components

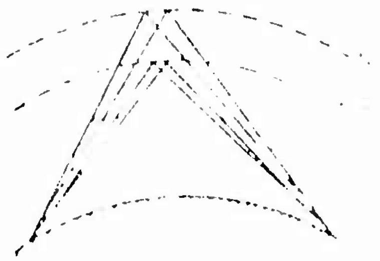


Figure 5.5. Selective Fading in Reflection from Different Layers of Ionosphere.

[illegible] ultra short wave range [illegible] Rayleigh [illegible] with a [illegible] regular [illegible] quasi Rayleigh [illegible] to do with the pro[illegible] are added [illegible] the interference [illegible]

We [illegible] fading [illegible] with general fading when condition $t_i < 1/$[illegible] is not fulfilled. [illegible] fading [illegible] indicates that the receiver is picking up beams which have been reflected from widely separated ionospheric (or tropospheric) regions. Thus, for example, in shortwave communications beams may come to the receiving antenna which were reflected from layers E and F_2 of the ionosphere (Figure 5.5), or which have undergone a number of different reflections (Figure 5.6), etc. For the most part, moreover, it is not the simple beam which traverses each of these pathways, but a beam consisting of a large number of individual components such as are shown in Figure 5.1. Therefore, each of the arriving beams which have undergone various reflections is also subject to fading.

In selective fading the phase shifts ψ_{ik} differ as the subscripts k differ. Therefore,

$$\begin{aligned} x'(t) &= \sum_{k=k_1}^{k_2} a_k \sum_{i=1}^{n} \mu_i \cos(\omega_k t - \varphi_k + \psi_{ik}) + n(t) = \\ &= \sum_{k=k_1}^{k_2} a_k \sum_{i=1}^{n} [\mu_i \cos\psi_{ik} \cos(\omega_k t - \varphi_k) - \mu_i \sin\psi_{ik} \sin(\omega_k t - \varphi_k)] + \\ &+ n(t) = \sum_{k=k_1}^{k_2} a_k \mu_k \cos(\omega_k t - \varphi_k + \theta_k) + n(t), \end{aligned} \tag{5.5}$$

where

$$\mu_k = \sqrt{\left(\sum_{i=1}^{n} \mu_i \cos\psi_{ik}\right)^2 + \left(\sum_{i=1}^{n} \mu_i \sin\psi_{ik}\right)^2},$$

$$\theta_k = \operatorname{arctg}\left(\sum_{i=1}^{n} \mu_i \sin\psi_{ik} \Big/ \sum_{i=1}^{n} \mu_i \cos\psi_{ik}\right).$$

Thus, in selective fading each of the frequency components of the signal has its own transmission coefficient μ_k and its own phase shift θ_k. The variables μ_k with different subscripts k are, of course, intercorrelated. This follows from the fact that identical coefficients μ_i enter into the expression for μ_k. This correlation is the greater, the smaller the difference between the frequencies of the components (or between the k subscripts) and the smaller the difference Δt_i in the paths of the beams. As for the unidimensional probability distributions of μ_k and θ_k, they are obviously the same as in general Rayleigh fading. Let us note that under identical propagation conditions fading may manifest itself as general if the signal frequency band is narrow or as selective if the signal is wide-band. With narrow-band signals when $k_2 - k_1$ is small condition (5.1a) is violated if the values of Δt_i are commensurate with T. In these cases the selective fading is accompanied by a related phenomenon in that the individual signal elements in the beams which have traversed different paths overlap (the phenomenon of echo-signals). Selective fading and echo-signals will be discussed in Chapter VII.

If the values of Δt_i remained constant, transmission factor μ and phase shift θ would be random, but constant for a given channel. Actually, wave reflection and dispersion conditions in the ionosphere or troposphere continually change. Therefore μ and φ are random processes. The changes of μ and φ in time may be characterized by a correlation factor [1] which depends on the physical processes in the ionosphere (troposphere) and may be determined only experimentally.

We will consider separately the cophasal μ_{cf} and quadrature μ_s components of the fluctuating part of the transmission factor which were determined above (Figure 5.4). Obviously $\mu_{cf}(t)$ and $\mu_s(t)$ are normal random processes with the same correlation coefficients:

$$R(\tau) = \frac{\overline{\mu_{cf}(t)\mu_{cf}(t+\tau)}}{\overline{\mu_{cf}^2}} = \frac{\overline{\mu_s(t)\mu_s(t+\tau)}}{\overline{\mu_s^2}}, \tag{5.6}$$

where the line indicates statistical averaging.

We will designate the coefficient of mutual correlation between μ_{cf} and μ_s, $\tilde{R}(\tau)$:

$$\tilde{R}(\tau) = \frac{\overline{\mu_{cf}(t)\mu_s(t+\tau)}}{\overline{\mu_s^2}} = -\frac{\overline{\mu_s(t)\mu_{cf}(t+\tau)}}{\overline{\mu_s^2}}. \tag{5.6a}$$

Here it is considered that $\overline{\mu_{cf}^2} = \overline{\mu_s^2}$.

As is known, $R(\tau)$ is an even and $\tilde{R}(\tau)$ an odd function of τ. Specifically, $\tilde{R}(0) = 0$.

We will set $R_0^2(\tau) = R^2(\tau) + \tilde{R}^2(\tau)$. Then the correlation coefficient is related to $R_0(\tau)$ by the relationship (see [1], formula 8.31):

$$R_\Gamma(\tau) = \frac{\pi}{4(4-\pi)}\left[R_0^2(\tau) + \sum_{n=2}^{\infty}\frac{[(2n-3)!!]^2}{2^{2n}(n!)^2}R_0^{2n}(\tau)\right] \tag{5.6b}$$

With sufficient inaccuracy for engineering calculations, when $R_0 \ll 1$, it can be assumed that

$$R_\Gamma(\tau) \approx R_0^2(\tau) = R^2(\tau) + \tilde{R}^2(\tau).$$

and with a large (close to unity) values of R_0

$$R_\Gamma \approx 1 - \frac{1}{4-\pi}[1 - R_0^2(\tau)]$$

Most authors suggest the following approximate formulas for correlation coefficient $R(\tau)$ with interference fading:

$$R_0(\tau) = \exp\left(-\frac{\tau^2}{2\tau_k^2}\right), \tag{5.6c}$$

or

$$R_0(\tau) = \exp\left(-\frac{|\tau|}{\tau_k}\right) \tag{5.6d}$$

The variable τ_k describes the rate of fading. Specifically, when $\tau = \tau_k$ we have respectively, from formulas (5.6c) and (5.6d), $R(\tau_k) = e^{-1}$ or $R_f(\tau_k) = e^{-1}$. Therefore, τ_k is often called correlation time or the average fading period. According to experimental data for ionospheric shortwave radio communication the variable τ_k ranges from 0.1 sec (over very long distances) to 2 sec (over relatively short distances) [2, 3, 4, 5]. For other channels τ_k may differ greatly from the values indicated.

For investigating conditions of signal transmission in fading channels what is important is not the absolute value of the fading correlation time but the relationship between the rate of fading and the rate of transmission. We will call the fading slow when $\tau_k \gg T$, where T is the duration of a signal element, and fast if τ_k is of the same order as T or less than T. In the limiting case when $T/\tau_k \to 0$, it can be considered that the variables μ_c and μ_s do not change at all for the duration of one of even several signal elements. Under these conditions we will call fluctuations in the transmission coefficient fading at a zero rate.[1]

In most channels which are used at the present time for the transmission of discrete messages, slow fading which can often be considered with good approximation as fading at a zero rate occurs. Still, in many cases, specifically in space radio communication and in several radio communication channels with tropospheric scattering fast fading is encountered. It should be remembered that the rate of fading is determined by the relative duration of signal element T and therefore one and the same physical communication line may be characterized by slow fading if signals with a small T are transmitted and fast if T is great.

Experience shows that selective fading predominates in the shortwave range if the signal frequency band is broader than several hundred cps. With narrower-band signals the selective nature of the fading does not manifest itself and in most cases fading may be considered general under these conditions. General fading is also often encountered in tropospheric scattering. We would remark that the nature of signal fading at receiver input depends on the directional pattern of incoming beams decreases and beams are principally received which have little course difference Δt_i. Therefore under identical conditions of propagation the signal received on a non-directional antenna may, for example, have selective Rayleigh fading, while that received on a narrowly directional antenna may have general quasi-Rayleigh fading.

---

[1] The classification of fading according to rate presented here is generally accepted. Specifically, it differs from the terminology used in the first edition of this book and in work [7] where slow fading means fading which we now call fading at a zero rate and fast fading (in the sense $\tau_k \leq T$) is not considered at all.

Interference phenomena are not the only cause of change in incoming signal strength. Relatively slow (hourly and daily) transmission coefficient fluctuations are caused by a number of other reasons, for example, changes in degree of absorption in the ionosphere, changes in tropospheric temperature gradients, etc. These fluctuations are sometimes called absorption fading. Their effect on communications reduces to the reception of some messages under better, and of others under worse conditions. Ordinarily these changes in transmission coefficient (which is averaged over a period of about an hour) are characterized by normally logarithmic probability distribution.

A description of fading in radio channels would be incomplete if we did not mention polarization of incoming waves. As a rule with reflection of radio waves the plane of polarization changes. If a transmitter emits waves with a certain polarity (linear or angular), then under conditions of interference fading a wave arriving at a receiver is unpolarized or partially polarized. When this occurs the fading of the polarized components of a received wave are weakly correlated with one another [5]. If an emitted wave is unpolarized the received wave also, as a rule, is unpolarized. Dividing it into two components which are orthogonal with respect to polarization, we are able to see that the fading in them is very weakly correlated. This phenomenon is usually called polarization fading. However, in our opinion, polarization fading should not be compared with interference fading inasmuch as these concepts describe, in essence, two sides of one and the same phenomenon. Now general fading will be considered in this chapter and the one following.

5.2. Coherent and Incoherent Reception Under Conditions of General Fading at a Zero Rate.

Reception in Channels with Rayleigh and Quasi-Rayleigh Fading

If general fading occurs so slowly that the changes in and in incoming signal elements in close proximity to each other are strongly intercorrelated, then analysis of the previously received signal elements can with a great degree of reliability predict the expected parameters of the next element. Under these conditions reception is effectuated just as if there were no fading and the same signal systems and the same decision systems that were studied in Chapter III are optimum, but with the difference that the system must be continuously regulated to conform with the expected values of and φ. This is usually accomplished by means of an automatic amplification control[1] and an automatic phase and frequency control.

---

[1] In active-interval systems the optimum decision principle does not depend on (see Chapters III and IV). Therefore such systems do not require automatic volume control in the receiver even during fading and at times use it only to maintain the linearity of the amplitude (volume) circuit.

In many cases automatic phase control results in complexity of equipment with the consequence that incoherent methods of reception not employing information about the expected initial phase of an incoming signal element have come into wide use.

The conditional probability of erroneous reception of a certain signal element in general slow fading (under the supposition that the expected signal parameters to be accounted for by the decision principle are accurately predicted) does not differ from error probability in a channel without fading, as computed for a given instantaneous value of the ratio of the signal element power to spectral noise density h^2. But during fading the value of h^2 changes in proportion to μ^2. Therefore in order to determine the entire probability of erroneous reception of a signal element we must average the conditional probability according to the probability distribution of μ.

Designating the mathematical expectation of variable h^2 by h_0^2 it is evident that

$$h^2 = \mu^2 h_0^2 \qquad (5.7)$$

Let the error probability in a channel without fading be expressed by function $f(h)$. Then the complete probability of error in a channel with slow general fading is defined as

$$p = \overline{p(h)} = \int_0^\infty w(\mu) f(\mu h_0)\, d\mu \qquad (5.8)$$

where $w(\mu)$ is the probability density of the transmission factor which describes the fading.

To illustrate, let us find the error probability in coherent reception of binary signals under conditions of slow Rayleigh fading. Substituting in expression (5.8) the expression for error probability in the absence of fading (5.61) and probability density $w(\mu)$ from expression [illegible] we get

[illegible] (5.9)

where coefficient [illegible] $\leq \sqrt{2}$ depends on the signal system selected.

Integrating by parts, we find [6]

[illegible]

[illegible] (5.10)

$$= \frac{1}{2}\left[1 - \sqrt{[illegible]}\right]$$

When $h_0^2 \gg 1$ formula (5.10) may be replaced by the approximate formula

$$p \approx \frac{1}{\ldots} \tag{5.10a}$$

In the particular case of a binary system with opposed signals (e.g., PM) $\gamma = \sqrt{2}$ and

$$p = \frac{1}{2}\left[1 - \sqrt{\frac{h_0^2}{h_0^2+1}}\right] \approx \frac{1}{4h_0^2}, \tag{5.11}$$

while with orthogonal signals when $\gamma = 1$,

$$p = \frac{1}{2}\left[1 - \sqrt{\frac{h_0^2}{h_0^2+2}}\right] \approx \frac{1}{2h_0^2}. \tag{5.11a}$$

Similarly, for a system of relative phase modulation (RPM) using coherent reception (polarity-comparison method) during Rayleigh fading we find, by substituting the value of f(h) from expression (4.99) into expression (5.8) and integrating by parts, that

$$\begin{aligned} p_{\text{rpm}} &= \int_0^\infty \frac{2\mu}{\mu_0^2}\exp\left(-\frac{\mu^2}{\mu_0^2}\right)\left[1-\Phi^2\left(\sqrt{2}\,\frac{\mu}{\mu_0}h_0\right)\right]d\mu = \\ &= \frac{1}{2} - \frac{2h_0}{\sqrt{\pi}}\int_0^\infty \exp[-x^2(1+h_0^2)]\,\Phi(\sqrt{2}\,h_0x)\,dx = \\ &= \frac{1}{2}\left[1 - \frac{4}{\pi}\sqrt{\frac{h_0^2}{h_0^2+1}}\operatorname{arctg}\sqrt{\frac{h_0^2}{h_0^2+1}}\right]. \end{aligned} \tag{5.12}$$

In the case of RPM signal reception using the method of phase comparison we may, by assuming that fading is so slow that the amplitudes and phases of two adjacent incoming signal elements are practically the same, compute the total probability of error by averaging expression (4.102). Let us solve this problem for quasi-Rayleigh fading. Substituting expressions (4.102) and (5.4) into expression (5.8), we find

$$\begin{aligned} p_{\text{rpm}} &= \int_0^\infty \frac{\mu}{\sigma_1^2}\exp\left(-\frac{\mu^2+\mu_r^2}{2\sigma_1^2}\right)I_0\left(\frac{\mu\mu_r}{\sigma_1^2}\right) \times \\ &\times \frac{1}{2}\exp\left(-\frac{\mu^2}{\mu_0^2}h_0^2\right)d\mu = \\ &= e^{-k^2}\int_0^\infty x\exp\left(-\frac{1+k^2+h_0^2}{1+k^2}x^2\right)I_0(2kx)\,dx, \end{aligned}$$

where

$$x^2 = \frac{\mu^2}{2\sigma_1^2}, \quad k^2 = \frac{\mu_r^2}{2\sigma_1^2}.$$

This integral is tabular. Taking its value into account, we find

$$p_{\text{rpm}} = \frac{1}{2}\,\frac{1+k^2}{1+k^2+h_0^2}\exp\left(-\frac{k^2h_0^2}{1+k^2+h_0^2}\right). \tag{5.13}$$

In the case of Rayleigh fading k = 0 and

$$P_{\text{RPM}} = \frac{1}{2 + 2h_0} . \tag{5.14}$$

When k tends toward infinity, as was to be expected, formula (5.13) converts into formula (4.102):

$$P_{\text{RPM}} = \frac{1}{2} e^{-h_0^2} , \tag{5.15}$$

which expresses the probability of error when there is no fading. Figure 5.7 depicts this relationship for various values of k.

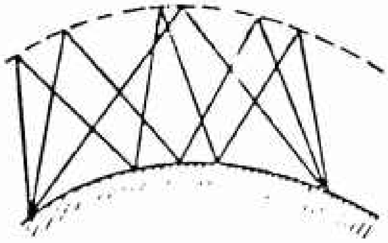


Figure 5.6. Selective Fading in Multiple Reflection.

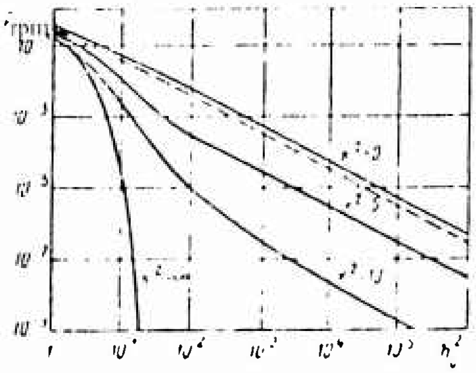


Figure 5.7. Error Probability in RPM In Channel with Fading.
——, incoherent reception;
----, coherent reception.

In incoherent reception the most interesting systems are those which are orthogonal in the intensified sense. For them may be found a complete error probability expression with any code base m in the general case of quasi-Rayleigh fading by starting with expression (4.48):

[illegible]

This integral is reduced by simple transformations to a sum of tabular integrals and finally

[illegible] (5.16)

When k = 0 we obtain, in the case of Rayleigh fading,

$$p = \sum [illegible] \qquad (5.16a)$$

This result may also be expressed by means of gamma-functions

$$p = 1 - [illegible] \qquad (5.16b)$$

If the fluctuating component is lacking (k → ∞) expression (5.16) turns into expression (4.48).

By substituting m = 2 into expression (5.16) we will derive

$$p = [illegible] \qquad (5.17)$$

for binary systems orthogonal in the intensified sense during quasi-Rayleigh fading and incoherent reception, while with Rayleigh fading (k = 0)

$$p = [illegible] \qquad (5.17a)$$

When k → ∞ expression (5.17) turns into expression (4.49). Figure 5.8 shows the relationship of error probability in incoherent reception to h_0.

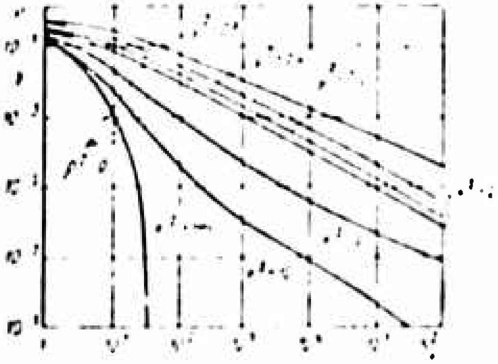


Figure 5.8. Error Probability for Active-Interval Binary Systems in the Case of Incoherent Reception.

In this way it is possible to determine the probability of error in the case of incoherent reception of binary signals with the same level of power if the condition of orthogonality in the intensified sense is not met. We will base ourselves on formula (4.61) which expresses the probability of error with a given value of h in the form of a series. Substituting in it (5.7) and assuming the distribution of [illegible] to be a Rayleigh distribution, we find

$$[illegible] \qquad (5.18)$$

where [illegible] is determined from formulas (4.57).

Thanks to uniform convergence of the series following the integral sign, the series can be integrated term-by-term. Indicating for brevity

$$[illegible]$$

we will express the probability of error by tabular integrals:

$$[illegible]$$

After substituting here the values of a, b, and c and after simple transformations we obtain

$$[illegible] \qquad (5.19)$$

In the particular case when [illegible] = 0 (signals are orthogonal in the intensified sense) formula (5.19) becomes (5.17a). In the other extreme case when [illegible] = 1 (signals differ only in the initial phase), p = 1/2 as should be expected in incoherent reception. If $h_0^2(1 - [illegible]) \gg 1$, then from (5.19) we obtain a convenient approximate expression for the probability of error:

$$p \approx \frac{1}{h_0^2(1 - [illegible])} \qquad (5.19a)$$

Comparing this result with (5.17a), we can assert that small deviations from orthogonality are equivalent to a decrease in signal power of $(1 - [illegible])^{-1}$ times.

The dependence of the probability of error on h_0^2 with various values of ρ^2 for Rayleigh fading ($k^2 = 0$) is shown in Figure 5.8.

An analysis of the results obtained shows that fading, especially Rayleigh fading, increases the probability of error greatly. The dependence of the probability of error on h^2 in the case of Rayleigh fading is in all cases close to inversely proportional, in distinction from a channel in which there is no fading. Therefore, to obtain a sufficiently high level of fidelity in reception in a channel with Rayleigh fading there must be a much higher ratio between average element power in an incoming signal and the spectral density of white noise than in the absence of fading. Quasi-Rayleigh fading is an intermediate case between the absence of fading and Rayleigh fading. We will note that the power gain does not exceed 3 db when coherent reception is used in the presence of general fading.

Reception with Unknown Values of μ and θ

The need to continuously measure the values of the channel parameters μ and θ greatly complicates a receiving device. As already indicated, for active-interval systems the optimal decision system does not depend on values of μ and in the case of incoherent reception knowledge of θ is not needed and, consequently, there is no longer any need to measure these parameters. However, it is possible to deduce the decision principle in the general case when the powers of the received signals are not the same, assuming that the values of μ and θ are unknown and only their probability distributions are known.

Based on the criterion of maximal likelihood we should adopt the hypothesis that signals $z_l(t)$ was transmitted if the conditional probability of arrival of signal $z'(t)w(z'|z_l)$ is greater than the conditional probability of $w(z'|z_r)$ for all $r \neq l$. Just as we determined the optimum incoherent decision principle in Chapter IV by using $w(z'|z_r)$, by averaging with respect to μ and with respect to θ, we are able to determine the conditional density $w(z'|z_r,\mu,\theta)$. If μ and θ are known, then according to (3.19) and (4.23)

$$w(z'|z_r,\mu,\theta) = \frac{1}{(2\pi)^N\sigma_0^N}\exp\left\{-\frac{1}{2\sigma_0^2}\sum_{k=1}^{N}[A_k-\mu(a_{rk}\cos\theta+b_{rk}\sin\theta)]^2+[B_k-\mu(b_{rk}\cos\theta-a_{rk}\sin\theta)]^2\right\}. \tag{5.20}$$

Averaging (5.20) with respect to μ and with respect to θ, we obtain

$$w(z'|z_r) = \int_0^\infty\int_0^{2\pi}\frac{1}{(2\pi)^N\sigma_0^N}\exp\left\{-\frac{1}{2\sigma_0^2}\sum_{k=1}^{N}[A_k-\mu(a_{rk}\cos\theta+b_{rk}\sin\theta)]^2+[B_k-\mu(b_{rk}\cos\theta-a_{rk}\sin\theta)]^2\right\}w(\theta)w(\mu)\,d\theta\,d\mu. \tag{5.21}$$

Substituting in (5.21) the values of the probability densities of $\omega(\vartheta)$ and $\omega(\mu)$, we can find $\omega(z'|z_r)$ for a given specific type of fading.

In the case of Rayleigh fading, by substituting (5.3) and considering that $\omega(\vartheta) = 1/2\pi$, after simple transformations we find [7]

$$w(z'|z_r) = \frac{1}{(2\pi\sigma_0)^k (h_{r0}^2+1)} \exp\left[\frac{V_r^2}{\nu_0^2 (h_{r0}^2+1)} - \frac{P_{\text{пр}}}{\nu_0^2}\right], \tag{5.22}$$

where similar to the notation introduced earlier, $h_{r0}^2 = \frac{\mu_0^2}{\nu_0}\int_0^T z_r^2\,dt$ is the ratio between the average value of the power of the incoming signal $z_r(t)$ and the interference spectral density; and

$$V_r = \frac{2\mu_0}{T}\left\{\left[\int_0^T z'(t) z_r(t)\,dt\right]^2 + \left[\int_0^T z'(t)\tilde{z}_r(t)\,dt\right]^2\right\}^{1/2}.$$

$$P_{\text{пр}} = \frac{1}{T}\int_0^T z'^2(t)\,dt.$$

It follows from (5.22) that the decision principle is: symbol y_l is registered if for all $r \neq l$

$$\frac{V_l^2}{h_{l0}^2+1} - \nu_0^2 \ln(h_{l0}^2+1) > \frac{V_r^2}{h_{r0}^2+1} - \nu_0^2 \ln(h_{r0}^2+1) \tag{5.23}$$

This principle may be realized using such systems as that expressed by the principle contained in (4.28) which was obtained for a channel without fading, the only change being in the functional transformations and the magnitudes of the thresholds. For active-interval systems (5.23) reduces to the inequality

$$V_l > V_r \tag{5.24}$$

which coincides with (4.30). Therefore, for active-interval systems the decision circuits under conditions of slow Rayleigh fading are the same as in the absence of fading.

The probability of error in the case of such reception principles is computed as the probability of nonfulfillment of inequalities (5.23) or (5.24) if signal $z_l(t)$. We will compute this probability for an active-interval system, assuming that all signals $z_1(t),\ldots,z_m(t)$ are mutually orthogonal in the intensified sense.

We can easily see that for all Rayleigh fading all values of V_r have a Rayleigh probability distribution

$$\begin{cases} w(V_l) = \dfrac{2V_l}{\sigma_l}\exp\left(-\dfrac{V_l^2}{\sigma_l}\right) & \text{when } V_l > 0, \\ w(V_l) = 0 & \text{when } V_l < 0, \end{cases} \tag{5.25}$$

where

$$\sigma_l = 2\mu_l^2 P_s + 2P_N;$$

$$\sigma_r = 2\mu_r^2 P_N \quad \text{when } r \neq l,$$

$P_s = \dfrac{\mu_0^2}{T}\int_0^T s_l^2(t)\,dt$ is the average power of the signal being received.

The probability of error is

$$p = 1 - P(V_l > V_r),\quad r = 1,\ldots,m,\ r \neq l$$

$$= 1 - \int_0^\infty \frac{2V_l}{\sigma_l}\exp\left(-\frac{V_l^2}{\sigma_l}\right)\left[\int_0^{V_l}\frac{2V_r}{\sigma_r}\exp\left(-\frac{V_r^2}{\sigma_r}\right)dV_r\right]^{m-1} dV_l$$

$$= 1 - \int_0^\infty \frac{2V_l}{\sigma_l}\exp\left(-\frac{V_l^2}{\sigma_l}\right)\left[1 - \exp\left(-\frac{V_l^2}{\sigma_r}\right)\right]^{m-1} dV_l$$

Successive integration by parts leads to the result:

$$p = 1 - \frac{(m-1)!}{\left(\dfrac{\sigma_r}{\sigma_l}+1\right)\left(\dfrac{\sigma_r}{\sigma_l}+2\right)\cdots\left(\dfrac{\sigma_r}{\sigma_l}+m-1\right)} = 1 - \frac{\Gamma(m)\,\Gamma\left(\dfrac{\sigma_r}{\sigma_l}+1\right)}{\Gamma\left(\dfrac{\sigma_r}{\sigma_l}+m\right)}.$$

Noting that $\dfrac{\sigma_r}{\sigma_l} = \dfrac{1}{1+h_0^2}$, we finally obtain

$$p = 1 - \frac{\Gamma(m)\,\Gamma\left(1+\dfrac{1}{1+h^2}\right)}{\Gamma\left(m+\dfrac{1}{1+h^2}\right)}, \tag{5.26}$$

which coincides with previously obtained formula (5.16b).

Such a result should be expected inasmuch as in the case of incoherent reception of signals with an active interval the values of channel parameters do not affect the decision system. The case is otherwise with signals having different power levels when the decision principle expressed in (4.28), which was obtained under the assumption that the transmission coefficient is known, differs greatly from the principle expressed in (5.25) which was deduced for a random transmission coefficient μ, about which is known only that it has a Rayleigh probability distribution and a mean-square value equal to μ_0.

We will explain this by citing the case of a binary AT system. There the optimal incoherent decision principle (4.50) in the case of a known transmission coefficient leads to a circuit with a threshold device in which the threshold level depends on the magnitude of . . Application of this rule in a channel with slow fading presupposes continuous adjustment of the threshold level based on measured values of . . The probability of error is determined by averaging the dependence depicted in Figure 4.12 with respect to . . This averaging can be done by the method of numerical integration and the curves shown in Figure 5.9 were drawn on this basis (curve 1).

If the transmission coefficient is considered unknown, then based on (5.23) the decision principle (registration of symbol v_1 corresponding to "send") is obtained in the following form:

$$V_1 > [illegible] \qquad (5.27)$$

Here the right side represents an unregulated threshold level the magnitude of which is determined by the average (and not instantaneous) power of the signal and the interference spectral density. With such a decision principle the probability of error in Rayleigh fading may be expressed in final form and is equal to

$$p = \frac{1}{2}\left[[illegible] \right] \qquad (5.28)$$

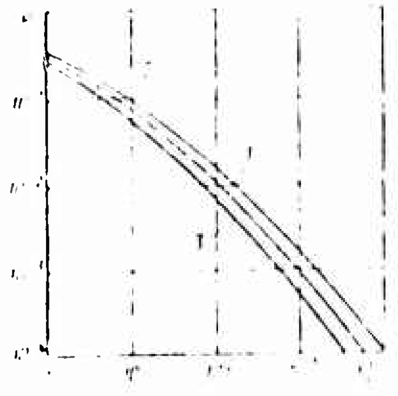


Figure 5.9. Probability of Error for a Binary AT System in the Case of Rayleigh Fading: 1, Known values of (incoherent reception); 2, With unknown values of and (incoherent reception); 3, With known and (coherent reception).

This dependence is also expressed in Figure 5.9 (curve 2). Thus, the absence of information about the instantaneous value of the transmission coefficient increases the probability of error somewhat. Curve 3 in the same figure is drawn from formula (5.1[illegible]) for coherent reception, i.e., under the assumption that the instantaneous values of as well as of * are known.

We will also consider a binary system with an active interval in the case of nonorthogonal signals. Since for active-delay systems, knowledge of the transmission coefficient .. does not affect the decision system, it is completely obvious that with unknown values of .. the probability of error is expressed by formula (5.19). Nevertheless, we will present the deduction of this formula based on decision principle (5.24), inasmuch as this example will be used to demonstrate several methods of computation useful in the investigation of more complex cases.

First of all we will rewrite principle (5.24) as applied to a binary system in the following way: symbol y_1 is registered in that case when

$$X_1^2 + Y_1^2 - X_2^2 - Y_2^2 > 0, \tag{5.29}$$

where as formerly

$$X_1 = \frac{2\mu_0}{T}\int_0^T z'(t)\, z_1(t)\, dt,$$

$$Y_1 = \frac{2\mu_0}{T}\int_0^T z'(t)\, \tilde{z}_1(t)\, dt,$$

$$X_2 = \frac{2\mu_0}{T}\int_0^T z'(t)\, z_2(t)\, dt,$$

$$Y_2 = \frac{2\mu_0}{T}\int_0^T z'(t)\, \tilde{z}_2(t)\, dt,$$

represent normally distributed random variables. The random variable

$$\xi = X_1^2 + Y_1^2 - X_2^2 - Y_2^2 \tag{5.30}$$

represents the quadratic form of normally distributed variables.

The probability of error can be computed as the probability of nonfulfillment of inequality (5.29) when signal $z_1(t)$ is transmitted

$$p = P\{\xi < 0 \mid z'(t) = \mu_c z_1(t) + \mu_s \tilde{z}_1(t) + n(t)\}. \tag{5.31}$$

For this purpose we will find the characteristic function of variable ξ from which it is possible to determine its density and compute its probability (5.31). In order to use this method in other cases we will demonstrate how to find the characteristic function of random variable ξ expressed in quadratic form of an arbitrary number of normally distributed variables $x_1, x_2, \ldots, x_{2n}$ with zero mathematical expectation:

$$\xi = \sum_{k=1}^{2n}\sum_{p=1}^{2n} a_{kp} x_k x_p, \qquad k, p = 1, \ldots, 2n. \tag{5.32}$$

where $a_{kp} = a_{pk}$ are actual constants defining quadratic forms.

An aggregate of normal variables with zero mathematical expectation is unambiguously described by a 2n × 2n correlation matrix

$$\mathbf{K} = (\overline{x_k x_p}), \qquad k, p = 1, \ldots, 2n, \tag{5.33}$$

where the horizontal line indicates statistical averaging.

It can be shown [8] that under these conditions the characteristic function $\theta(v)$ of random variable ξ is equal to (see Note 4 to this chapter)

$$\theta(v) = \frac{1}{\prod\limits_{k=1}^{n}(1-2iv\lambda_k)^{1/2}}, \tag{5.34}$$

where λ_k represent the eigenvalues of matrix KA, i.e., the roots of the equation

$$|\mathbf{KA} - \lambda \mathbf{I}| = 0, \tag{5.35}$$

A is a 2n × 2n square matrix (5.32);

$$\mathbf{A} = (a_{cr}).$$

I is a 2n × 2n matrix.

Solving equation (5.35) we find values of λ_k and determine the characteristic function $\theta(v)$ by knowing which we are able to find the distribution density of the random variable:

$$\begin{aligned} w(\xi) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\theta(v)\exp(-iv\xi)\,dv \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\exp(-iv\xi)}{\prod\limits_{k=1}^{n}(1-2iv\lambda_k)^{1/2}}\,dv. \end{aligned} \tag{5.37}$$

The integral obtained usually (although not always) can be computed using the residue method.

We will use the method described to find the distribution density of random variable ξ (5.30) [6]. Assuming

$$z'(t) = \mu_c z_1(t) + \mu_s \tilde{z}_1(t) + n(t),$$

where μ_c and μ_s in the case of Rayleigh fading are normally distributed variables with zero mathematical expectation, it is easy to see that X_1, Y_1, X_2, and Y_2 also have a normal distribution and zero mathematical expectation. It is easy to find their correlation matrix:

$$K = \begin{bmatrix} h_{\cdot}+1 & 0 & p_1(h'+1) & p_2(h'+1) \\ 0 & h+1 & p_2(h'+1) & p_1(h'+1) \\ p_1(h'+1) & p_2(h'+1) & p'h'+1 & 0 \\ p_2(h'+1) & p_1(h'+1) & 0 & p'h'+1 \end{bmatrix} \frac{\mu' \cdot r}{l}, \qquad (5.38)$$

where [illegible]$_1$ and [illegible]$_2$ are determined by formulas (4.57).

The square matrix is

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \qquad (5.39)$$

We will represent matrix K in the form

$$K = \begin{bmatrix} K_1 & K_2 \\ \dot{K}_2 & K_3 \end{bmatrix} \frac{\mu' r}{l},$$

where

$$K_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (h'_0+1), \quad K_2 = \begin{bmatrix} \beta_1 & \beta_2 \\ \beta_2 & \beta_1 \end{bmatrix} (h'_0+1),$$

$$K_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (h'_0+1).$$

$\dot{K}_2$ is the transpose of K_2.

Then

$$KA = \begin{bmatrix} K_1 & K_2 \\ \dot{K}_2 & K_3 \end{bmatrix} \frac{1}{c},$$

where for brevity we [illegible] 2 for the constant factor $\frac{\mu' \omega r_s}{l}$, which has no great significance for what follows.

Now equation (5.37) assumes the form

$$\left\| \begin{bmatrix} K_1 - \omega I & K_2 \\ \dot{K}_2 & K_3 - \omega I \end{bmatrix} \right\| = 0,$$

where I is a 2 × 2 singular matrix, or

$$|(K_1 - \omega I)(K_3 - \omega I) - K_2\dot{K}_2| = 0$$

Performing multiplication of matrices and substituting in their values we obtain the equation

$$\{[-(h_*^2+1)(h_*^2\rho^2+1)-c\lambda(h_*^2+1)+c\lambda(h_*^2\rho^2+1)+$$
$$+c^2\lambda^2+\rho^2(h_*^2+1)^2]\}=0$$

or

$$[c^2\lambda^2-c\lambda h_*^2(1-\rho^2)-(h_*^2+1)(1-\rho^2)]^2=0. \tag{5.40}$$

Solving this equation, we find its four roots:

$$\lambda_{1,2}=\frac{1}{c}\left[\frac{h_*^2(1-\rho^2)}{2}\pm\sqrt{\frac{h_*^4(1-\rho^2)^2}{4}+(h_*^2+1)(1-\rho^2)}\right],\quad \lambda_3=\lambda_1;\ \lambda_4=\lambda_2 \tag{5.41}$$

Inasmuch as the roots are multiple, the characteristic function (5.34) of variable ξ is

$$\theta(z)=\frac{1}{(1-2iz\lambda_1)(1-2iz\lambda_2)}$$

and its distribution density (5.37)

$$w(\xi)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\exp(-iz\xi)}{(1-2iz\lambda_1)(1-2iz\lambda_2)}dz \tag{5.42}$$

To compute the probability of error we are interested only in the value of $w(\xi)$ in the case when $\xi\le 0$. Considering that when $\xi<0$ and Im $z>0$ the integrand when $|z|\to\infty$ approaches zero, and by using the Jordan lemma, we may assert that this integral is equal to the sum of the residues of the integrand at the poles of the upper semi-plane multiplied by $2\pi i$. In the given case the integrand has two poles:

$$z_1=\frac{1}{2i\lambda_1}\quad\text{and}\quad z_2=\frac{1}{2i\lambda_2}.$$

But, as is apparent from (5.41), $\lambda_1>0$ and $\lambda_2<0$. Therefore, only pole z_2 lies in the upper semi-plane of the complex plane and the residue at this point is equal to

$$\frac{\exp(\xi/2\lambda_2)}{2\pi i\lambda_2(1-\lambda_1/\lambda_2)}.$$

Thus, when $\xi<0$

$$w(\xi)=-\frac{\exp\left(\frac{\xi}{2\lambda_2}\right)}{2(\lambda_2-\lambda_1)} \tag{5.43}$$

It follows that the probability of error is

$$p=\int_{-\infty}^{0}w(\xi)d\xi=\frac{1}{2(\lambda_1-\lambda_2)}\int_{-\infty}^{0}\exp\left(\frac{\xi}{2\lambda_2}\right)d\xi=\frac{\lambda_2}{\lambda_2-\lambda_1}. \tag{5.44}$$

Substituting the values of [illegible] from (5.41), we obtain

$$r\ [illegible]$$

which coincides with (5.19).

We will not take up in detail the reception of signals with unknown values of μ and θ in the case of quasi-Rayleigh fading. In work [9] a decision principle is obtained under the assumption that the regular component μ_r of the transmission coefficient and the phase of the regular component θ_r are known. The ratio k^2 between the powers of the regular and fluctuating components is also assumed to be known. A simpler decision principle is obtained there under the assumption that all values of phase shift θ are equiprobable.

For active-interval systems this principle is greatly simplified and reduces to the condition

$$[illegible] \tag{5.45}$$

which coincides with the optimal incoherent principle for channels without fading and with Rayleigh fading. In this case the probability of error in the case of orthogonal signals coincides with (5.16).

We will point out that the decision principles described in this section are optimal also for that hypothetical channel in which the transmission coefficient μ changes by jumps when there is a shift in signal element and remains constant for the length of an element. Although such channels do not actually exist, the idea is suitable for use in Section 5.4.

Reception with an Unknown Fading Law

In actual practice the distribution of probabilities of a transmission coefficient in a fading channel is often unknown. In some cases it may differ greatly from the usual and generalized Rayleigh distribution. If fading is very slow and it is possible to measure μ and θ continuously and with sufficient accuracy, lack of knowledge of the law of fading has no effect on the structure of the decision system. However, as already noted, the need to evaluate μ and θ greatly complicates a receiving device. Therefore, obtaining a decision principle for that case when the values of μ and θ are not known, and even their probability distributions are not known, is of interest.

One possible way to construct such a principle is to use the generalized criterion of maximal likelihood [10]. For the case of fading at a zero rate we will consider that in the reception of a certain signal element μ and θ are constant (not random) but unknown parameters. According to the generalized criterion of maximal likelihood [11] of several hypotheses, that one is selected for which the maximum of the likelihood function is greater than for all remaining hypotheses and the maximum pertains to all unknown parameters.

In the given case the likelihood function of signal $z_r(t)$ for several values of μ and φ is expressed by formula (5.20). For simplification of the following derivation we will switch from parameters μ and φ to $\mu_c = \mu\cos\varphi$ and $\mu_s = \mu\sin\varphi$. Then

$$w(z'|z_r) = N\exp\left\{-\frac{1}{2\sigma_0^2}\sum_{k=1}^{K}\left[(A_k - \mu_c a_{rk} - \mu_s b_{rk})^2 + (B_k - \mu_c b_{rk} + \mu_s a_{rk})^2\right]\right\}, \tag{5.46}$$

where N is a certain constant not dependent on z_r, μ_c, and μ_s.

Instead of seeking a maximum for this function, we will find the maximum of its logarithm

$$\ln w(z'|z_r) = \ln N - \frac{1}{2\sigma_0^2}\sum_{k=1}^{K}\left[(A_k - \mu_c a_{rk} - \mu_s b_{rk})^2 + (B_k - \mu_c b_{rk} + \mu_s a_{rk})^2\right]. \tag{5.47}$$

For this purpose we will equate the partial derivatives (5.47) with respect to μ_c and μ_s to zero, as a result of which we obtain the system of equations

$$\sum_{k=1}^{K} a_{rk}(A_k - \mu_c a_{rk} - \mu_s b_{rk}) + \sum_{k=1}^{K} b_{rk}(B_k - \mu_c b_{rk} + \mu_s a_{rk}) = 0,$$

$$\sum_{k=1}^{K} b_{rk}(A_k - \mu_c a_{rk} - \mu_s b_{rk}) - \sum_{k=1}^{K} a_{rk}(B_k - \mu_c b_{rk} + \mu_s a_{rk}) = 0,$$

by solving which with respect to μ_c and μ_s we find the values of these parameters which determine the maximum of the likelihood function for z_r:

$$\left.\begin{aligned}\mu_c &= \frac{\sum_{k=1}^{K}(A_k a_{rk} + B_k b_{rk})}{\sum_{k=1}^{K}(a_{rk}^2 + b_{rk}^2)},\\ \mu_s &= \frac{\sum_{k=1}^{K}(B_k a_{rk} - A_k b_{rk})}{\sum_{k=1}^{K}(a_{rk}^2 + b_{rk}^2)}.\end{aligned}\right\} \tag{5.48}$$

Substituting (5.48) in (5.47) we find

$$\ln \Lambda [illegible] \quad (5.49)$$

$$\ln [illegible]$$

Signal z_1 thus has the greatest maximum of likelihood function if

$$\frac{V_1}{P_1} \geq \frac{V_i}{P_i} \quad (i = 1, \ldots, m), \quad (5.50)$$

which is the rule for registering symbol z_1 based on the criterion of maximal likelihood when the fading law is unknown.

For active-interval systems when $P_1 = P_r$ this principle, as might be expected, coincides with (5.24) [10] and does not depend on whether the fading law is known. As far as the probability of error when using a decision system based on (5.24) is concerned, it, of course, depends on the probability distribution of μ.

5.3. Memory in a Channel with Slow Fading. Several Problems in Coding.

In the preceding section the probabilities of incorrect reception of a signal element were computed for different communication systems when fading occurs. A comparison of these results with those obtained in Chapters III and IV (no fading) shows that the probability of error in a fading channel (especially in the case of Rayleigh fading) greatly exceeds the probability of fading in a channel without fading when the ratio between average power of a signal and interference spectral density remains the same. For example, in order to provide a probability of error on the order of 10^{-4} in the case of incoherent reception in a binary orthogonal system with an active interval, it is sufficient to have $h^2 \approx 18$ in a channel without fading and $h^2 \approx 10,000$ in a channel with Rayleigh fading

However, it should not be thought that a channel without fading in the case where $h^2 = 10,000$ will be equivalent in fidelity of reception or in carrying capacity. The fact is that a channel without fading when the interference is approximated by white noise is (in the discrete representation) a channel without memory while a channel with slow fading is a channel with memory and this memory encompasses a larger number of elements, the slower is the fading.

Thus, the probability of error computed for a channel without fading does not change if it is known how the preceding signal elements were received. In a channel with slow fading the probabilities of error computed above are only unconditional probabilities which may differ greatly from the conditional probabilities of error if the result of reception of one or several preceding elements is given.

By way of example we will compute the conditional probability of error in incoherent reception of a signal element in a binary orthogonal active interval system, considering only the result of reception of one preceding signal element. In so doing the fading will be considered Rayleigh and so slow that the values of μ in adjacent signal elements will be practically the same.

We will assume that a preceding signal element is received correctly. Then, obviously, the conditional probability of incorrect reception of the next element is

$$p(\text{incor}|\text{cor}) = \int_0^\infty p(\text{incor}|\mu)\, w(\mu|\text{cor})\, d\mu ,$$

where $p(\text{incor}|\mu)$ is the conditional probability of error for a certain value of μ, and $w(\mu|\text{cor})$ is the conditional density of probability of μ if the preceding element was received correctly.

To find $w(\mu|\text{cor})$ we use the Bayes formula

$$w(\mu|\text{cor}) = \frac{w(\mu)p(\text{cor}|\mu)}{p(\text{cor})} \tag{5.51}$$

where

$$p(\text{cor}|\mu) = 1 - \frac{1}{2}\exp\left(-\frac{\mu^2}{2\bar{\mu}^2}h_0^2\right)$$

is the probability of correct reception with a given value of μ; and

$$p(\text{cor}) = 1 - \frac{1}{h_0^2+2}$$

is the unconditional probability of correct reception.

Substituting $w(\mu)$ from (5.3), we obtain

$$w(\mu|\text{cor}) = 2\frac{h_0^2+2}{h_0^2+1}\frac{\mu}{\bar{\mu}^2}\exp\left(-\frac{\mu^2}{\bar{\mu}^2}\right)\left[1 - \frac{1}{2}\exp\left(-\frac{\mu^2}{2\bar{\mu}^2}h_0^2\right)\right]. \tag{5.52}$$

Whence, considering that

$$p(\text{incor}|\mu) = \frac{1}{2}\exp\left(-\frac{\mu^2}{2\bar{\mu}^2}h_0^2\right),$$

we obtain

$$p(\text{incor}|\text{cor}) = \frac{h_0^2+2}{h_0^2+1}\int_0^\infty \frac{\mu}{\bar{\mu}^2}\left\{\exp\left[-\frac{\mu^2}{\bar{\mu}^2}\left(1+\frac{h_0^2}{2}\right)\right] - \frac{1}{2}\exp\left[-\frac{\mu^2}{\bar{\mu}^2}(1+h_0^2)\right]\right\}d\mu = \frac{1}{h_0^2+1} - \tag{5.53}$$

$$= \frac{h_0^2+2}{4(h_0^2+1)^2} = \frac{3h_0^2+2}{4(h_0^2+1)}.$$

When $h_0^2 \gg 1$ this conditional probability of error is 3/4 of the unconditional probability of error.

Similarly, we can compute the conditional probability of incorrect reception of an element if a preceding element is received incorrectly.

$$p(\text{incor}|\text{incor}) = \frac{h_0^2+2}{4(h_0^2+1)}. \qquad (5.54)$$

With an increase in h_0^2 from zero to infinity the conditional probability p(incor|incor) decreases from 0.5 only to 0.25. Therefore, even with a very great preponderance in signal over interference the probability of error is great if a preceding element is received incorrectly. Consequently, in such a channel errors will with great probability be grouped. With an increase in signal power, these flashes or packets of errors occur more rarely but the average duration of a packet changes very slowly and depends mainly on the average period of fading.

As already noted in Chapter II, this circumstance must be taken into account in the selection of a correcting code. Such a code in a channel with slow fading must permit detection or correction of a packet of errors which are longer, the slower is the fading in the channel. One of the simplest methods (although far from optimal) is construction of a code providing for decorrelation of errors, i.e., such an arrangement of symbols entering into a general parity check that they are separated by time into intervals which exceed the time of correlation of fading.

We will consider by way of example conditions under which use is made of the simplest binary correcting codes, i.e., the three-element code of (3.1) which permits correcting one error. Such a code amounts to repeating every signal element three times and registering the one that occurs two out of three times. In this case uncorrected errors will occur in that case when at least two elements are received incorrectly.

We will assume that such a code is used without decorrelation of errors, i.e., all three elements of a code combination are transmitted one after the other. We will consider the fading to be so slow that μ changes practically not at all over the length of three elements. Let the coefficient of transmission μ adopt a certain value. Then the conditional probability of incorrect reception of an element (incoherent reception of signals which are orthogonal in the intensified sense is assumed) is equal to

$$p = p(\text{incor}|\mu) = \frac{1}{2}\exp\left(-\frac{\mu^2}{\mu_1^2}\frac{h_0^2}{2}\right).$$

The conditional probability of an uncorrected error in a combination consisting of three elements is

$$\begin{aligned} p(\mathrm{incor}|\mu) &= p^3 + 3p^2(1-p) = 3p^2 - 2p^3 = \\ &= \frac{3}{2}\exp\left(-\frac{\mu^2}{\mu_0^2}h_0^2\right) - \exp\left(-\frac{3}{2}\frac{\mu^2}{\mu_0^2}h_0^2\right). \end{aligned} \tag{5.55}$$

Averaging this expression with respect to μ, we obtain for Rayleigh fading

$$p(\mathrm{incor}) = \frac{5h_0^2 + 2}{(2h_0^2 + 2)(3h_0^2 + 2)}. \tag{5.56}$$

When $h_0^2 \gg 1$, this probability is approximately equal to $5/6h_0^2$, i.e., only 17% less than the average probability of error in the case of primary coding. If we consider a code combination as one element of duration three times as long, then h_0^2 increases by three times and the probability of error will be approximately equal to $1/3h_0^2$, i.e., to 2.5 times less than $5/6h_0^2$. Refraining from interference-resistant coding and maintaining the rate of transmission, it is possible to triple the length of a signal element. Thus, the use of a code (3.1) without decorrelation in a channel with slow fading does not increase but decreases fidelity of reception.

In the case where we use the same code but with decorrelation of errors, the probabilities of incorrect reception of elements entering into a code combination are independent and are determined by expression (5.17a). Then the probability of an uncorrected error is

$$\begin{aligned} p(\mathrm{incor}) &= 3p^2 - 2p^3 = \\ &= \frac{3}{(h_0^2 + 2)^2} - \frac{2}{(h_0^2 + 2)^3} = \frac{3h_0^2 + 4}{(h_0^2 + 2)^3}, \end{aligned} \tag{5.57}$$

in the case where $h_0^2 \gg 1$

$$p(\mathrm{incor}) \approx 3/h_0^4,$$

i.e., it is significantly less than $1/3h_0^2$.

Thus, in the case of decorrelation of errors in a channel with slow Rayleigh fading code (3.1) yields a significant increase in fidelity of reception.

In the more general case it is necessary to compute the probability that with reception of an n-element code combination there will be k errors ($k \leq n$) [12]. Specifically, in many cases the probability of error-free reception of an n-element code combination ($k = 0$) is of interest.

We will assume that the fading is Rayleigh in nature and so slow that the transmission coefficient μ changes practically not at all over the length of n elements of a code combination. The probability of correct reception $q_n(\mu)$

of an n-element code combination with a given value of μ for a binary system with an active-interval orthogonal in an intensified sense is equal to

$$q_n(\mu) = [1 - p(\mu)]^n, \tag{5.58}$$

where $p(\mu)$ is the probability of error with a fixed value of μ

$$p = \frac{1}{2}\exp\left(-\frac{\mu^2 [illegible]}{2 [illegible]}\right)$$

Averaging (5.58) with respect to μ we find the probability q_n of correct reception of the entire n-element combination with slow Rayleigh fading:

$$q_n = \int_0^\infty w(\mu)\, q_n(\mu)\, d\mu =$$

$$= \int_0^\infty \frac{[illegible]}{\mu} \exp\left(-\frac{\mu^2}{\mu [illegible]}\right)\left[1 - \frac{1}{2}\exp\left([illegible] \right)\right]^n d\mu = \tag{5.59}$$

$$= \sum [illegible]$$

Formula (5.59) may be represented in a form more convenient for computation using an incomplete beta-function:

$$q_n = [illegible]\, 2^n\, B [illegible] \left([illegible],\ n + 1 \right) \tag{5.59a}$$

A recurrent formula easily obtainable from (5.59) may also be useful:

$$q_{n+1} = \frac{1 + 2^n [illegible]}{2^n \left(1 + n [illegible]\right)} \tag{5.59b}$$

With an increase in the length of code combination n the probability of error-free reception q_n, of course, decreases. However, it decreases much more slowly than in a channel with independent errors.

Using formulas (5.59), it is possible, for example, to calculate that when $h_0^2 = 20$ and when the unconditional probability of error is equal to $1/h_0^2 + 2 \approx 4.5\cdot10^{-2}$, the probability of error-free reception of a code combination ten symbols in length is equal to $q_{10} \approx 0.79$. We will note for comparison that in a channel without fading, i.e., with independent errors where $p = 4.5\cdot10^{-2}$, the probability of correctly receiving a ten-element code combination is equal to $q_{10} = (1 - p)^{10} = 0.955^{10} \approx 0.63$. It is true that for this it is sufficient to have $h^2 \approx 4.82$. Nevertheless, it is apparent from this that if we compare channels with respect to probability of correct reception of relatively long code combinations the existence of slow fading does not impair the quality of a channel to the same degree as when comparing a single symbol with respect to probability of correct reception.

We will present one more example to confirm this. As already noted, a probability of error-free reception of a symbol of $p = 10^{-4}$ in the case of an orthogonal active-interval system and incoherent reception is provided in a channel without fading if $h^2 \approx 18$. Under these conditions the probability of incorrect reception of a code combination from among 100 symbols is approximately equal to 10^{-2}. In order to provide such a probability of error-free reception of one symbol in the case of Rayleigh fading it is necessary to have $h^2 \approx 10^4$, i.e., the average power of a transmitted symbol must be increased by approximately 550 times. If it is necessary to provide only for a probability of error free reception of a 100-digit code combination equal to 10^{-2}, $h_0^2 \approx 900$ is sufficient, i.e., the power of a transmitted signal must be increased only by 50 times to compensate for the effects of fading. Understandably it is assumed that the coefficient of transmission changes practically not at all over the length of 100 symbols.

5.4. Effect of Rate of Fading on Probability of Error

In this section we will, as formerly, assume that fading is slow in the sense that in formula (5.6a) or (5.6b) τ_k greatly exceeds the length of a signal element T. However, we will not consider this preponderance so great that μ_c and μ_s change practically not at all over T. In this case the decision principle deduced for the absence of fading or for fading at a zero rate will, generally speaking, be no longer optimal. Nevertheless, with respect to slow fading it can be assumed that these decision principles remain sufficiently close to optimal. Therefore, we will assume that incoherent reception occurs in accordance with the rule determined for fading at a zero rate which, in the case of a system with an active interval coincides with the optimal for a channel without fading, and we will try, if only approximately, to evaluate how much the probability of error changes if it is considered that the magnitude of μ over time T changes within small limits.

And so, let the signal element being received be

$$z'(t) = \mu_c(t)\, z_r(t) + \mu_s(t)\, \tilde{z}_r(t) + n(t), \quad 0 \le t \le T,\ r = 1, \ldots, m \qquad (5.60)$$

where $z_r(t)$ is a signal element which has been transmitted, and $\mu_c(t)$ and $\mu_s(t)$ are slowly changing functions which are realizations of two conjugate gaussian processes.

We will designate the average value over time T of $\mu_c(t)$ and $\mu_s(t)$ by $\langle\mu_c\rangle$ and $\langle\mu_s\rangle$

$$\left.\begin{aligned} \langle\mu_c\rangle &= \frac{1}{T}\int_0^T \mu_c(t)\,dt \\ \langle\mu_s\rangle &= \frac{1}{T}\int_0^T \mu_s(t)\,dt \end{aligned}\right| \qquad (5.61)$$

It is obvious that symbols $\bar{\mu}_s$ and $\bar{\mu}_s$ are normal random variables inasmuch as they are obtained as a result of a linear operation of integrating gaussian processes. We will determine their mathematical expectations and dispersions:

$$\left.\begin{aligned} \langle \bar{\mu}_s \rangle &= \frac{1}{T}\int_0^T \langle \mu_s(t) \rangle\, dt = 0, \\ \langle \bar{\mu}_c \rangle &= \frac{1}{T}\int_0^T \langle \mu_c(t) \rangle\, dt = \mu_0. \end{aligned}\right\} \tag{5.62}$$

$$\begin{aligned} \langle \bar{\mu}_s^2 \rangle &= \frac{1}{T^2}\int_0^T\int_0^T \langle \mu_s(t_1)\mu_s(t_2) \rangle\, dt_1\, dt_2 \\ &= \frac{1}{T^2}\int_0^T\int_0^T \overline{\mu_s^2}\, R(t_2 - t_1)\, dt_1\, dt_2 \\ &= \frac{\overline{\mu_s^2}}{2T^2}\int_0^T\int_{-t_1}^{T-t_1} R(\tau)\, dt_1\, d\tau \\ &= \frac{\overline{\mu_s^2}}{2T^2}\left\{\int_0^T\int_0^{T-\tau} R(\tau)\, d\tau\, dt_1 + \int_{-T}^0\int_{-\tau}^{T} R(\tau)\, d\tau\, dt_1\right\} \\ &= \frac{\overline{\mu_s^2}}{2T^2}\left\{\int_0^T R(\tau)(T-\tau)\, d\tau \right. \\ &\quad \left. + \int_{-T}^0 R(\tau)(T+\tau)\, d\tau\right\} = \frac{\overline{\mu_s^2}}{T}\int_0^T \left(1 - \frac{\tau}{T}\right) R(\tau)\, d\tau \end{aligned} \tag{5.63}$$

Here the exchange of variables and consideration of parity of correlation coefficient are done as when deducing equation (4.90).

Similarly,

$$\langle (\bar{\mu}_c - \mu_0)^2 \rangle = \frac{\overline{\mu_c^2}}{T}\int_0^T \left(1 - \frac{\tau}{T}\right) R(\tau)\, d\tau. \tag{5.63a}$$

$$\begin{aligned} \langle (\bar{\mu}_c - \mu_0)\bar{\mu}_s \rangle &= \frac{1}{T^2}\int_0^T\int_0^T \langle [\mu_c(t_1) - \mu_0]\, \mu_s(t_2) \rangle\, dt_1\, dt_2 \\ &= \frac{1}{2T^2}\int_0^T\int_0^T \overline{\mu_s^2}\, \hat{R}(t_2 - t_1)\, dt_1\, dt_2 = \frac{\overline{\mu_s^2}}{2T^2}\int_0^T\int_{-t_1}^{T-t_1} \hat{R}(\tau)\, dt_1\, d\tau \end{aligned}$$

$$= \frac{\mu_s^2}{2T}\left\{\int_\tau^T\int_0^{\tau}\tilde{R}(\tau)\,d\tau\,dt_1 + \int_0^T\int_{T-\tau}^{T}\tilde{R}(\tau)\,d\tau\,dt_1\right\}$$

$$= \frac{\mu_s^2}{2T}\left\{\int_0^T\tilde{R}(\tau)(T-\tau)\,d\tau + \int_{-T}^{0}\tilde{R}(\tau)(T+\tau)\,d\tau\right\}$$

$$= \frac{\mu_s^2}{2T}\int_0^T[\tilde{R}(\tau)+\tilde{R}(-\tau)](T-\tau)\,d\tau = 0, \qquad (5.63b)$$

inasmuch as, as was shown in Section 5.1, R(τ) is an odd function.

We will also introduce the notation

$$\begin{aligned}\Delta_c(t) &= \mu_c(t) - \langle\mu_c\rangle,\\ \Delta_s(t) &= \mu_s(t) - \langle\mu_s\rangle.\end{aligned} \qquad (5.64)$$

Now a signal being received (5.60) in the transmission of $z_1(t)$ can be written as follows:

$$\begin{aligned}z'(t) &= \{\langle\mu_c\rangle z_1(t) + \langle\mu_s\rangle \tilde{z}_1(t)\} +\\ &+ [\Delta_c(t) z_1(t) + \Delta_s(t)\tilde{z}_1(t)] + n(t),\end{aligned} \qquad (5.65)$$

and we may consider it as the sum of the useful signal included in the braces where the components of the coefficient of transmission remain constant over $0 \le t \le T$, the interference n(t), and the additional term which is included in the brackets.

This additional term

$$\delta_1(t) = \Delta_c(t) z_1(t) + \Delta_s(t)\tilde{z}_1(t)$$

is a random process. It arrives at the decision system and, generally speaking, influences the probability of error. But this influence can vary greatly depending on the signal. In order to explain this we will consider two extreme cases.

a) The additional term is added to the useful signal. This occurs in that case when the coefficient of mutual correlation between useful signal and additional term is equal to unity, e.g., in that system where signals $z_r(t)$ represent very short pulses occurring at various instants of time t_r (counting from instant of beginning of reading of a signal element). It can easily be seen that under these conditions the additional term acts on the decision system just as does the useful signal. Obviously, in this case the probability of error is expressed by the same formulas as in the case of a zero rate of fading.

b) The additional term is added to the interference. This occurs in those cases when the additional term is statistically independent of the useful signal and can be considered to be random noise. A typical example is provided by the system in which signal $z_r(t)$ represent different realizations of normal noise with a uniform spectrum in a certain rather wide frequency band F. Then $\xi(t)$ will also be normal noise in practically the same frequency band. In the first approximation PT and RPT systems and also FK systems with a small spread of frequencies of adjacent signals can be relegated to this case.

Various intermediate cases between these two extremes are possible. Thus, the additional term may be orthogonal with respect to all realizations of the signal. Then its power is subtracted from the power of the signal but not added to the interference. Cases are also possible when only part of the additional term should be assigned to interference. However, inasmuch as we are interested only in an approximate evaluation of the influence of rate of fading, we may limit ourselves to a consideration of case b). The components of the transmission coefficient for the "useful signal" μ_c and μ_s over one element do not change and may change only in jumps at the instant of element change. As indicated in the preceding section in this case dependences obtained for fading at a zero rate remain valid. It is also necessary to consider that the power of the "useful signal" in (5.65) decreased by the magnitude of the power of the process $\xi(t)$ which was added to the interference.

On the basis of (5.63) we can easily see that the average power P_s' of useful signal in (5.65) is equal to

$$P_s' = P_s\frac{\mu_r^2 + I\mu_\phi^2}{\mu_r^2 + \mu_\phi^2} = P_s\frac{k^2 + I}{k^2 + 1}, \tag{5.66}$$

where P_s is the initial signal power; k^2 is the ratio between the powers of the regular and fluctuating components; and I is a dimensionless magnitude dependent only on the coefficient of correlation of fading $R(\tau)$ and the duration of signal element T and is equal to

$$I = \frac{2}{T}\int_0^T\left(1 - \frac{\tau}{T}\right)R(\tau)\,d\tau. \tag{5.67}$$

In the case of fading at a zero rate when we may assume $R(\tau) = 1$ in the entire interval of integration, from (5.67) it follows that $I = 1$.

Inasmuch as a decrease in power of useful signal as a consequence of a finite rate of fading concerns only the fluctuating part, then coefficient k is somewhat increased and becomes equal to

$$k'^2 = \frac{\mu_r^2}{I\mu_\phi^2} = \frac{k^2}{I}. \tag{5.68}$$

Simultaneously the power of interference increases by a magnitude of $P_s(1 - L)/1 + k^2$ and its spectral density may be considered in the first approximation to be equal to

$$\nu'^2 = \nu^2 + P_s\frac{(1-L)}{F(1+k^2)} = \nu^2 + \frac{2P_sT}{B(1+k^2)}(1-L) = \nu^2\left[1+\frac{2h_0^2(1-L)}{(1+k^2)B}\right], \tag{5.69}$$

where F is the frequency band accepted for the system; and B = 2FT is its base.

Thus, all formulas obtained for fading at a zero rate remain valid for this case if in them we replace k with k' and h_0^2 with h'^2_0 where

$$h'^2_0 = \frac{P'_sT}{\nu'^2} = \frac{P_sT}{\nu^2}\,\frac{(k^2+L)B}{(k^2+1)\left[B+\frac{2h_0^2(1-L)}{(k^2+1)}\right]} = h_0^2\frac{(k^2+L)B}{(k^2+1)B+2h_0^2(1-L)}. \tag{5.70}$$

Specifically, for a binary orthogonal FK system with a frequency spacing of 1/T, considering that B = 4, we obtain from formula (5.17)

$$p = \frac{1}{2}\,\frac{h_0^2(1-L)+2k^2+2}{h_0^2+2k^2+2}\exp\left(-\frac{k^2h_0^2}{h_0^2+2k^2+2}\right). \tag{5.71}$$

when L = 1, i.e., in the case of fading at a zero rate, this formula becomes (5.17). With a decrease in L the probability of error grows rather rapidly, especially with large values of h_0 and small k.

If signal power is increased, i.e., if h_0^2 is increased, then when $h_0^2 \to \infty$, in distinction from all cases considered above, the probability of error approaches not zero, but a finite value

$$\lim_{h_0^2\to\infty} p = \frac{1-L}{2}e^{-k^2}. \tag{5.72}$$

This result should not be considered unexpected. Fading at a finite rate amounts to multiplicative interference which, with a certain probability, can make signal $z_i(t)$ more like another signal $z_r(t)$ even in the complete absence of additive interference. This probability approaches zero when the rate of fading decreases, i.e., L approaches unity.

In the case of Rayleigh fading (k = 0) we obtain from formula (5.71)

$$p = \frac{1}{2}\,\frac{h_0^2(1-L)+2}{h_0^2+2}, \tag{5.71a}$$

and the limit which the probability of error approaches when h_0^2 approaches infinity is equal to

$$\lim_{h_0^2 \to \infty} p = \frac{1}{2}(1 - L). \tag{5.72a}$$

In such a binary FK system if the frequency spacing is much greater than 1/T, the limiting probability of error is much less than (5.72). If $B \to \infty$ is substituted in (5.70), then $h_0'^2 = h_0^2 \frac{k^2 + L}{k^2 + 1}$ and instead of (5.71) we obtain

$$p = \frac{k^2 + 1}{h_0^2 L + 2(k^2 + 1)} \exp\left[- \frac{k^2 h_0^2}{h_0^2 L + 2(k^2 + 1)} \right], \tag{5.73}$$

and with Rayleigh fading

$$p = \frac{1}{h_0^2 L + 2}. \tag{5.73a}$$

Here with an increase in h_0^2 the probability of error approaches zero, although more slowly than in the case of fading at a zero rate.[1]

Figure 5.10 shows curves which are computed from formulas (5.71) and (5.73) for different values of L in the case when B = 4 and $B = \infty$. The curves plainly show that attempts to decrease the frequency spacing in FK systems should be handled with great circumspection if the rate of fading in the channel differs markedly from zero.

We may view other systems in the same way. We will discuss a binary RPM system somewhat more in detail. We will use the same approach as in Chapter IV, i.e., we will base ourselves on the fact that this system can be considered orthogonal if we consider the signal over the interval $-T < t < +T$. Therefore, we will perform averaging of the components of a coefficient of transmission over the indicated interval, assuming

---

[1] Strictly speaking, the discussion presented above is not applicable to an FK system with a large frequency spacing. Here we cannot consider the additional term $\xi_l(t)$ as interference with a uniform spectrum in a frequency band of F. The assumption that with sufficiently slow fading the spectrum lies close to the spectrum of $z_l(t)$ and does not overlap with the spectrum of other signals is closer to actuality. Therefore, the power of process $\xi_l(t)$ should be subtracted from the power of the signal and not added to the power of the interference. This pertains to any system in which the spectra of signals $z_r(t)$ are far removed from one another. It can easily be seen that based on such an idea we again arrive at formulas (5.73) which, apparently, correctly evaluate the probability of error with relatively moderate values of B.

$$\left.\begin{aligned} \langle \mu_c \rangle &= \frac{1}{2T} \int_{-T}^{T} \mu_c(t)\,dt, \\ \langle \mu_s \rangle &= \frac{1}{2T} \int_{-T}^{T} \mu_s(t)\,dt. \end{aligned}\right\} \qquad (5.74)$$

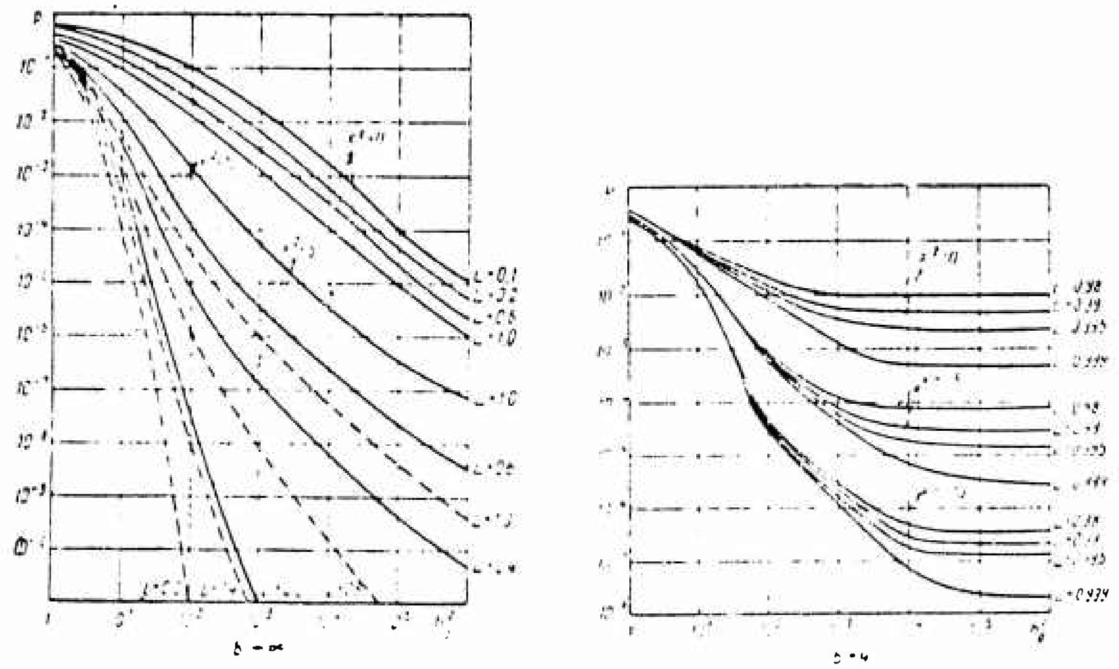


Figure 5.10. The Effect of Rate of Fading on Probability of Error for Binary Orthogonal Active-Interval Signals.

Reasoning as previously, we find that it is possible to consider the rate of fading in incoherent reception of binary RPM signals in the first approximation by replacing in formula (5.13) h_0^2 with

$$h_0'^2 = h_0^2 \frac{k^2 + M}{(k^2 + 1)[1 + h_0^2(1 - M)]},$$

and k'^2 with

$$k'^2 = \frac{k^2}{M},$$

where

$$M = \frac{1}{T} \int_0^{2T} \left(1 - \frac{\tau}{2T}\right) R(\tau)\,d\tau \qquad (5.75)$$

As a result of this exchange we obtain

$$p = \frac{1}{2}\left[\frac{h_0^2(1-M)}{h_0^2+k^2+1}+1\right]^{-1}\exp\left(-\frac{k h_0^2}{h_0^2+k^2+1}\right). \qquad (5.76)$$

This dependence is shown in Figure 5.11 for different values of M and k^2.

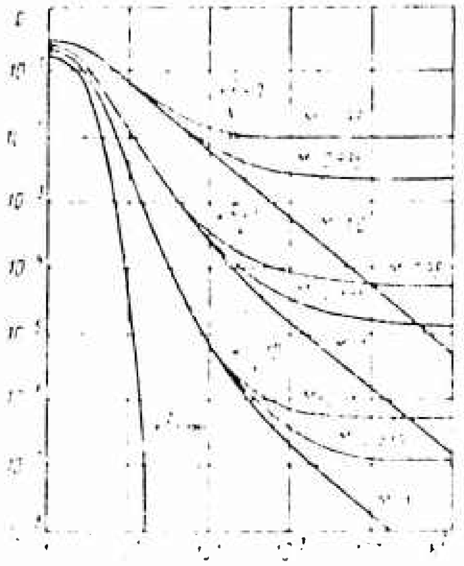


Figure 5.11. Effect of Rate of Fading on Probability of Error for Binary FFT System.

In the case of Rayleigh fading, $k^2 = 0$

$$p = \frac{1}{2}\,\frac{1}{h_0^2(1-M)+1}. \qquad (5.76a)$$

The limiting probability of error when h_0 approaches infinity in the case of quasi-Rayleigh fading is

$$\lim_{h_0\to\infty} p = \frac{1}{2}(1-M)e^{-k^2}, \qquad (5.77)$$

and in the case of Rayleigh fading

$$\lim_{h_0\to\infty} p = \frac{1}{2}(1-M). \qquad (5.77a)$$

Since it is always M [illegible] in the case of fading at a zero rate when [illegible] is ignored, the limiting probability of error in a RPM system is greater than it in FM system with a spacing of 1/T when operating with the same channel and at the same rate.

If an FM system with a frequency spacing of 1/T permits obtaining in a certain channel with slow fading a given probability of error p for all values of P_s and T, then the same probability of error will be obtained in an RPM system for the same value of P_s and a value half that of T (i.e., twice the specified rate of transmission). We can see this easily by comparing formula (5.76) with (5.71) and formula (5.75) with (5.67).

The values of L and M can be computed if we know the coefficient of correlation R(τ) of fading and the length of a signal element T. Specifically, if R(τ) is approximated by a bell curve (5.6c), then

$$L = \frac{2}{T}\int_0^T \left(1 - \frac{\tau}{T}\right)\exp\left(-\frac{\tau^2}{2\tau_k^2}\right)d\tau =$$

$$= \frac{\tau_k}{T}\sqrt{2\pi}\,\Phi\left(\frac{T}{\tau_k}\right) - 2\frac{\tau_k^2}{T^2}\left(1 - e^{-\frac{T^2}{2\tau_k^2}}\right)$$

In the case when $\tau_k \gg T$ (and the approach used here is applicable only to this particular case)

$$L \approx 1 - \frac{T^2}{4\tau_k^2}. \quad (5.78)$$

Similarly,

$$M \approx 1 - \frac{T^2}{\tau_k^2}. \quad (5.78a)$$

If it is considered that under these conditions the value of the coefficient of correlation for τ = T is

$$R(T) = \exp\left(-\frac{T^2}{2\tau_k^2}\right) \approx 1 - \frac{T^2}{2\tau_k^2},$$

then the result obtained can be written as follows:

$$1 - M \approx 2[1 - R(T)].$$

$$1 - L \approx \frac{1}{2}[1 - R(T)].$$

For an exponential coefficient of correlation (5.6d)

$$L = \frac{2}{T}\int_0^T \left(1 - \frac{\tau}{T}\right)\exp\left(-\frac{\tau}{\tau_k}\right)d\tau =$$

$$= 2\frac{\tau_k}{T} - 2\frac{\tau_k^2}{T^2}\left[1 - e^{-\frac{T}{\tau_k}}\right]$$

or in the case when $T \ll \tau_k$

$$L \approx 1 + \frac{T}{3\tau_k}. \tag{5.79}$$

Similarly,

$$M \approx 1 + \frac{4T}{3\tau_k}.$$

In an RPM system (and also in other systems in which the power of the additional term ·.(t) is added to interference power, e.g., FK with a small frequency spacing) the increase in duration of signal element T in the case of unchanged power does not always increase the fidelity of reception. With an increase in T the variable M (or L) changes and this can sometimes lead to an increase in the probability of error despite the increase in h_0^2. Therefore, in such systems there must be an optimal value of T which provides for the most effective possible transmission of information.

Finding an optimal duration of signal element in light of the many variable factors is a very difficult task. We will limit ourselves to a more particular setting of the problem in that we will find for what value of T the power of the signal essential for obtaining a given probability of error p is minimal. For an RPM system with Rayleigh fading from (5.76a):

$$h_0^2 \approx \frac{1-2p}{M\cdot 2p-1}.$$

Assuming that $R(\tau)$ is expressed by formula (5.6c) and substituting the value of M from (5.78a) and also expressing h_0^2 by P_s, we obtain

$$P_s \approx \frac{\nu^2}{T}\,\frac{1-2p}{\left(1+\frac{T^2}{\tau_k^2}\right)2p-1} \approx \frac{\nu^2(1-2p)}{2pT-\frac{T^3}{\tau_k^2}}.$$

Whence we easily find the value $T = T_{opt}$ for which P_s is minimal:

$$T_{opt\,RPT} \approx \tau_k\sqrt{\frac{2}{3}p}. \tag{5.80}$$

If τ_k = 0.5 sec (a rather typical value for an extended shortwave channel), then with values of p from 10^{-5} to 10^{-4} the variable T_{opt} changes from 1.5 to 4 msec and this agrees well with experimental data.

For an FK system with a frequency spacing of 1/T, from (5.71a) and (5.78) we find in the same way

$$T_{opt\,FK} \approx \tau_k\sqrt{8p}. \tag{5.80a}$$

In the case where $\tau_k = 0.5$ sec and $p = 10^{-5} \div 10^{-4}$, the optimal value of T lies between 16 and 44 msec.

It should be stressed that expression (5.80) determines only a relative optimum which provides a minimum of signal power for a given probability of error. The result will be different if the rate of information transmission (depending on p as well as directly on T) and not probability of error is given.

§.5. Carrying Capacity of Channel with Slow General Fading

A fading channel amounts to a typical example of a channel with variable parameters. Computing its carrying capacity in general form is a difficult task. At the present time only a few approximate expressions and evaluations [15-22] have been found which, incidentally, are sufficient for cases of practical importance.

The states of a channel are determined by the values of μ and ϑ (or μ_c and μ_s). By analogy with (2.74) and (2.75) it can be shown [23] that

$$I'(z,z') = \overline{[I'(z,z'|\mu,\vartheta)]}_{\mu\vartheta} - [I'(\mu,\vartheta, z|z')]_{z'} \qquad (5.81)$$

$$[I'(z,z'|\mu,\vartheta)]_{\mu\vartheta} - H'(\mu,\vartheta) \leq I'(z,z') \leq$$
$$\leq [I'(z,z'|\mu,\vartheta)]_{\mu\vartheta} \qquad (5.82)$$

For determining the carrying capacity of a channel it is necessary to find a probability distribution of signal z(t) such that (with certain limitations, for example, a given power) a maximum of expression (5.81) is provided. This is the most difficult part of the problem.

We will discuss the simplest case when the rate of fading is very slow in comparison with the rate of transmission of a message, i.e., when by analyzing previously received elements of a signal it is possible with great precision to predict the values of μ and ϑ. Under these conditions, obviously, $H'(\mu,\vartheta) < I'(z,z')$ and the extreme terms of inequality (5.82) differ little from one another. Therefore, the rate of transmission of information may be obtained in the first approximation by averaging with respect to μ and ϑ the value of $I'(z',z|\mu,\vartheta)$ which amounts to nothing more than the rate of transmission of information in a channel without fading for given values of μ and ϑ. The maximum of this averaging of rate occurs in that case when signal z(t) is so selected as to provide a maximum of $I'(z',z|\mu,\vartheta)$ and for this, as shown in Chapter III, the signal with a given average power $\mu^2 P$ must be gaussian. In this case the conditional (for given μ and ϑ) the maximal rate of information transmission or conditional carrying capacity, in accordance with (3.84), does not depend on a fixed value of ϑ and is equal to

$$C_{\mu} = F \ln\left(1 + \frac{\mu^2 P}{P_n}\right) \text{ natural units/sec} \qquad (5.83)$$

where P is the emitted signal power.

Changing to average signal power at the input of the receiving device, $P_s = \overline{\mu_0^2} P$ and averaging with respect to μ, we find the carrying capacity of a channel with slow fading

$$C = \int_0^\infty w(\mu) F \ln\left(1 + \frac{\mu^2}{\mu_0^2}\frac{P_s}{P_n}\right) d\mu \qquad (5.84)$$

For the case of Rayleigh fading [17], after substituting $w(\mu)$ from (5.3) and exchanging variables based on the formula $x = 1 + \frac{P_s}{P_n}\frac{\mu^2}{\mu_0^2}$, it is possible to bring (5.84) to the tabular integral

$$C = F\frac{P_n}{P_s}\exp\left(\frac{P_n}{P_s}\right)\int_1^\infty \ln x \exp\left(-\frac{P_n}{P_s}x\right)dx = -F\exp\left(\frac{P_n}{P_s}\right)Ei\left(-\frac{P_n}{P_s}\right) \qquad (5.85)$$

where

$$Ei(x) = \int_{-\infty}^{x} \frac{e^t}{t}\,dt$$

In the case where $P_s \gg P_n$ the integral exponential function $Ei(-P_n/P_s)$ is well approximated by the expression $\ln P_n/P_s + c$, where $c \approx 0.5772$ is the Euler constant. Considering that $\exp P_n/P_s \approx 1 + P_n/P_s$ we obtain

$$C \approx F\left(1 + \frac{P_n}{P_s}\right)\left(\ln\frac{P_s}{P_n} - c\right) \quad (P_s \gg P_n) \qquad (5.86)$$

In the case when $P_s \ll P_n$, by using an asymptotic expression of the integral exponential function $Ei\left(-\frac{P_n}{P_s}\right) \sim -\frac{P_s}{P_n}\exp\left(-\frac{P_n}{P_s}\right)$, we find

$$C \approx F\frac{P_s}{P_n} \quad (P_s \ll P_n), \qquad (5.87)$$

which coincides with the approximate expression for the carrying capacity of a channel without fading. The dependence of C/F on P_s/P_n for a channel with slow Rayleigh fading is shown in Figure 5.12. The same dependence in the absence of fading is shown for comparison. It follows from Figure 5.12 that slow Rayleigh fading reduces the carrying capacity of a channel by not more than 17%. For small P_s/P_n ratios the curves practically coincide.

In the case of slow quasi-Rayleigh fading it is apparent that the carrying capacity must assume a certain average value between the carrying capacity of a channel with Rayleigh fading and a channel in which fading is absent. An approximate expression of the carrying capacity for the case when $k < 1$ had been obtained in work [18]. In our notation in the case of a uniform noise spectrum it may be written in the following form:

[illegible]

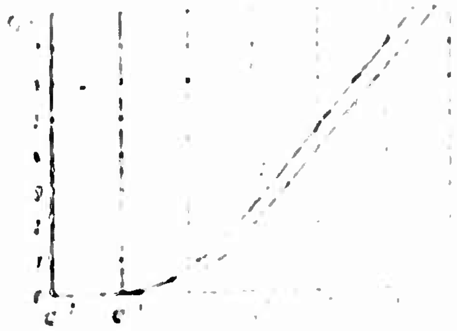


Figure 5.[illegible] Dependence of carrying capacity on signal-to-noise ratio [illegible] Absence of [illegible] Rayleigh fading

For the other extreme case when [illegible]

[illegible]

For channels with relatively [illegible] the [illegible] carrying capacity is made difficult [illegible] of rate of information transmission [illegible] gaussian and its spectral [illegible] evaluation has been [illegible] carrying [illegible] the fact that it has [illegible] in the case of an arbitrary [illegible] of information transmission [illegible] such an evaluation [illegible] are not presented here. They [illegible] carrying capacity [illegible] relatively general fading has [illegible] carrying capacity [illegible]

Based on this it can [illegible] fading channels in the case [illegible] by a signal can be expressed [illegible] for a channel with constant parameters

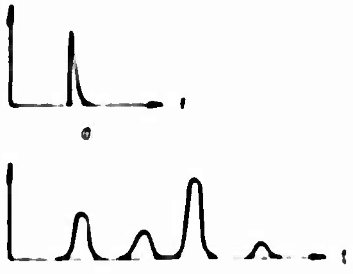
[illegible]

Incidentally, it should be [illegible] increase in [illegible] in any actual channel fading [illegible] However, [illegible] (5.90) still has real meaning [illegible] of the total rate of information transmission [illegible] separate [illegible] "channels" based on frequency [illegible] each channel occupies a limited frequency [illegible] general

Notes

1. (See Section 5.1) The physical picture described of the origin of selective fading is, of course, approximate. It may be used only in a relatively small width of the signal power spectrum. Signals of very wide spectra also display the directly dispersive properties of the ionosphere (or troposphere) in that the coefficients of reflection (or scattering) depend on frequency. The depth of wave penetration into the ionosphere also depends on frequency. The result of these phenomena is that even in "single-path" channels the transmission coefficient and phase shift prove to be nonidentical for the different frequency components of the signal.

The fading's frequency relationship occurring because of the dispersive properties of the medium is significantly more weakly expressed than the relationship determined by the interference phenomena in "multipath" channels. Thus, for example, in shortwave ionospheric propagation the dispersion phenomena cause perceptible differences in transmission coefficients only for frequencies which differ by tens of kilocycles and more, while in interference selective fading the correlation coefficient between the values of $_i$ are substantially less than unity, and at times even close to zero for frequency components differing only by hundreds of cps.



b)

Figure 5.13. Effect of Ionospheric Dispersivity and Multipath Propagation in Transmission of Short Pulse.

The difference between dispersion and interference phenomena in ionospheric reflection of radio waves may be graphically shown by scrutinizing the transmission of a very short (e.g., duration on the order of a microsecond) radio pulse whose envelope is shown in Figure 5.13a. Figure 5.13b represents the envelope of the incoming signal in multipath propagation. Here the case is illustrated where four "beams" of different strength arrive (each of which in turn is the sum of several beams (Figure 5.11)). Each of the "beams" which have arrived has been subjected to general fading because of the interference of its components. The difference in course between these individual components which make up the "beam" has a value on the order of small fractions of a microsecond. Therefore on the scale of our figure they are not distinguished. The difference in speed between the individual "beams" reflected a varying number of times from differing layers of the atmosphere is from hundreds of microseconds to entire milliseconds (in rare cases up to tens thereof).

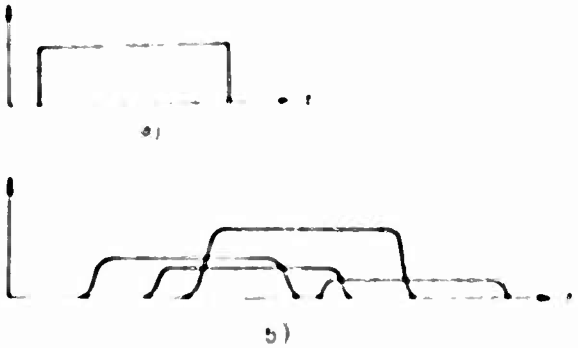


Figure 5.14. Overlap of Signals Arriving Over Several Paths in Transmission of Long Pulse.

Because of ionospheric scattering the pulse of the individual beam is extended and the shape of its envelope is distorted. Interference phenomena between the separate beams is not directly perceptible here because the incoming beams are separated in time. The multipath nature of the propagation here announces itself in the reception of four pulses instead of the single transmitted pulse.

Let a longer pulse of the order of several milliseconds be transmitted in the same channel (Figure 5.14a). Now the incoming pulses corresponding to the different beams mutually overlap (Figure 5.14b), and since they arrive with different phases interference phenomena causing selective fading occur along with the stretching of the received pulse.

2. (See Section [illegible]) As was mentioned in Chapter IV, real systems of communication most often utilize non-optimum decision systems which are less sensitive to frequency inexactitude, e.g., systems of narrow-band envelope reception, those with post-detection integration, etc. Everything which has been said about these systems remains valid likewise in slow general fading with the sole exception that the error probability expressions derived in Chapter IV must be averaged with respect to h to conform to the nature of the fading. We will present the result of such averaging for narrow-band envelope reception in the case of a binary FM system (4.74) and Rayleigh fading

$$P = \frac{2}{h_0^2 + 4}. \qquad (5.91)$$

In this case the probability of error when the values of h_0^2 are sufficiently large is practically twice as great as in the case of optimal incoherent reception (5.17a).

We may calculate the probability of error for other systems in the very same way. However, it must be remembered that many formulas obtained for non-optimal reception systems in Chapter IV are approximate and are valid only when $h^2 \gg fT$ (e.g., formulas (4.93), (4.95), and (4.97)). In the case of fading h^2 may become as small as desired no matter what the value of h_0^2. Therefore, formal averaging [illegible] may yield an approximately correct result if the rate of error is occurring when [illegible] is not great. The probability of error as computed in [24] for [illegible] of signals in a binary FM system when there is Rayleigh fading with respect to instantaneous fading for that case when a pi-response filter with [illegible] passband [illegible] is connected in front of the discriminator, the probability of error is

$$P = [illegible]$$

where q_1^2 [illegible] is the ratio between average signal power and interference power at filter output, [illegible] is the angular frequency deviation.

In the same work this problem is solved in a more general form with the rate of fading taken into account. Use is made of the results found in work [25] in which a distribution for the probability of instantaneous frequency of a sum of two stationary gaussian processes with different spectra is obtained. Obviously, in the case of Rayleigh fading a signal is a narrow-band gaussian process (although not stationary) and the rate of fading can be characterized by the mean-square width of the band [illegible]. In this case the probability of error is expressed by the following formula

$$P = \frac{1}{2}\left[[illegible] \right]$$

which in the case of fading at a zero rate [illegible] becomes (5.93). Here [illegible] is the mean-square filter passband (4.77). From [24] it is apparent that formula (5.93) holds for any filter and not only for a pi-response filter if [illegible] is understood to be the mean-square passband of the filter and if we set [illegible]. In light of this, and also expressing q_0 by h_0, we obtain

$$P = \frac{1}{2}\left[[illegible] \right] \qquad (5.93a)$$

We should keep in mind that these formulas are obtained by considering a signal to be a stationary process. In actuality, in the case of frequency keying a signal is not stationary and the formulas presented can be considered approximately true only on condition that the filter passband is rather wide that at the instant of reading an instantaneous frequency the oscillations in it can be considered steady, i.e., $\tau f l \gg 1$. Here all stipulations made with respect to formula (4.76) remain valid.

In the case of fading at [illegible] we get [illegible] and [illegible] formula (5.94) yields [illegible] the probability of error turns out to be the same as in the case of optimal incoherent reception; however, with such a small value of [illegible] the tolerance of such [illegible], apparently, is rather great. For the [illegible] values of [illegible] with which [illegible] the probability of error [illegible] to be much greater than [illegible] the case of optimal incoherent reception [illegible] fading [illegible] of error [illegible].

[illegible] Many experimental [illegible] connected with [illegible] changes in field potential [illegible] fading [illegible] scattering [illegible] rather well [illegible] the assumption that the [illegible] probability of transmission [illegible] at least [illegible] observe [illegible] general [illegible] in many works [illegible] significant [illegible] distributions and other [illegible] are suggested for the density of transmission coefficient probability [illegible] these works [illegible] distribution [illegible] in which the transmission coefficient [illegible] density

[illegible]

where [illegible] is a parameter which describes the fading in a channel.

This distribution was suggested [26] as an approximation of the probability density of a sum of a finite number of interference signals. It was obtained somewhat earlier in another form in [27] in the case of whole-number values of 2m as a result of [illegible] experimental investigation into several extended shortwave radio lines. When m = 1/2 a truncated normal distribution is obtained, and when m = 1, a Rayleigh distribution.

The probabilities of error with an "m-distribution" of the transmission coefficient are computed in [10, 28, 29] and in many other works. For an active interval system which is orthogonal in the intensified sense and with a code base of M the following expression was obtained in [29] for the probability of error in the case of optimal incoherent reception

$$p = \sum \text{[illegible]} \qquad (5.95)$$

which coincides with (5.16a) in the case where m = 1.

In work [30] it is found that measurements over short intervals of time (on the order of tens of minutes) show that in most radio channels fading has a Rayleigh distribution, however, the mean-square value of the transmission coefficient μ_0 amounts also to a random process with very slow (hourly) changes which obey an m-distribution. Incidentally, most authors assert that such hourly changes are best described by a normal logarithmic distribution.

Further generalization of Rayleigh fading with gaussian coefficients μ_c and μ_s with different average values and dispersions are considered in detail in [39].

Among other possible physical models of interference fading mention should be made of the dual-beam model in accordance with which a signal being received is the sum of two regular components with a constant amplitude ratio of k arriving along different paths and with a random uniformly distributed phase difference of φ. The transmission coefficient μ in such a channel is

$$\mu = \mu_0\sqrt{1 + \frac{2k\cos\varphi}{1+k^2}} \tag{5.96}$$

In the case of incoherent reception of binary orthogonal signals with an active interval, the probability of error is

$$p(\varphi) = \frac{1}{2}\exp\left[-\frac{\mu^2}{\mu_0^2}\frac{h_0^2}{2}\right] = \frac{1}{2}\exp\left[-\left(1 + \frac{2k\cos\varphi}{1+k^2}\right)\frac{h_0^2}{2}\right],$$

and the average probability of error in accordance with (5.[illegible]) may be found by averaging the expression obtained with respect to φ:

$$\bar{p} = \frac{1}{2\pi}\int_0^{2\pi} p(\varphi)\,d\varphi = \frac{1}{2}e^{-\frac{h_0^2}{2}}\,I_0\left(\frac{k h_0^2}{1+k^2}\right) \tag{5.96a}$$

When $\frac{k h_0^2}{1+k^2} \gg 1$, using the asymptotic representation of a modified Bessel function, we obtain

$$\bar{p} \approx \frac{1}{2}e^{-\frac{h_0^2}{2}}\frac{1}{h_0\sqrt{\frac{2\pi k}{1+k^2}}}\exp\left(\frac{k h_0^2}{1+k^2}\right) = \frac{1}{2h_0\sqrt{\frac{2\pi k}{1+k^2}}}\exp\left[-\frac{(1-k)^2}{2(1+k^2)}h_0^2\right] \tag{5.96b}$$

From this it is apparent that when $k \neq 1$ and sufficiently large h_0 the probability of error decreases exponentially with an increase in the power of the signal as in the case of quasi-Rayleigh fading. However, if $k = 1$, then

$$\bar{p} \approx \frac{1}{2\sqrt{\pi}\,h_0},$$

and the probability of error decreases only inversely proportionally to the square root of the signal power, i.e., much more slowly than even in the case of Rayleigh fading.

4. (See Section 5.2)[1] The matrix method of finding a characteristic function of the quadratic form of gaussian variables was first used in [8].

---

[1] Note 4 was written by I. S. Andronov.

It is a powerful means for computing the probability of error in the case of Rayleigh fading when the optimal incoherent decision principle [illegible] and also many suboptimal decision principles, amount to a comparison of the value of such a quadratic form with a certain (often zero) threshold. We will present here a basis for expression (5.34) and also several conclusions flowing from it.

A characteristic function of quadratic form [illegible] distribution is equal to the 2n-fold integral

$$\Theta(t) = \int_{-\infty}^{\infty}\cdots\int w(x_1, \dots, x_{2n}) \times \exp\left(jt\sum_{k=1}^{2n}\sum_{l=1}^{2n} a_{kl}x_kx_l\right)dx_1 \cdots dx_{2n} \qquad (5.97)$$

where $w(x_1, \dots, x_{2n})$ is the 2n-fold density distribution

$$w(x_1, \dots, x_{2n}) = \frac{1}{(2\pi)^n K^{1/2}} \exp\left(-\frac{1}{2}\sum\sum [illegible]\right) \qquad (5.98)$$

Here K is the discriminant of correlation matrix [illegible] and K_{kl} are elements of matrix K^{-1} which is the inverse of K.

In matrix notation

$$w(x_1, \dots, x_{2n}) = \frac{1}{(2\pi)^n K^{1/2}} \exp\left(-\frac{1}{2}\tilde{x}K^{-1}x\right) \qquad (5.99)$$

$$\Theta(t) = \int_{-\infty}^{\infty}\frac{1}{(2\pi)^n K^{1/2}} \exp\left[-\frac{1}{2}\tilde{x}(K^{-1} - 2jtA)x\right]dx \qquad (5.100)$$

where $\tilde{x}$ and x are the vector row and vector column [illegible] of x_k. Further, the symbol ~ always indicates a transposed matrix.

We will set

$$G = K^{-1} - 2jtA \qquad (5.101)$$

Then

$$\Theta(t) = \int_{-\infty}^{\infty}\frac{1}{(2\pi)^n K^{1/2}} \exp\left[-\frac{1}{2}\tilde{x}Gx\right]dx \qquad (5.102)$$

Matrix G is symmetrical inasmuch as matrices K and A are symmetrical. In this case there is a linear transformation of variables x_k which transforms quadratic form $\tilde{x}Gx$ into canonical form, i.e., into a sum of squares [31]. A linear orthogonal transformation is a transformation

$$x = Qy, \qquad (5.103)$$

when

$$y = Q_1^{-1}x. \qquad (5.104)$$

where Q_1 is a transformation matrix of the order 2n × 2n, satisfying the condition

$$Q_1 Q_1 = I \qquad (5.105)$$

I is a singular matrix of the order 2n × 2n or

$$\bar{Q}_1 = Q_1^{-1} \qquad (5.106)$$

Substituting [illegible] and considering that the Jacobian transform is equal to [illegible], we obtain

$$\Phi(t) = \frac{1}{(2\pi)^{2n}} \int [illegible] \, ds \qquad (5.107)$$

[illegible]

[illegible] that

$$[illegible] \qquad (5.108)$$

[illegible]

$$\Phi(t) = \frac{1}{(2\pi)^{2n}} \int [illegible] \, ds \qquad (5.109)$$

[illegible]

$$\Lambda = \bar{Q}_1 G Q_1 \quad [illegible] \qquad (5.110)$$

[illegible]

Multiplying from the left both sides of the first equality (5.110) [illegible] (5.106), we find

$$GQ_1 = Q_1\Lambda \qquad (5.111)$$

This last equality permits finding elements of transforming matrix Q_1. [illegible] if the matrix is represented in form of a vector row

$$Q_1 = (q_1^1, \ldots, q_{2n}^1) \qquad (5.112)$$

where q_p^1 is the p-th column of matrix Q_1, then for elements of the p-th column, from (5.111), we have the following equation:

$$Gq_p = \lambda_p q_p \quad (p = 1, 2, \ldots, 2n) \qquad (5.113)$$

For element q_{kp}^1 which stands at the intersection of the k-th row and the p-th column it is easy to find from (5.113)

$$\sum_{i=1}^{2n} g_{ki} q_{ip} = \lambda_p q_{kp} \quad (k, p = 1, 2, \ldots, 2n). \tag{5.114}$$

where g_{ki} is an element of matrix G standing at the intersection of the k-th row and i-th column.

From condition (5.105) it follows that elements of matrix Q must, furthermore, satisfy the following equation:

$$\sum_{i=1}^{2n} q_{ik} q_{il} = \delta_{kl}, \tag{5.115}$$

which provides for meeting the requirement of orthogonality and normalizing a linear transformation.

Finally, the eigenvalues λ_k of matrix G are found as solutions of the characteristic equation

$$|G - \lambda I| = 0 \tag{5.116}$$

Making the indicated transformations, we find that the multiple integral in (5.109) becomes the product of single integrals

$$\theta(z) = \frac{1}{(2\pi)^n \sqrt{|K|}} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}\tilde{y}\Lambda y\right] dy$$
$$= \frac{1}{(2\pi)^n \sqrt{|K|}} \prod_{k=1}^{2n} \int_{-\infty}^{\infty} \exp\left[-\frac{\lambda_k}{2} y_k^2\right] dy_k \tag{5.117}$$

The integral after the product symbol is tabular and, computing it, we obtain

$$\theta(z) = \frac{1}{(2\pi)^n \sqrt{|K|}} \frac{(2\pi)^n}{|\Lambda|^{1/2}} \tag{5.118}$$

Since the determinant of the matrix is invariant in the case of an orthogonal linear transformation,

$$|\Lambda| = |G| \tag{5.119}$$

Recalling (5.101) and substituting in (5.118), we have

$$\theta(z) = \frac{1}{(|K|\,|K^{-1} - 2izA|)^{1/2}} \tag{5.120}$$

It is known that a determinant of a product of two square matrices is equal to the product of the determinant, i.e.:

$$\theta(z) = \frac{1}{|I - 2izKA|^{1/2}}. \tag{5.121}$$

Since matrix KA is real and symmetrical, it is possible to select a new orthogonal linear transformation Q which will make it diagonal. Using the invariance of the determinant to such a transformation, we have

$$\theta(v) = \frac{1}{|Q^{-1}(I - 2ivKA)Q|^{1/2}} = \frac{1}{|Q^{-1}IQ - 2ivQ^{-1}KAQ|^{1/2}}. \tag{5.122}$$

But

$$Q^{-1}IQ = Q^{-1}Q = I, \tag{5.123}$$

and matrix

$$Q^{-1}KAQ = \Lambda = (\lambda_k \delta_{kp}), \; (k, p = 1, 2, \ldots 2n) \tag{5.124}$$

is diagonal, so that

$$|I - 2iv\Lambda|^{1/2} = \prod_{k=1}^{2n} (1 - 2iv\lambda_k)^{1/2}. \tag{5.125}$$

Thus, we finally obtain

$$\theta(v) = \frac{1}{\prod\limits_{k=1}^{2n} (1 - 2iv\lambda_k)^{1/2}},$$

where λ_k are found as roots of the equation

$$|KA - \lambda I| = 0,$$

which is what we wanted to find.

3. (See Section 5.2) Many problems in computing the probability of error in the case of slow Rayleigh fading, e.g., by the method described in the preceding note) are solved much more simply than the same problems for the case of which fading is absent. If a system with an active interval is considered, by knowing the probability of error P_f in the case of Rayleigh fading, it is possible to calculate the probability of error p_0 without fading by using the method suggested by N. P. Khvorostenko [14]. On the basis of (5.8) in the case of Rayleigh fading

$$p_f(h_0^2) = \int_0^\infty \frac{2h}{h_0^2} \exp\left(-\frac{h^2}{h_0^2}\right) p_0(h^2)\, dh. \tag{5.126}$$

We will set $x = h^2$ and $S = 1/h_0^2$. Then

$$p_f(s) = \int_0^\infty s p_0(x) e^{-sx} dx, \tag{5.127}$$

whence it follows that function $p_z(s)$ is a Carson-Laplace transform of function $p_0(x)$ [32]. It follows that by knowing the "image" $P_f(s)$ it is possible to find the "original" $p_0(x)$ by methods of operational calculus and using, for example, the Heavyside expansion theory, the Rieman-Mellin conversion formula, etc.

Finally, in most cases $p_0(x)$ can be found using one of the many reference books on operational calculus, for example [33].

Thus, from formula (5.17a) for a binary orthogonal system we have

$$p_1(s) = \frac{1}{\frac{1}{s}+2} = \frac{1}{2}\,\frac{s}{s+\frac{1}{2}}.$$

and from a reference book we can easily find

$$p_0(x) = \frac{1}{2}e^{-\frac{x}{2}}$$

or

$$p_0 = \frac{1}{2}e^{-\frac{h^2}{2}}.$$

Thus, from formula (5.19) which can easily be deduced from a correlation matrix (see pp. 354-357) it is possible to obtain formula (4.61) more simply than was done in Chapter IV.

6. (See Section 5.3) In many works (e.g., [34, 35]) the probability of error is calculated in the case of joint action of interference (usually Rayleigh) and absorption fading, i.e., fluctuations of absorption in the medium in which the signal is propagated. Various distribution laws are suggested for absorption fading and most frequently, as already indicated, the normal logarithmic law and sometimes the m-distribution law. In this case it is not considered that the rate of even exceedingly "slow" (in the sense $\tau_k \gg T$) interference fading is usually several tens of orders greater than the rate of absorption fading. Inasmuch as the average probability of correct reception of a symbol is calculated, such a consideration is not necessary. However, it cannot but be noted that such an "average" probability of error characterizes a channel no better than the average temperature of a man over a period of ten years characterizes his state of health.

The average probability of incorrect reception of a symbol is a useful (although not completely so) characteristic of a fading channel only in that case when the duration of a finished (in sense) message being transmitted exceeds the average period of fading. In this case it permits evaluating the expected number of incorrectly received symbols in a message. For a more complete characteristic, as already noted in Section 5.3, how errors are grouped in a channel should be indicated and this is determined by rate of fading. Such a situation usually occurs in the case of interference fading. Thus, in shortwave channels τ_k fluctuates within limits of several tens of fractions of a second to several seconds while the transmission of a finished message usually lasts not longer than several tens of seconds.[1] Similar relationships

---

[1] If a shortwave channel is used for the transmission of very short messages (e.g., commands in a telecontrol system), the average probability of error due to interference fading loses practical meaning. It is necessary to compute, instead of it, the probability of distortion of a command.

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

BIBLIOGRAPHY

1. Levin, B. R., [illegible] [Theoretical Fundamentals of Statistical Radio Technology], Vol. 1, "Sovetskoye radio" Press, 1966.

2. McNicol, R. W. E., "The Fading of Radio Waves of Medium and High Frequencies," JIEE, Pt. 3, No. 44, 1949.

3. Dolukhanov, M. P., [illegible] [Propagation of [illegible]], second edition, Svyaz'izdat, [illegible].

4. Kotov, V. N., [illegible] Distortions in Radiotelegraphy], Radiotekhnika, No. 6, [illegible].

5. Grisdale, G. L., J. G. Morris and D. S. Palmer, "Fading of Long-Distance Radio Signals and a Comparison of Space and Polarization Diversity Reception in the 6-18 Mc. Range," JIEE, Pt. B, No. 13, 1957.

6. Turin, G. L., "Communication Through Noisy Multipath Channels," IRE Conv. Rec., Vol. 4, Pt. 4, 1956.

7. Fink, L. M., [illegible] [Potential Resistance to Interference with Signal Fading], Radiotekhnika, No. 9, 1959.

8. Middleton, D., [illegible] [Introduction to Statistical Communication], Vol. II, Chapters 17, 19, 20, "Sovetskoye radio" Press, 1962.

9. Klovskiy, D. D., [illegible] [Problems in Potential Resistance to Interference with Signal Fading], Radiotekhnika, No. 5, 1960.

10. Voronin, V. A., [illegible] [Problem of Potential Resistance to Interference in Channels with a Random Change in Parameters], Elektrosvyaz', No. 10, 1960.

11. Davenport, W. B. and W. L. Root, [illegible] [Introduction to Theory of Random Signals and Noise], Foreign Literature Press, 1960.

12. Korzhik, V. I., [illegible] [Several Problems in Coding in Nonuniform Channels with Feedback], Theses of Report at the First Symposium on the Use of Redundancy in Information Systems, 1964.

13. Khvorostenko, N. P., [illegible] [Statistical Theory of Demodulation of Discrete Signals], "Svyaz'" Press, 1968.

14. Khvorostenko, N. P., [illegible] [Interference Resistance in Multiple Phase Telegraphy], Elektrosvyaz', No. 7, 1964.

15. Siforov, V. I., [illegible] [Carrying Capacity of Communication Channels with a Random Change in Absorption], No. 5, 19[illegible].

16. Siforov, V. I., [illegible] [Conditions for Attaining a High Level of Carrying Capacity with a Random Change in Parameters], Elektrosvyaz', No. [illegible], 19[illegible].

17. Tsybakov, B. S., [illegible] [Carrying Capacity of Channels with a Large Number of Beams], Radiotekhnika i elektronika, No. 9, 19[illegible].

18. Tsybakov, B. S., [illegible] [Carrying Capacity of Several Multi-Beam Communication Channels], Radiotekhnika i elektronika, No. [illegible], 19[illegible].

19. [illegible] [Evaluation of Carrying Capacity of Several Communication Channels with [illegible]], Radiotekhnika, No. [illegible], 19[illegible].

[illegible] [Evaluation of Carrying Capacity of [illegible] Channels with [illegible] Parameters], [illegible] Radiotekhnika, No. [illegible], 19[illegible].

21. [illegible] [Carrying Capacity of Channels with General and [illegible] Fading], Radiotekhnika, No. [illegible].

22. [illegible], [illegible] and M. S. Pinsker, [illegible] [Evaluation of Carrying Capacity of Communication Channel the Parameters of Which Are Random Functions of Time], Radiotekhnika, No. [illegible].

23. Dobrushin, R. L., [illegible] [Transmission of Information over a Channel with Feedback], [illegible], Vol. III, [illegible] edition, 19[illegible].

24. Fedin, [illegible] [Resistance to Interference of Reception of Fading Signals in Frequency Radiotelegraphy], Elektrosvyaz', No. 8, 1965.

25. Zhuk, [illegible] [Probability Density of a Derivative Phase of a Sum of Narrow-Band Signal and Gaussian Noise], Radiotekhnika i elektronika, No. 9, 1964.

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible] [Transmission [illegible] Messages by Radio Channels] [illegible]

CHAPTER VI

CHANNELS WITH [illegible] GENERAL FADING
DIVERSITY RECEPTION

6.1. Methods of Diversity Reception

[illegible] [1], which is very convenient [illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

transmitted signals by means of analysis of the received specimens; and this increases reliability in diversity reception.

We will regard the noise active in the different branches mutually independent. This is of course true if the noise is internal to the receiver and in most cases it is also true for the noise which enters the receiving unit from without.

The transmission factors . and phase shifts differ in the various branches in a channel with general fading. Diversity reception provides the greatest noise-resistance gain when the transmission factors in the different branches are mutually uncorrelated, but, as will be shown below, a substantial gain may be realized even with a certain correlation among the transmission coefficients. We will measure this correlation by the magnitude of the correlation factor between the cophasal ($_{cf}$) and quadrature ($_{s}$) portions of the fluctuating component of the transmission factor, like the definition of R(τ) in expression (5.6), but with the difference that here we are considering the factor of mutual correlation between two fading processes at identical moments in time:

$$[illegible] \qquad (6.1)$$

where the subscripts m and n refer to the m-th and the n-th branches of the diversity reception system.

To simplify the problem we will at times limit ourselves to the case of $R_0 = 0$. The justification for this is that with small values of R (approximately $R \leq 0.3$) calculation of mutual correlation introduces only a small correction (this will be illustrated by several examples), and also that when a system of diversity reception in a channel with fading is being designed the smallest possible values of R are striven after.

It is apparent that the greatest reliability in diversity reception may be obtained when the a priori information about the signals expected in each branch is fully utilized. In sufficiently slow fading where the values of . and can be predicted in all the receiving branches the optimum system of diversity reception is that of coherent summation. The optimum system of incoherent summation will be determined for fast fading and likewise for cases where, in order to simplify instrumentation, control devices utilizing the opportunities of predicting . and are dispensed with.

In addition, we will examine several simpler nonoptimal systems of diversity reception which are employed in practice, including the system of discrete summation.

It should be noted that of all methods of diversity reception only reception with spaced antennas does not involve losses either of signal power or of the system's actual carrying capacity. Decreasing the rate of information transmission (e.g., in time diversity) is equivalent to a power loss because at the same rate it would be possible in single-transmission reception to increase the duration of the element and correspondingly increase the average ratio of signal element power to specific noise power, denoted by h_0.

This power loss must be taken into account in order to compare the noise resistance of different systems of diversified reception. We will designate by h_0^2 the average ratio of signal power to specific noise power which would occur if the same transmitting unit were used for single transmission reception. The real value of the signal power to specific noise power ratio, however, depends, in the case of frequency or time diversity on the number Q of diversity receiving branches.

In time diversity the element length decreases by a factor of Q. Therefore, the average ratio of signal power to specific noise power is in this case $h_Q^2 = h_0^2/Q$. This ratio will have the same value likewise in the case of frequency diversity if each branch has its own transmitter. If, however, the frequencies are radiated by a single transmitter, then its power is known to be utilized significantly more poorly. The multichannel signal is most often formed in the transmitter by tone single-band modulation. In order to avoid large cross-over noise linear operating conditions must be secured in the transmitter. In a strictly linear regime the amplitude in each of the Q branches in frequency diversity transmission must be Q times less than the maximum permissible amplitude for telegraphic operation of the transmitter. Thus the power peak for each branch will be Q^2 times less than the peak power of the transmitter and then $h_Q^2 = h_0^2/Q^2$.

In many cases we can dispense with strict linearity in transmitter operating conditions or have reserve peak power which permits us to tolerate overmodulation of the transmitter for short periods of time. In these cases h_Q^2 lies between h_0^2/Q and h_0^2/Q^2.

For greater generality of analysis let us assume

$$h_Q^2 = \frac{h_0^2}{Q^\nu}, \qquad ([illegible])$$

where the exponent ν can assume different values from 0 to 2. During reception with spaced antennas ν = 0. In both time and frequency diversity ν = 1 if each branch uses its own transmitter. If, however, in frequency diversity all the branches are radiated by one transmitter the value of ν lies between 1 and 2 and characterizes the "reserve linearity" of the transmitter.

6.2. Coherent Diversity Reception

Let us find the optimum (in the sense of the ideal observer criterion) decision principle in diversity reception of a signal element, supposing that for every receiving branch the values of μ_c and μ_s are exactly known on the basis of analysis of the preceding elements and interference is independent.

Let the system use the signals $z_r(t)$ ($r = 1, \ldots, m$) and let the i-th branch receive the signal (5.2) ($i = 1, \ldots, Q$):

$$z_i(t) = \mu_c^{(i)} z_r(t) + \mu_s^{(i)} \tilde{z}_r(t) + n^{(i)}(t), \quad 0 < t < T, \tag{6.3}$$

where $n^{(i)}(t)$ is the interference in the i-th branch.

From (6.3) it follows that

$$n^{(i)}(t) = z_i(t) - \mu_c^{(i)} z_r(t) - \mu_s^{(i)} \tilde{z}_r(t), \tag{6.4}$$

which allows us to write in light of (5.15) and (5.18) the conditional probability density of the incoming signals in all the branches if signal $z_r(t)$ is transmitted

$$\begin{aligned} w(z_1, \ldots, z_Q/z_r, \mu_c^{(i)}, \mu_s^{(i)}) = {} & \text{[illegible]} \prod_{i=1}^{Q} \text{[illegible]} \\ & \times \exp\left\{ -\sum_{i=1}^{Q} \frac{1}{N_i} \int_0^T n^{(i)2}(t)\,dt \right\} = \text{[illegible]} \prod_{i=1}^{Q} \text{[illegible]} \\ & \times \exp\left\{ -\sum_{i=1}^{Q} \frac{1}{N_i} \int_0^T [z_i(t) - \mu_c^{(i)} z_r(t) \right. \\ & \left. - \mu_s^{(i)} \tilde{z}_r(t)]^2 dt \right\}, \end{aligned} \tag{6.5}$$

where N is an arbitrarily large number.

According to the maximal likelihood criterion the symbol c_k must be registered if

$$w(z_1, \ldots, z_Q/z_k) > w(z_1, \ldots, z_Q/z_r) \quad (r \neq k)$$

or

$$\begin{aligned} & \sum_{i=1}^{Q} \frac{1}{N_i} \int_0^T [z_i(t) - \mu_c^{(i)} z_k(t) - \mu_s^{(i)} \tilde{z}_k(t)]^2 dt < \\ & < \sum_{i=1}^{Q} \frac{1}{N_i} \int_0^T [z_i(t) - \mu_c^{(i)} z_r(t) - \mu_s^{(i)} \tilde{z}_r(t)]^2 dt \quad (r \neq k) \end{aligned} \tag{6.6}$$

Removing the parentheses following the integral signs in [illegible] and transforming in light of the fact that[1]

[illegible]

[illegible] to the equivalent inequality

[illegible]

[illegible]

[illegible]

[illegible]

[illegible] in another form

[illegible]

---

[1] The integral equalities can easily be checked by representing the functions [illegible] and [illegible] with Fourier series.

where $z_r(\vartheta^{(i)},t)$ represents signal $z_r(t)$ in which all components have been shifted in phase by $\vartheta^{(i)}$. In this form the principle is realized by a decision system consisting of Q individual systems of coherent reception and summators which add the results obtained in each branch.

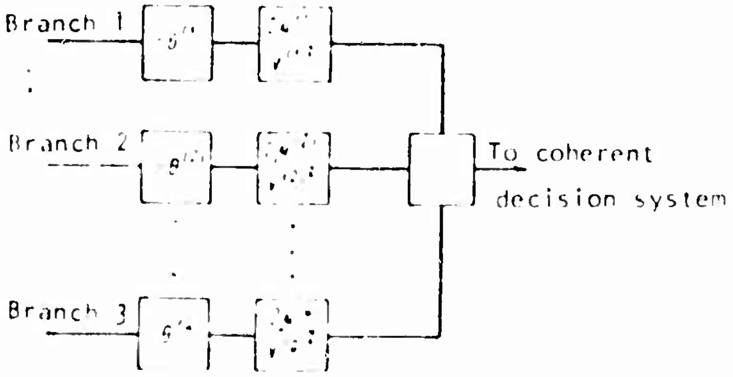


Figure 6.1. System of Coherent Addition.

For an active interval principle (6.7) may be simplified, taking into consideration that

$$\sum [illegible] \quad \sum [illegible]$$

$$\sum_{i} \mu^{(i)} \int_0^T z'_1(t - \vartheta^{(i)},t) z_1(t)\,dt \; [illegible]$$
$$\sum_{i=1}^{Q} \mu^{(i)} \int_0^T z'_1(t - \vartheta^{(i)},t) z_2(t)\,dt \qquad (6.7b)$$

If we use high-frequency active-interval signals which are mutually orthogonal in the intensified sense, coherent summation with good approximation can be done using the diagram shown in Figure 6.2. The diagram shows a binary system and duplex reception in order to avoid being cumbersome. The reader can easily generalize it for any code base and any number of diversity branches.

Here $m_1(t)$ and $m_2(t)$ are signals created by a local oscillator and differing from $z_1(t)$ and $z_2(t)$ respectively only in the shift by a certain "intermediate" frequency ω_{int} and also, of course, in power as in the diagram of synchronous heterodyning (see Note 5 to Chapter IV). Thus, if

$$z(t) = Z(t)\cos\Phi(t),$$

then

$$m(t) = kZ(t)\cos[\Phi(t) - \omega_{int}t],$$

where k is an arbitrary constant.

The arriving signal in the i-th branch

$$z'_i(t) = \mu^{(i)} A_i(t)\cos[\Phi_i(t) + \vartheta_i] + n^{(i)}(t)$$

is multiplied by the sum of the signals of the local oscillators and the product $z'_i(t)\sum_l m_l(t)$ goes to filter F_{int} which separates the component with a frequency of ω_{int}. Obviously,

$$\begin{aligned} z'_i(t)\sum_l m_l(t) = {} & \frac{k}{2}\mu^{(i)}\Big\{\sum_{l\neq i} A_i(t)A_l(t)\cos[\Phi_i(t) + \\ & + \Phi_l(t) - \omega_{int}t + \vartheta_{(i)}] + \sum_{l\neq i} A_i(t)A_l(t)\cos[\Phi_i(t) - \\ & - \Phi_l(t) + \omega_{int}t + \vartheta_{(i)}] + A_i^2(t)\cos[2\Phi_i(t) - \omega_{int}(t) + \vartheta_{(i)}] + \\ & + A_i^2(t)\cos[\omega_{int}t + \vartheta_{(i)}]\Big\} + kn^{(i)}(t)\sum A_l(t)\cos[\Phi_l(t) - \omega_{int}t]. \end{aligned} \qquad (6.8)$$

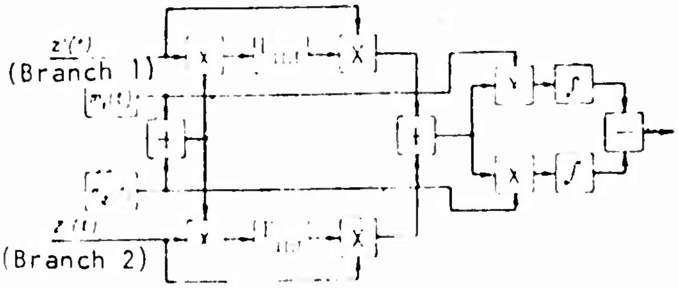


Figure 6.2. Variation of a Diagram for Coherent Summation for Orthogonal Signals with an Active Interval.

The frequency ω_{int} is so selected that it is lower than the least frequency of the spectrum of any of the signals $z_l(t)$ and the passband of the filter (F_{int}) so that its time constant τ_f is much greater than the duration of signal element T and at the same time much less than the fading correlation time τ_k. Obviously this condition can be met if fading is sufficiently slow.

It is easy to see that the first and third terms in the braces (6.8) have a spectrum beyond the limits of the filter passband. The spectrum of the second filter, generally speaking, occupies a passband including frequency ω_{int}, however, it can be shown that in the case of orthogonality of signals $z_i(t)$ the value of its spectral density at frequency ω_{int} is equal to zero. Thus, the potential at the filter output is determined only by the fourth and fifth terms. Obviously the potential at the filter output due to the fourth term is proportional to

$$U_1(t) = \mu^{(i)} \overline{A_r^2(t)} \cos(\omega_{int} t + \varphi^{(i)}),$$

Inasmuch as $\overline{A_r^2(t)} = 1/2P_r$ and in an active interval system the powers of all signals are the same $P_r = P_s$, then this potential

$$U_f(t) = \frac{1}{2} \mu^{(i)} P_s \cos(\omega_{int} t + \varphi^{(i)}), \qquad (6.8a)$$

does not depend on what signal was transmitted and is determined entirely by the values of $\mu^{(i)}$ and $\varphi^{(i)}$ in the given branch.

The potential created by the fifth term amounts to noise from which the filter separates that part lying in its passband. When the filter has a sufficiently narrow band this potential can be neglected in the first approximation.

Indeed, it can be shown that its power is less than the average power of potential (6.8a) by $h_Q^2 \tau_f/T$ times. Inasmuch as the magnitude of h_Q^2 with any satisfactory reception whatever is much greater than unity and the ratio τ_f/T in the case of slow fading is usually not less than 100, then the addition of the noise term to (6.8a) causes a change in its amplitude by only a few percentage points and in its phase by not more than 5°. A consideration of this shows that in the diagram of Figure 6.2 coherent summation is accomplished not quite accurately but the effect of this inaccuracy on the probability of error is negligible.

And so, we will assume that the potential at filter output is proportional to (6.8a). This potential goes to the second multiplier (frequency converter) together with the input signal. At its output the potential is proportional to

$$U_2(t) = 2\mu^{(i)}[z_i'(\varphi^{(i)}, \omega_{int}, t) + z_i'(-\varphi^{(i)}, -\omega_{int}, t)], \qquad (6.8b)$$

where $z_i'(\varphi, \omega_{int}, t)$ is the signal being received which is shifted in frequency by ω_{int} and in phase by $\varphi^{(i)}$. The first term in (6.8b) usually can easily be cut off by a filter since its spectrum lies $2\omega_{int}$ higher than the spectrum of the second term. The second term with an accuracy to the shift in frequency coincides with the expression figuring in decision principle (6.7). In other words, at the output of the multipliers the initial phases of the signals do not depend on the number of a branch and coincide with the phases of the local signals $m_i(t)$. This permits performing coherent summation and then using local signals as references for coherent reception.

We will now proceed to computation of probability of error for several cases of coherent diversity reception.

The conditional probability of incorrect reception of a signal element in coherent summation is defined for given values of $\mu^{(i)}$ and $\varphi^{(i)}$ as the probability of non-fulfillment of inequality (6.7) during transmission of symbol y_i. Total error probability is found by averaging the conditional probability with respect to $\mu^{(i)}$ and $\theta^{(i)}$ in conformity with the nature of the fading. Computation of this probability in the general form is laborious. Let us limit ourselves to several particular cases which will permit us to judge the order of increase in noise-resistance gain in diversity reception by the method of coherent summation.

We will consider an active-interval binary system for which the principle of expression (6.7) for receiving symbol y_1 may be written as:

$$\sum_{i=1}^{Q}\mu^{(i)}\int_0^T z'_i(-\theta^{(i)},t)\,z_1(t)\,dt > \sum_{i=1}^{Q}\mu^{(i)}\int_0^T z'_i(-\theta^{(i)},t)\,z_2(t)\,dt \qquad (6.9)$$

If symbol y_1 has, in fact, been transmitted, then[1]

$$z'_i(-\theta^{(i)},t) = \mu^{(i)}z_1(t)+n^{(i)}(t), \qquad (6.10)$$

where $n_i(t)$ is the noise active in the i-th branch.

Substituting expression (6.10) in (6.9) we derive the fact that the conditional error probability with given values of $\mu^{(i)}$ and in transmission of symbol y_1 amounts to the probability of fulfilling inequality

$$\begin{gathered}\sum_{i=1}^{Q}\mu^{(i)2}\int_0^T z_1^2(t)\,dt + \sum_{i=1}^{Q}\mu^{(i)}\int_0^T z_1(t)\,n^{(i)}(t)\,dt \\ < \sum_{i=1}^{Q}\mu^{(i)2}\int_0^T z_1(t)\,z_2(t)\,dt + \sum_{i=1}^{Q}\mu^{(i)}\int_0^T z_2(t)\,n^{(i)}(t)\,dt.\end{gathered} \qquad (6.11)$$

Taking into account that

$$\mu_0^2\int_0^T z_1^2(t)\,dt = P_s T,$$

we may write expression (6.11) as

$$\sum_{i=1}^{Q}\mu^{(i)}\int_0^T [z_2(t)-z_1(t)]\,n^{(i)}(t)\,dt > \gamma^2 P_s T\sum_{i=1}^{Q}\frac{\mu^{(i)2}}{\mu_0^2}. \qquad (6.11a)$$

Here γ is determined by expression (3.61a). We will point out that with orthogonal signals $\gamma = 1$ and with opposed signals $\gamma = \sqrt{2}$. The integrals on the left side of this inequality represent independent* normally distributed random variables with a mathematical expectation of zero. Their dispersion may be computed in a way similar to that employed in chapter III and is [illegible] (P's). Therefore the dispersion of the sum written on the left side of (6.11a) equals

---

[1] Expression (6.10) follows from the fact that the phase shift of $-\theta^{(i)}$ in the signal being received $z'_i(t)$ compensates for the phase shift in the channel.

$$[illegible] \; \mu_i \sum_{i=1}^{Q} [illegible]$$

The right side of this inequality with fixed values of [illegible]$^{(i)}$ is also fixed. Hence the probability of fulfilling this inequality, i.e., the conditional probability of error is

$$p_e = \frac{1}{2}\left[1 - \Phi\left(\frac{[illegible] \sum [illegible]}{[illegible] \sum [illegible]}\right)\right] \tag{6.12}$$

$$= \frac{1}{2}\left[1 - \Phi\left(\sqrt{\sum [illegible] h_i}\right)\right]$$

The complete probability of error (taking into account the symmetry of a binary channel in coherent reception)* may be derived by averaging expression (6.12) with respect to all values of [illegible]

$$p = \frac{1}{2}\int_0^{\infty} \cdots \int_0^{\infty} w([illegible]) \times \left[1 - \Phi\left(\sqrt{\sum [illegible] h_i}\right)\right] d\mu^{(1)} \cdots d\mu^{(Q)} \tag{6.13}$$

Let us take a look at several particular cases.

a) We shall suppose that the values of [illegible]$^{(i)}$ are random, but do not change throughout the reception of a message, i.e., there is practically no fading, and that the transmission factors for the different branches also differ. Here the magnitude [illegible] represents the ratio of the signal power of the i-th branch to the spectral noise density. The complete error probability in the absence of fading agrees with the conditional probability of expression (6.12)

$$p = \frac{1}{2}\left[1 - \Phi\left(\sqrt{\sum_{i=1}^{Q} [illegible]}\right)\right] = \frac{1}{2}[1 - \Phi(h_{res})] \tag{6.14}$$

where $h_{res}^2 = \sum_{i=1}^{Q} [illegible]$ is the resultant ratio of signal power to spectral noise density after coherent summation with the optimum weight factors.

This result may be briefly formulated as follows: optimum coherent summation results in a signal-to-noise ratio which equals the sum of the signal-to-noise ratios in each branch*. We will note that expression (6.14) may be easily derived without imposing any condition of equality between the spectral noise densities in the different branches [3].

---

*Independence follows from our examination here of mutually uncorrelated noise in the different branches. (See note 1 to Chapter VI).

The gain in noise resistance (in comparison to single transmission reception when [illegible]$_0$) in this case is obtained only during reception on spaced antennas when exponent [illegible] in expression (6.2) is zero. In frequency diversity $h^2_{res} = h^2_0$ if [illegible] 1, i.e., no gain is made. If, however, [illegible] 1, then diversity reception without fading gives decreased noise resistance in comparison with single-transmission reception.

b) We shall suppose that fading in all branches is completely correlated ($R_0 = 1$). Since [illegible]$_0$ is identical in all branches it follows from our supposition that all [illegible]$^{(i)}$ are also identical at every given moment[1] [illegible]. From expression (6.13) we get

$$r = \frac{1}{2} \int_0^\infty [illegible] \left[[illegible] \right] [illegible] \qquad [illegible]$$

In the case of Rayleigh fading of [illegible] from (5.3) and integrating, we obtain

$$r = \frac{1}{2} \left[1 - \sqrt{\frac{Q [illegible]}{Q [illegible]}} \right] \qquad [illegible]$$

By comparing the result with expression (5.10) we may observe that in this case coherent reception with spaced antennas (when [illegible] and $h_i = h_0$) provides a power gain of Q times. A similar result may be obtained even without introducing assumptions as to the equality of the average transmission coefficients in all the receiving branches if h_0 is understood to mean the value of the ratio between signal power and spectral noise density averaged with respect both to time and to all branches. When [illegible] and R_0 [illegible] 1 coherent diversity reception gives no gain.

c) Now let us assume that the transmission factors in the different branches are pairwise uncorrelated ($R_0 = 0$). We shall give the notation [illegible] to the positive random variable

$$[illegible] \sum_{i=1}^{Q} [illegible] = \frac{1}{[illegible]} \sum_{i=1}^{Q} ([illegible]) \qquad (6.16)$$

Instead of expression (6.13) we may now write

$$r = \frac{1}{2} \int_0^\infty [illegible] \qquad (6.17)$$

---

[1] See the deviation of formula (5.10).

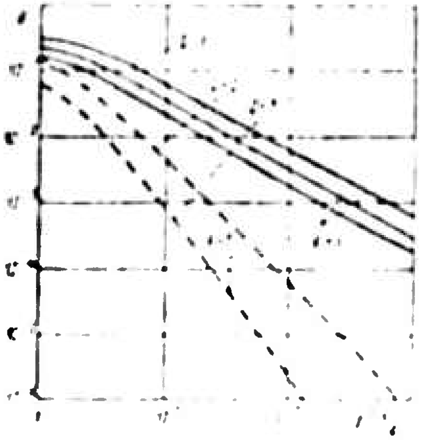


Figure [illegible] Error Probability [illegible]
[illegible] Reception

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

$$[illegible] \qquad (6.24)$$

It can easily be seen that when $R_0 = 1$ this formula coincides with (6.15a). When $R_0 = 0$ (6.19a) is obtained as a limiting case. When $\nu^2 h^2 (1-R) \gg 1$

$$[illegible] \qquad (6.24a)$$

By comparing the obtained asymptotic expression with (6.19b) we see that if $R_0 \leq 0.6$, the power gain occurring because of correlation does not exceed 1 dB, i.e., under these conditions correlation has no practical effect on the effectiveness of diversity reception.

A more general case of coherent diversity reception is considered in [4].

6.3. Incoherent Optimal Diversity Reception

Let us consider what sort of optimum decision system for diversity reception we should have in the case where it is impossible to predict the expected phase and transmission coefficient values either because of great fading rate, or indeterminate phase during transmission, or, finally, limited instrumentation potentialities. Under these conditions the joint probability of receiving signals $z_i'(t)$ $(i = 1,\ldots,Q)$ when signal $z_r(t)$ is transmitted and at certain values of $\mu_c^{(i)}$ and $\mu_s^{(i)}$ in the receiving branches is expressed by formula (6.5). Since, however, the values of $\mu_c^{(i)}$ and $\mu_s^{(i)}$ are considered to be unknown in advance, to find the decision principle we will use the generalized criterion of maximal likelihood, i.e., require that the receiver decide that signal $z_r(t)$ was transmitted if

$$\max_{\mu_c^{(i)},\mu_s^{(i)}} w(z_1',\ldots,z_Q'|z_r,\mu_c^{(1)},\mu_s^{(1)},\ldots,\mu_c^{(Q)},\mu_s^{(Q)}) > \max_{\mu_c^{(i)},\mu_s^{(i)}} w(z_1',\ldots,z_Q'|z_l,\mu_c^{(1)},\mu_s^{(1)},\ldots,\mu_c^{(Q)},\mu_s^{(Q)}) \qquad (6.25)$$

$$l = 1,\ldots,m,\quad r \neq l$$

But with fixed values of $\mu_c^{(i)}$ and $\mu_s^{(i)}$ functions of $z_i'(t)$ are mutually independent (inasmuch as the interference in the different branches is considered independent). Furthermore, each z_i' depends only on $\mu_c^{(i)}$ and $\mu_s^{(i)}$ with the same superscript. Therefore

$$\max_{\mu_c^{(i)},\mu_s^{(i)}} w(z_1',\ldots,z_Q'|z_r,\mu_c^{(1)},\mu_s^{(1)},\ldots,\mu_c^{(Q)},\mu_s^{(Q)}) = \prod_{i=1}^{Q} \max_{\mu_c^{(i)},\mu_s^{(i)}} w(z_i'|z_r,\mu_c^{(i)},\mu_s^{(i)}) \qquad (6.26)$$

Substituting (6.26) in (6.25) and taking logarithms, we obtain the decision principle that signal $z_l(t)$ was transmitted in the following form

$$\sum_{\nu=1}^{Q} \max \ln w(\text{[illegible]}) > \sum_{\nu=1}^{Q} \max \ln w(\text{[illegible]}) \qquad (6.27)$$

$$r = 1, \ldots, m, \; r \neq l$$

According to (5.48a)

$$\max \ln w(\text{[illegible]}) = \ln N + \frac{V_{\nu}^{2}}{\text{[illegible]}}$$

where N is a constant which does not depend on z_r.

Substituting this expression in (6.27) we find the decision principle

$$\sum_{\nu=1}^{Q} \frac{V_{l\nu}^{2}}{\text{[illegible]}} > \sum_{\nu=1}^{Q} \frac{V_{r\nu}^{2}}{\text{[illegible]}} \qquad (6.28)$$

$$r = 1, \ldots, m, \; r \neq l$$

For an active-interval system when $E_l = E_r$ for all r we obtain

$$\sum_{\nu=1}^{Q} V_{l\nu}^{2} > \sum_{\nu=1}^{Q} V_{r\nu}^{2} \qquad (6.29)$$

$$r = 1, \ldots, m, \; r \neq l$$

It is easy to construct a decision system which is called a summation system from this principle. As was shown in Chapter IV, the values of V_r (or values proportional to them) may be derived by means of a quadrature system or matched filters plus envelope detectors. Figure 6.4 shows a decision system with matched filters in duplex reception.

In the particular case of duplex reception of active-interval binary signals the decision system may be somewhat different. Inequality (6.29) reduces to the following rule for recording symbol x_1

$$V_1^{(1)2} + V_1^{(2)2} > V_2^{(1)2} + V_2^{(2)2},$$

which may be transformed as follows

$$V_1^{(1)2} - V_2^{(1)2} + V_1^{(2)2} - V_2^{(2)2} > 0 \qquad (6.30)$$

It is easy to see that in order to fulfill this inequality it is necessary and sufficient that one of the two differences $(V_1^{(1)2} - V_2^{(1)2})$ and $(V_1^{(2)2} - V_2^{(2)2})$

which has the greatest absolute value be positive. Hence it follows that the optimum decision system may be constructed as follows (Figure 6.5). The values of $V^{\prime}$ are derived in the same way as in the system of Figure 6.4, but the indicated differences are formed and that one of them is chosen which is absolutely larger than the other. The decision is made in accordance with the sign of this difference. Here each signal element is in fact received by one branch, but this branch is the one in which the absolute value of the difference $V_1 - V_2$ is maximum. We will call this method of diversity reception the method of [illegible] selection. [illegible] that when [illegible], and also when [illegible], for binary systems, the method is not optimum.

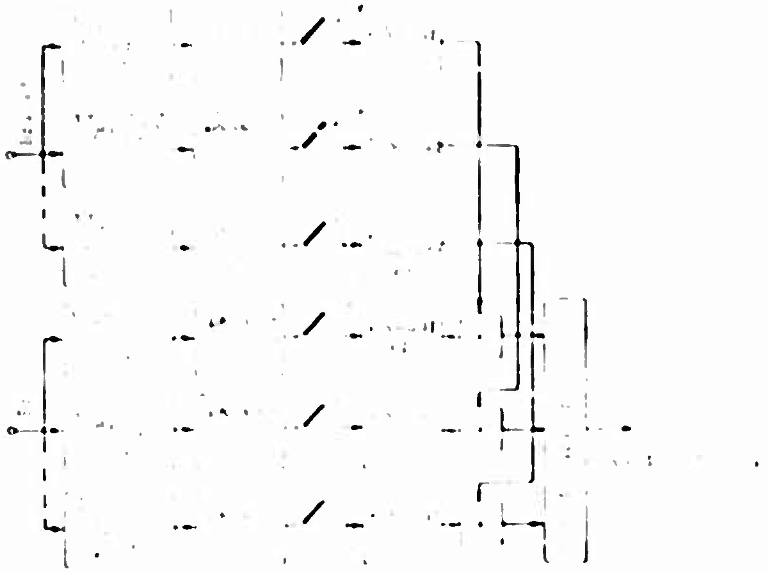


Figure 6.4. Quadratic Incoherent Summation for an Active-Interval System.

We will find the probability of error in optimal quadratic summation for the case when there is Rayleigh fading in a binary system with an active interval and orthogonal signals. The probability of correct reception in transmitting a certain symbol y_j is the probability of fulfilling inequality (6.29). This probability can be computed by knowing the probabilities of the sum of the squares of values of V.

From (6.31) it can easily be seen that the random variables V_r^{1} and Y_r^{2} in (4.25) and (4.29) have a normal probability distribution, are pairwise independent for a particular superscript, have zero average values, and their dispersions are equal to $E_c^2 \sigma_0^2 I$ when r ≠ [illegible] and $E_c^2 \sigma_0^2 I \cdot (1 + h_r^2)$ when r = [illegible]. Then the decision principle can be rewritten in the following form for reception

$$\sum_{i=1}^{q} ([illegible]) > \sum_{i=1}^{q} ([illegible]).$$

Dividing both sides of the inequality by [illegible], we obtain

$$\sum_{i=1}^{q} ([illegible]) > \sum_{i=1}^{q} ([illegible])$$

Here

$$[illegible]$$

and

$$[illegible].$$

We will introduce the following notation

$$X_1 = \sum_{i=1}^{q} ([illegible])$$

$$X_2 = \sum_{i=1}^{q} ([illegible])$$

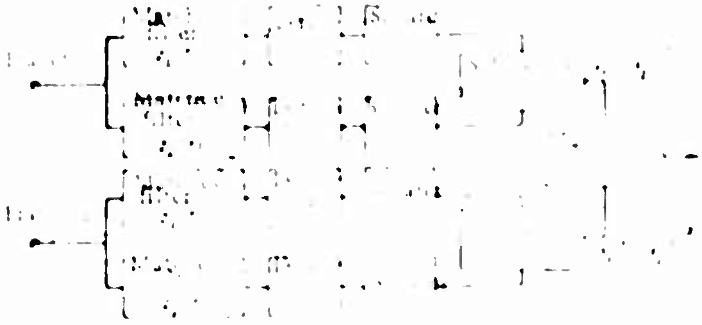


Figure 6.5. Maximum Likelihood Selection System

These magnitudes are [illegible] to the supposition of orthogonality [illegible] and [illegible] these variables are statistically independent. [illegible] correct reception is determined [illegible] is equal to

$$p = P\{X_1 > X_2\}$$

The distribution of variable [illegible] will be determined [illegible]

$$w(X_r) = \sum_{i=1}^{q} [illegible], \quad X_r > 0$$

$$w(X_r) = 0, \quad X_r < 0$$

where λ_i are the eigenvalues of the matrix $D = KV$ and are determined as the solution of the equation

$$\det(KV - \lambda I) = 0 \qquad (6.32)$$

The square matrix in the case under consideration will be $V = I$ and the matrix of correlation coefficients is

$$K = \|\overline{\xi_r \xi_l}\| \ldots, \quad r, l = 1, 2, \ldots, Q$$

computation of the elements of the correlation matrix yields:

$$\overline{\xi_r \xi_l} = [illegible] \begin{cases} 1 + [illegible], & r = l, \\ [illegible], & r \neq l, \end{cases} \qquad (6.33)$$

where $\delta_{rl} = \begin{cases} 1 \text{ when } r = l \\ 0 \text{ when } r \neq l \end{cases}$ is the Kronecker symbol, and R_{rl} are the coefficients of correlation of the quadrature components of the transmission coefficients r and l of reception branches determined by formula (6.1).

It is easy to see that when $r \neq l$ matrix $KV = I$, and $\lambda_i = 1$ when $i = 1, 2, \ldots, Q$ and by successive finding of the indeterminancies in (6.31) it is possible to obtain the distribution of λ_r in the form of a χ^2-distribution with 2Q degrees of freedom

$$\begin{aligned} w(\lambda) &= \frac{1}{2^Q (Q-1)!} \lambda^{Q-1} \exp\left(-\frac{\lambda}{2}\right), & \lambda \geq 0 \\ w(\lambda) &= 0 & \lambda < 0 \end{aligned} \qquad (6.34)$$

The probability of incorrect reception now can be found in light of (6.31) and (6.34) from the expression

$$p = P(\lambda_1 < \lambda_2) = \int_0^\infty w(\lambda_1)\, d\lambda_1 \int_{\lambda_1}^\infty w(\lambda_2)\, d\lambda_2 =$$

$$= \int_0^\infty \sum [illegible] \frac{\exp[illegible]}{\prod_{i=1} \left([illegible] \right)} d\lambda_1 \int_{\lambda_1}^\infty [illegible] \exp\left[-\frac{\lambda_2}{2}\right] d\lambda_2$$

By changing the order of summating and integrating and then integrating with respect to λ_2 and λ_1 we obtain

---

We will remind the reader that transmission of signal $z_1(t)$ is being discussed.

$$P = \sum\sum\prod \ldots$$ [illegible] (6.35)

For the most interesting case when independent Rayleigh fading of signals occurs in the receiving branches, by solving (6.32) in light of (6.33) when r = [illegible] we find that [illegible]$_{i1}$ = 1 + h_Q^2 for all i = 1, 2, ..., Q. By substituting [illegible]$_{i1}$ in (6.35) and successively removing the indeterminancies found in it, it is possible to find the following expression for the probability of error

$$P = \sum \ldots$$ [illegible] (6.36)

In the case of duplex reception of binary signals we find by substituting Q = 2 in expression (6.36) that

$$P = \ldots$$ [illegible] (6.37)

Formula (6.36) may be represented in more convenient form by using the notation k = Q + i − 1:

$$P = \ldots \sum C \ldots$$ [illegible] (6.38)

or

$$P = \sum C \ldots P_1 (1 - P_1) \ldots$$ [illegible]

where $P_1 = 1/(h_Q^2 + 2)$ after expression (5.17a) is the probability of error in single-transmission optimum incoherent reception if $h_0^2 = h_Q^2$.

When $h_Q^2 \gg 1$ we may assume $(h_Q^2 + 1)/(h_Q^2 + 2) \approx 1$, and taking into account the well-known identity

$$\sum C \ldots$$ [illegible]

derive a simple approximation of error probability

$$P \approx \ldots P_1 \ldots$$ [illegible]

Figure 6.6 represents the dependence of error probability on h_Q^2 in binary systems in duplex, triplex, and quadruplex reception.

Error probability is thus approximately inversely proportional to signal power in degree of Q.

In reception with spaced antennas when $h_Q = h_0$ an increase in branch number Q decreases the probability error. In frequency or time diversity the probability of error first decreases when Q is increased, but then increases because of the decrease in h_Q.

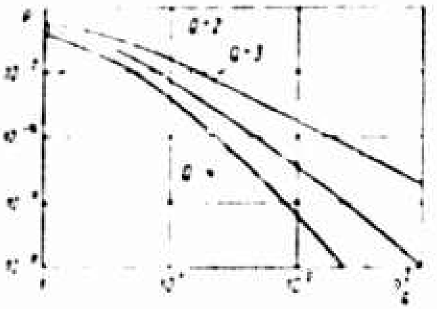


Figure 6.6. Error Probability in Quadratic Summation in Binary Orthogonal Systems (Rayleigh Fading).

The author of work [5] gives the results of computer computation of the optimal number of branches for time or frequency diversity with the assumption that = 1. This optimal value of Q is higher, the greater is the power of the signal. Thus, for $h_0 = 27.5$ $Q_{opt} = 10$ and for $h_0 = 100$ $Q_{opt} = 30$. Obviously when 1 the optimal number of branches will be less than that indicated.

Until now we have been concerned with resistance to interference in optimal incoherent reception, assuming no correlation among the transmission coefficients in the different branches. In fact, there always is some correlation which is measured by the coefficient of mutual correlation between the quadratic components R_{ij} as determined from formula (6.1).

As a result of this there is a correlation between the values of $V_r^{(i)}$ and $V_r^{(j)}$ in receiving signal r = which can be described by the correlation coefficients of their quadratic components as determined from formula (6.35).

Principle (6.29) was obtained with the assumption of previously unknown values of $_c^{(i)}$ and $_s^{(i)}$. It is altogether natural that with a previously known joint distribution of these values it would be possible without particular difficulty by averaging to obtain the conditional probability of signals received, to realize a decision principle optimal in this case, and thereby to somewhat improve the resistance to interference in reception. However, such a determination of the optimal decision system as far as practical value is concerned does not go beyond the framework of mathematical equations. In fact, if coherent reception is impossible, i.e., $_c^{(i)}$ and $_s^{(i)}$ cannot be predicted, then it is also impossible in practice to predict the values of R_{ij} and we cannot talk seriously of using these values in optimal processing.

Based on what has been said, how the existence of correlation between transmission coefficients in different branches of diversity reception affects the probability of error in an incoherent decision system designed in accordance with (6.29) [6] is of practical importance.

We will solve this problem for the case of greatest practical interest when there is duplex reception of binary signals orthogonal in the intensified sense in an active-interval system. The probability of error in this case is characterized by a general formula (6.35) for $Q = 2$. We will find the eigenvalues λ_{ij} by solving (6.32) which for the case under consideration can be put in the form

$$\begin{vmatrix} 1 + h_*^2 - \lambda & R_0 h_*^2 \\ R_0 h_*^2 & 1 + h_*^2 - \lambda \end{vmatrix} = 0,$$

where $R_0 = R = R_0$.

Solving this we find

$$\begin{aligned} \lambda_{11} &= 1 + h_*^2(1 + R_0) \\ \lambda_{21} &= 1 + h_*^2(1 - R_0) \end{aligned} \qquad (6.39)$$

Substituting (6.39) in (6.35) after transformation we obtain

$$p = \frac{3h_*^4(1 - R_0^2) + 10h_*^2 + 8}{[h_*^4(1 - R_0^2) + 4h_*^2 + 4]^2}. \qquad (6.40)$$

In the case of uncorrelated transmission coefficients when $R_0 = 0$

$$p = \frac{3h_*^2 + 4}{(h_*^2 + 2)^3}, \qquad (6.40a)$$

which coincides with the previously obtained formula (6.37).

If $R_0 = 1$, then

$$p = \frac{1}{8}\,\frac{5h_*^2 + 4}{(h_*^2 + 1)^2}. \qquad (6.40b)$$

For small probabilities of error when $h_*^2(1 - R_0^2) \gg 1$, it is possible to use instead of (6.40) the approximate expression

$$p \approx \frac{3}{h_*^4(1 - R_0^2)}. \qquad (6.40c)$$

Figure 6.7 shows the dependence of probability of error in the case of different values of R_0^2. As indicated above, the total coefficient of mutual correlation of the fluctuating part of the transmission coefficients is $R_f = R_0^2$. As can be seen from the figure, when $R_f < 0.6$, the existence of correlation has almost no effect on the effectiveness of duplex diversity reception. Even when $R_f = 0.8$ duplex reception provides for a power gain on the order of 10 db in comparison with single-transmission reception and only when $R_f > 0.8$ is the effectiveness of duplex reception significantly reduced.

Thus, the existence of correlation has a noticeable effect on duplex reception only when R_f if greater than 0.7-0.8. Therefore, for an approximate evaluation of the probability of error in the case of diversity reception it is possible to ignore the correlation between transmission coefficients in different branches if the coefficient of correlation is not very great.

In actual diversity reception systems the magnitudes of R_f usually do not exceed 0.6 although they very rarely are less than 0.2 [1]. The results obtained shew that further decrease in the coefficient of correlation by greatly increasing the spatial diversity of the antennas or partial separation does not yield a significant gain.

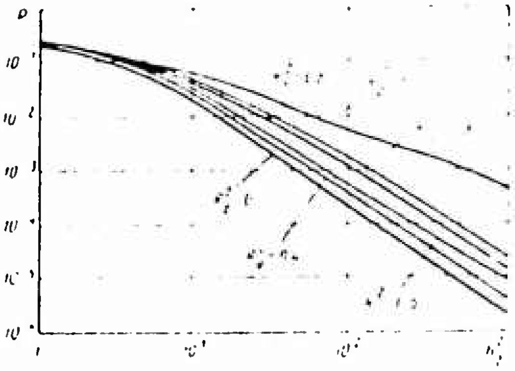


Figure 6.7. Probability of Error in the Case of Duplex Reception of Binary Orthogonal Signals with Account Taken of the Fading Correlation.

6.4. Nonoptimal Methods of Diversity Reception

Radio communications practice with diversity reception most often uses incoherent decision systems which differ from optimum. There are many variants of these systems. Knowing the probability distribution of the transmission coefficients we can in principle compute the error probability for any system and compare it with the optimum system. Here we will limit ourselves to a few examples which have to do with active-interval systems orthogonal in the intensified sense. We will consider the fading in the different receiving branches to be uncorrelated.

Maximum Likelihood Selection System

As applicable to binary systems this system (Figure 6.5) is based on the fact that each element is received in a branch for which the difference $V_1^{(i)2} - V_2^{(i)2}$ is maximum. This difference is proportional to the logarithm of the likelihood ratio, from which the system also gets its name.

It has already been pointed out that in duplex reception this system is completely equivalent to that of quadratic addition and is therefore optimum. When $Q > 2$ this system differs from the optimum.

We will use the notation

$$u_i = \frac{\mu_0^2}{\beta_S^2 \nu_0 l} \left(V_i^{(1)2} - V_i^{(2)2} \right)$$

to calculate the probability of error. It is easy to show that in the transmission of symbol y_1 the random variables u_i have the same probability density:

$$w(u) = \left\{ \begin{array}{ll} \dfrac{1}{2(h_Q^2+2)} e^{u/2} & \text{when } u < 0 \\ \dfrac{1}{2(h_Q^2+2)} e^{-\frac{u}{2(h_Q^2+1)}} & \text{when } u > 0 \end{array} \right\} \qquad (6.41)$$

An error in signal element reception occurs in one of Q incompatible events which are that the value of u_i in the i-th branch is negative and exceeds in absolute value the magnitude of u in the other (Q - 1) branches. The probability of such an event is

$$\int_{-\infty}^{0} w(u_i) \left[\int_{u_i}^{-u_i} w(u)\, du \right]^{Q-1} du_i$$

Consequently the probability of error is

$$p = Q \int_{-\infty}^{0} w(u_i) \left[\int_{u_i}^{-u_i} w(u)\, du \right]^{Q-1} du_i.$$

From considerations of symmetry it is obvious that error probability will also be the same during transmission of symbol y_2. Substituting expression (6.41) here and changing the notation of the variables we find

$$\begin{aligned} p &= \frac{Q}{(2h_Q^2+4)^Q} \int_{-\infty}^{0} e^{\frac{x}{2}} \left[\int_{x}^{0} e^{u/2} du + \int_{0}^{-x} e^{-\frac{u}{2(h_Q^2+1)}} du \right]^{Q-1} dx = \\ &= \frac{Q}{2h_Q^2+4} \int_{-\infty}^{0} e^{\frac{x}{2}} \left[1 - \frac{h_Q^2+1}{h_Q^2+2} e^{\frac{x}{2h_Q^2+2}} - \right. \\ &\qquad \left. - \frac{1}{h_Q^2+2} e^{\frac{x}{2}} \right]^{Q-1} dx. \end{aligned} \qquad (6.42)$$

This integral is easily computed for any value of Q. Its expression is not quoted in the general form because of its awkwardness. When Q = 2 the error probability agrees with expression (6.37), which corroborates the optimality of the selection network for duplex reception in a binary system. It follows for Q = 3 from expression (6.42) that

$$p = \frac{23h_1^4 + 57h_1^2 + 36}{(h_1^2+2)^2 (h_1^2+3)(2h_1^2+3)}.$$

To compare this system with the optimum one in triplex reception we will calculate p for Q = 3 from expression (6.36):

$$p = \frac{1}{(h_1^2+2)^3} \sum_{n=0}^{2} C_{n+2}^{n} \left(\frac{h_1^2+1}{h_1^2+2} \right)^n = \frac{10h_1^4 + 25h_1^2 + 16}{(h_1^2+2)^5}.$$

When $h_3^2 \gg 1$ in the optimum system of quadratic addition $p \approx 10/h_3^6$, while for the maximum likelihood selection system $p \approx 11.5/h_3^6$. Power loss in the selection system as compared to the optimum system is 0.2 dB. Therefore in triplex reception the maximum likelihood selection system is scarcely inferior to the optimum.

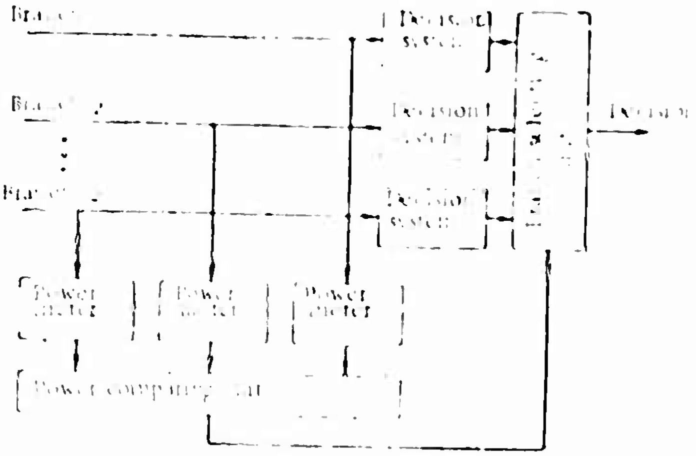


Figure 6.8. Maximum Power Selection System.

Maximum Power Selection System

One of the most widespread systems of diversity reception is the selection system with respect to maximum power (Figure 6.8). Here each branch has its own decision system (the same as in single-transmission reception), but the final decision is determined from the branch in which the power of the received signal is greater than in the others. There are many varieties of these systems which differ in the method of comparing signal powers in the branches and in the method of switching the branches. We will not pause over these. The basic underlying idea of these systems is that during fading the most reliable decision can be made in the branch in which transmission coefficient [illegible] assumes its greatest value at a given moment. It is, however, impossible to get a direct measurement of the transmission coefficient because of noise. Therefore this measurement is replaced by that of the power of the incoming signals (in conjunction with noise).

To estimate the noise resistance of such a circuit we will calculate the error probability in duplex reception of binary signals.

We will estimate the power of the incoming signal from the sum of $V_1^2 + V_2^2$. If symbol y_1 was transmitted, then an error may occur as a result of one of two incompatible events:

1) $V_1^{(1)2} + V_2^{(1)2} > V_1^{(2)2} + V_2^{(2)2}$ and moreover $V_1^{(1)} < V_1^{(2)}$

or

2) $V_1^{(2)2} + V_2^{(2)2} > V_1^{(1)2} + V_2^{(1)2}$ and moreover $V_2^{(1)} < V_2^{(2)}$.

Starting from considerations of symmetry the probability of error may be defined as

$$p = 2P\{V_1^{(1)} + V_1^{(1)} < V_1^{(2)} + V_2^{(2)}\} \qquad (6.43)$$

Let us designate

$$\frac{\mu_2}{R\sigma_0}(V_1^{(1)} + V_1^{(2)}) = \xi_2,$$

$$\frac{\mu_2}{R\sigma_0} V_1^{(1)} = \eta_0,$$

$$\frac{\mu_2}{R\sigma_0} V_1^{(2)} = \xi_1$$

and find the probability that $\xi_1 + \eta_1 < \xi_2$ which obviously agrees with the probability of the inequality $V_1^{(1)} + V_1^{(1)} < V_1^{(2)} + V_2^{(2)}$. The probability densities of ξ_1 and η_1 may be derived from expression (6.18)

$$\left.\begin{aligned} w_-(\xi_1) &= \begin{cases} \frac{1}{2}\exp\left(-\frac{\xi}{2}\right) & \text{when } \xi \geq 0, \\ 0 & \text{when } \xi < 0, \end{cases} \\ w_+(\eta_1) &= \begin{cases} \frac{1}{2(h_2^2+1)}\exp\left(-\frac{\eta}{2(h^2+1)}\right) & \text{when } \eta \geq 0, \\ 0 & \text{when } \eta < 0, \end{cases} \end{aligned}\right\} \qquad (6.44)$$

(the subscripts "+" and "-" correspond to the presence or absence of the appropriate signals), whence

$$P(\xi_2 \geq \eta_1 + \xi_1) = \int_0^k w_-(\xi)\int_0^{k-\xi} w_+(\eta)\,d\eta = 1 + \frac{1}{h_2^2}\exp\left(-\frac{k}{2}\right) - \frac{h_2^2+1}{h_2^2}\exp\left(-\frac{k}{2(h_2^2+1)}\right).$$

Substituting here the value $k = \xi_1 + \eta_1$, we find

$$P(\xi_2 \geq \eta_1 + \xi_1 + \eta_1) = 1 + \frac{1}{h_2^2}\exp\left(-\frac{\xi_1+\eta_1}{2}\right) - \frac{h_2^2+1}{h_2^2}\exp\left(-\frac{\xi_1+\eta_1}{2(h_2^2+1)}\right). \qquad (6.45)$$

That we may find the probability written in the right side of expression (6.43) expression (6.45) must be averaged with respect to all the values of ξ_1 and η_1 which satisfy condition $\xi_1 < \eta_1$. By doubling this probability we find error probability

$$p = 2\int_0^\infty\int_{\eta_1}^\infty \frac{1}{4(h_2^2+1)}\exp\left(-\frac{\xi_1}{2} - \frac{\eta_1}{2(h_2^2+1)}\right)\times.$$

$$\times\left[1 + \frac{h_1'}{h}\exp\left(-\frac{\xi_1 \ldots}{2}\right) - \frac{h_2'+1}{h}\exp\left(-\frac{\xi_1 \ldots}{2(\ldots+1)}\right)\right]d\xi_1\,d\xi_2 = \frac{4 + \ldots + 16}{(h \ldots)\ldots(h\ldots+4)} \tag{6.46}$$

Figure 6.9 (curve 2) represents this relationship. Comparing it to curve 1 plotted from formula (6.37) for quadratic addition we may satisfy ourselves that the difference between them is not very significant.

If $h_2^2 \gg 1$, then in quadratic addition $p \approx 3/h_2^4$, while in maximum power selection $p = 15/5h_2^4$, i.e., the power gain in the maximum power selection network (as compared to the optimum system of incoherent addition) is no more than 0.8 db.

General Comparison System

In a general comparison network (Figure 6.10) incoherent diversity reception is accomplished as follows. Values of $V_r^{(i)}$ are formed in each branch by quadrature reception or with matched filters. All of these values are compared to each other in a single comparator and the decision is made in accord with the subscript of the maximum value of $V_r^{(i)}$.

Let us, as in the preceding case, pass from the values of $V_r^{(i)}$ to those $\xi_r^{(i)} = \frac{\mu_i}{P_{r,i}} V_r^{(i)}$ $(r = 1, \ldots, m,\ i = 1, \ldots, Q)$ whose probabilities densities are expressed by formula (6.44).

Correct reception during transmission of some symbol y_1 occurs when one of the Q incompatible and equiprobable events which consist in some value of $\xi_1^{(i)}$ $(i = 1,\ldots,Q)$ exceeding each of the other $(mQ - 1)$ values of $\xi_r^{(i)}$ also occurs.

The probability of correct reception therefore is

$$1 - p = Q\int_0^\infty w_1(x)\left[\int_0^x w_1(y)\,dy\right]^{Q-1}\left[\int_0^x w_0(z)\,dz\right]^{(m-1)Q} dx =$$

$$= \frac{Q}{2(h_Q'+1)}\exp\left(-\frac{x}{2(h_Q'+1)}\right)\times$$

$$\times\left[\int_0^x \frac{1}{2(h_Q'+1)}\exp\left(-\frac{y}{2(h_Q'+1)}\right)dy\right]^{Q-1}\times$$

$$\times\left\{\left[\int_0^x \frac{1}{2}\exp\left(-\frac{y}{2}\right)dy\right]^{(m-1)Q} dx = \frac{Q}{2(h_Q^2+1)}\times$$

$$\times\int_0^\infty \exp\left(-\frac{x}{2(h_Q^2+1)}\right)\left[1-\exp\left(-\frac{x}{2(h_Q^2+1)}\right)\right]^{Q-1}\times$$

$$\times\left[1-\exp\left(-\frac{x}{2}\right)\right]^{(m-1)Q}dx =$$

$$= \frac{Q}{2(h_Q^2+1)}\sum_{k=0}^{Q-1}\sum_{n=0}^{(m-1)Q}(-1)^{k+n}C_{Q-1}^{k}C_{(m-1)Q}^{n}\times$$

$$\times\int_0^\infty \exp\left[-x\left(\frac{k+1}{2(h_Q^2+1)}+\frac{n}{2}\right)\right]dx =$$

$$= Q\sum_{k=0}^{Q-1}\sum_{n=0}^{(m-1)Q}(-1)^{k+n}C_{Q-1}^{k}C_{(m-1)Q}^{n}\frac{1}{n(h_Q^2+1)+k+1},$$

while probability of error is

$$p = 1 - Q\sum_{k=0}^{Q-1}\sum_{n=0}^{(m-1)Q}(-1)^{k+n}C_{Q-1}^{k}C_{(m-1)Q}^{n}\times \tag{6.47}$$

$$\times\frac{1}{n(h_Q^2+1)+k+1}.$$

This error probability is, of course, greater than in the case of the optimum system of quadratic addition (6.38), but computations show that the difference between these probabilities is very insignificant.

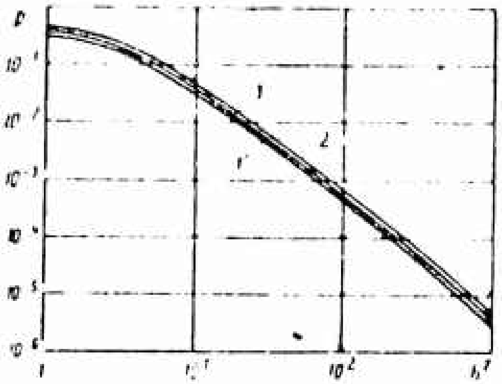


Figure 6.9. Probability of Error in Orthogonal Binary Systems for Different Duplex Reception Networks: 1, Quadratic addition; 2, Maximum power selection; 3, General comparison.

In the particular case of duplex reception in a comparison network we will, by substituting Q = 2 in expression (6.47), derive

$$p = 2\sum_{n=1}^{2m-2}(-1)^{n+1}C_{2m-2}^{n}\frac{1}{(nh_2^2+n+1)(nh_2^2+n+2)}. \tag{6.48}$$

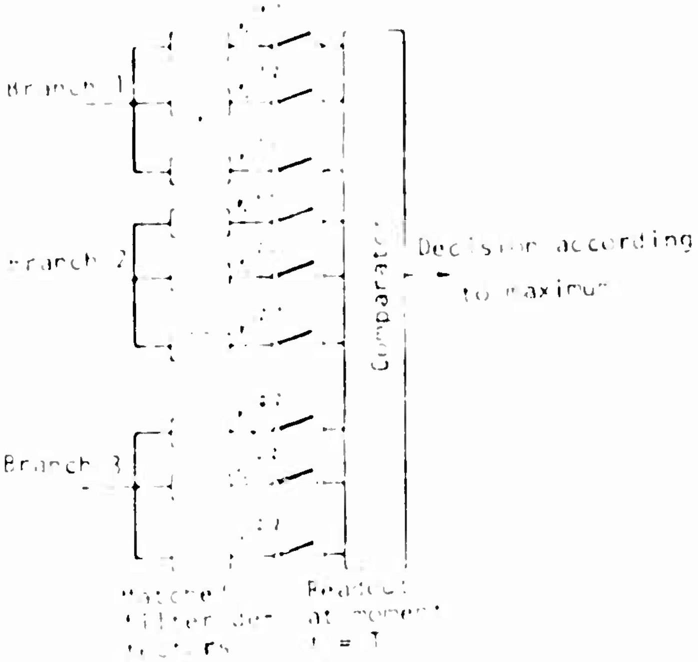

Branch 1
Branch 2
Branch 3
Comparator
Decision according to maximum
Matched filter detectors
Readout at moment t = T

Figure 6.10. Diversity Reception System Using General Comparison Method.

In duplex reception of binary [illegible] we find

$$P [illegible] \qquad (6.48)$$

[illegible]

$$P [illegible] \qquad (6.49)$$

Relationship (6.48) [illegible] difference in noise resistance between the [illegible] quadratic addition [illegible] practically imperceptible. The power gain in the general comparison in relation to the optimum system of quadratic addition [illegible] reception of binary signals is no more than 0.4 db.

For binary systems when [illegible] the general comparison network from expression (6.47) we find

$$p = Q\sum\sum [illegible] \qquad (6.50)$$

This probability differs little from (6.48). The power gain in comparison with the quadratic addition system when [illegible] 4 does not exceed 0.8 db. The general comparison system is realized most simply when diversity reception is according to frequency.

Linear Addition System

One of the simplest systems of diversity reception is the linear addition system which realizes the following decision principle for reception of symbol y:

$$\sum_{i=1}^{q} [illegible] \gtrless \sum_{i=1}^{q} [illegible] \qquad (6.[illegible])$$

Here in each branch there is a network for computing the values of λ_i [illegible] as in single transmission reception, and the final decision is made based on the results of linear addition of the values of [illegible]. Such a principle can be realized in the system shown in Figure 6.4 in which the operation of squaring is excluded. Unfortunately, in the general case the probability of error in a linear addition system cannot be expressed in elementary functions. Nevertheless, this probability can be obtained rather simply in the case of double reception of binary systems with independent Rayleigh fading.

The density of distribution of random variable [illegible]

$$[illegible]$$

can be obtained in the following form [illegible]

$$w(x) = [illegible] \quad \text{when } x \geq 0,$$
$$w(x) = 0 \quad \text{when } x < 0.$$

$$w(u) = [illegible] \exp[illegible] \cdot \exp[illegible]\,\Phi([illegible]) \quad \text{when } u \geq 0,$$
$$w(u) = 0 \quad \text{when } u < 0,$$

where Φ [illegible] function.

Now the probability of error can be found in the usual way:

$$P = \int_0^\infty w(x)\,dx \int_x^\infty w(u)\,du \qquad (6.[illegible])$$

Substituting (6.[illegible]) in (6.52) and performing simple but rather cumbersome calculations, we obtain

$$P = \frac{1}{h_2^2+1}\left[1 - \frac{2}{h_2^2+2}\sqrt{\frac{h_2^2+1}{h_2^2+3}}\,\mathrm{arctg}\sqrt{\frac{h_2^2+1}{h_2^2+3}} - \frac{2(h_2^2+1)}{h_2^2+2}\,\frac{1}{\sqrt{h_2^2+3}}\,\mathrm{arctg}\frac{1}{\sqrt{h_2^2+3}}\right] \qquad (6.54)$$

With high reception fidelity when $h_2^2 \gg 1$, formula (6.54) allows a simple asymptotic representation

$$P \approx \frac{3.21}{h_2^4} \qquad (6.55)$$

Comparing this result with (6.53) we see that the power gain in the transition from quadratic addition to linear does not exceed 0.2 db. Thus, the simpler system of linear addition has practically the same resistance to interference as an optimal system.

To summarize the conclusions of this section we may affirm that all practically usable methods of diversity reception provide a noise resistance which differs little from potential. Therefore it is impossible to get any perceptible noise resistance gain by the use of any new networks of diversity reception. It should not, however, be forgotten that all the results derived are applicable to actual instrumentation only on condition that it actually operates in conformity with the decision principles studied. The noise resistance of actual equipment for diversity reception is in fact often considerably poorer than theoretical because of divergences from the decision principle. These divergences are caused in particular by an amplification differential in the diversity receiving branches. Further discussion of this question goes beyond the bounds of this work [4].

6.5. Discrete Addition

In some cases it is convenient to use the simple, though far from optimum, method of diversity reception which is based on the utilization by each branch of an independent decision system which reduces a decision to determining the probable transmitted symbol from the signal in a given branch. The final decision is made from a comparison of the "particular" decisions reached in each of the branches. Here neither the differences in likelihood of the particular decisions nor the power differentials between the received signals are taken into consideration, as they were in a selection system. Since all the branches are considered to be equally right the most likely symbol is the one which is registered in the greatest number of branches. This method of diversity reception is especially convenient in time variant diversity reception because it requires only that discrete values be remembered.

Generally speaking, this decision principle can lead to indeterminancy if two or more different symbols are registered in the same number of branches. In the particular case, however, where the system is binary, while the number of branches is odd this indeterminancy cannot arise. We shall compare the noise resistance of this method of discrete addition and limit ourselves to this particular case.

If the number of receiving branches is $Q = 2q - 1$, then error probability p equals the probability that an erroneous symbol has been registered in q or more branches. If the error probability in one branch is denoted by p_1, then

$$p = \sum_{k=q}^{2q-1} C_{2q-1}^{k} p_1^{k} (1 - p_1)^{2q-1-k} \qquad (6.56)$$

This formula may be interpreted as follows. Let us assume that a series of some experiments or other is carried out with a probability of successful issue p_1 in each test. The series consists of $2q - 1$ tests. Let us stipulate that event A has occurred if there have been q or more positive outcomes in this series. Then, obviously, the probability of occurrence of event A will be expressed by the binomial law

$$P(A) = \sum_{k=q}^{2q-1} C_{2q-1}^{k} p_1^{k} (1 - p_1)^{2q-1-k},$$

which coincides with expression (6.56).

But by a somewhat different line of reasoning another expression may be derived for P(A). In actuality, in order to determine that event A has happened it is not at all obligatory to carry the series of tests to its conclusion. It suffices to continue the tests until we have obtained q positive outcomes, and it may then be asserted that event A has occurred because any succeeding tests cannot change this fact. Only if $2q - 1$ tests have been performed and a positive outcome has not occurred q times need we draw the conclusion that event A has not occurred.

From this point of view we will determine the probability that the occurrence of event A has been ascertained after the m-th test. This means that in the preceding $(m - 1)$ tests there were $(q - 1)$ positive outcomes and the m-th test also gave a positive outcome. The probability of this is

$$C_{m-1}^{q-1} p_1^{q-1} (1 - p_1)^{m-q} p_1 = C_{m-1}^{q-1} p_1^{q} (1 - p_1)^{m-q}.$$

Event A may be ascertained no sooner than the q-th trial. Therefore the probability of the occurrence of an event may be represented as the sum of the probabilities of confirming event A after the m-th trial, taken with respect to all values of m from $m = q$ to $m = 2q - 1$, i.e.,

$$P(A) = \sum_{m=q}^{2q-1} C_{m-1}^{q-1} p_1^{q} (1 - p_1)^{m-q} \qquad (6.57)$$

If now we denote by n the number of tests performed after the q-th test, i.e., $n = m - q$, then

$$P(A) = \sum_{n=0}^{q-1} C_{q+n-1}^{n} p_1^{q} (1 - p_1)^{n} \qquad (6.58)$$

Hence ensues the identity

$$\sum_{k=q}^{2q-1} C_{2q-1}^{k} p_1^{k} (1 - p_1)^{2q-1-k} = \sum_{n=0}^{q-1} C_{q+n-1}^{n} p_1^{q} (1 - p_1)^{n} \qquad (6.59)$$

Let us now assume that in every branch of a discrete addition system there exists optimum incoherent reception; then, according to expression [illegible]:

[illegible]

and the right side of expression [illegible] agreed with expression [illegible] for the probability of error in optimum quadratic addition [illegible] because [illegible]

This is all the more [illegible] the error [illegible] system where [illegible] that of [illegible] addition system where [illegible]. Therefore the [illegible] increasing the number of branches [illegible]. It [illegible] course, here assumed that as the number of branches increases the signal power per branch is [illegible].

Thus, for example, [illegible] reception [illegible] spaced antennas with [illegible] which is equivalent [illegible] reception [illegible] separate receivers with selection of the best [illegible] the receivers [illegible] separate receivers with [illegible].

These findings make it possible to compare the discrete addition [illegible] with [illegible] which [illegible]

Notes

[illegible]

[illegible]

[illegible]

[illegible] to the generalized criterion [illegible] maximal likelihood and therefore are optimal under conditions when the distribution [illegible] of transfer [illegible] coefficients are unknown. [illegible] for a channel in which the distribution of probabilities of transfer coefficients is known exactly, [illegible] maximal likelihood and [illegible] the optimal [illegible] taking a maximum by averaging with [illegible] principle obtained with [illegible] for greater accuracy than [illegible]

[illegible] signals [illegible] Figure [illegible] shows the [illegible]

[illegible] based on [11] [illegible]

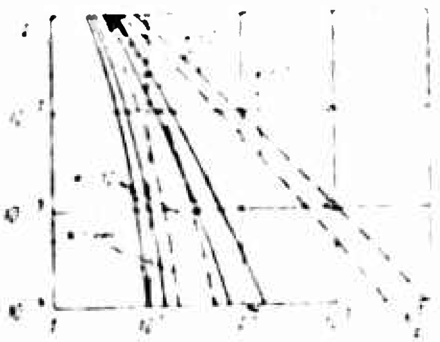


Figure [illegible]

[illegible]

Expressions for the probability of error in the case of correlation fading and Q = 2 are obtained in [6, 9, 11, 13, 15] and in many other works. In [11] the resistance to interference in the case of diversity reception of nonorthogonal signals is also determined.

Attention should be drawn to the fact that the coefficient of mutual correlation of two Rayleigh values (just as the coefficient of autocorrelation of a Rayleigh process) is a non-negative value. Therefore, diversity reception in the case of Rayleigh fading is most effective when the fading in the branches is mutually uncorrelated. In the general case when the fading is not Rayleigh a negative fading correlation is possible. Obviously, diversity reception in the case of negatively correlated fading is more effective than in the case of independent fading in branches.

Unfortunately, an analysis of diversity reception in the case of correlated fading which is not Rayleigh entails great difficulty inasmuch as a multidimension of coefficients of transmission in this case is not unambiguously determined by two moments. The results obtained for duplex reception in the dual beam channel model ([illegible]) and also several generalizations of them will be published in work [4].

5. (See Section 6.3) Incoherent diversity reception in a binary RPM system can be viewed from the position set forth in Chapter II, i.e., based on the fact that a decision is reached through an analysis of a signal in the interval (0, 2T). In this case the signals are orthogonal in the intensified sense and the power of a signal element should also be determined over a double interval of time. Therefore, for an RPM system the probability of error is also determined by formulas (6.36), (6.35), (6.40), (6.42), (6.30), (6.38), (6.33) depending on the diversity reception system and the existence of fading correlation in which, however h^2 should be replaced with $2\bar{h}_0^2$. Diversity reception of RPM signals is considered in more detail in [3, 13, 14 and 15].

6. (See Section 6.3) Formula (6.35) for Rayleigh fading, as shown in [15], holds not only in the case of incoherent diversity reception by the method of quadrature addition, but in the case of optimal coherent diversity reception of binary signals if p_1 is understood to mean the probability of error for single-transmission coherent reception. Specifically, it holds for diversity reception of FM signals.

7. (See Section 6.3) All expressions for the probability of error introduced in the text are obtained under the supposition that the average values of the ratio between signal power and spectral noise density in all diversity branches are the same and equal to $\bar{h}_0^2$. In several cases this does not take place. The optimal system of incoherent diversity reception when in the q-th diversity branch the ratio between the signal power and the spectral noise density is equal to $\bar{h}_q^2$ (q = 1, ..., Q) and the values of h_q are unknown is obtained in [8]. The probability of error in the case of duplex reception in this system and with Rayleigh fading is equal to

$$P=\frac{2(h_1^2+h_2^2)+3h_1^2h_2^2}{(2+h_1^2)(2+h_2^2)(h_1^2+h_2^2+h_1^2h_2^2)}. \tag{6.61}$$

If, however, the values of h_q are unknown, then the quadrature addition systems remains optimal (in the sense of the generalized criterion of maximal likelihood). The probability of error in this case is [12]

$$P=\sum_{q=1}^{Q}\sum_{l=1}^{Q}\frac{(h_q^2+1)^{Q-l}}{(h_q^2+2)^{l}\prod\limits_{\substack{i=1\\ i\neq q}}^{Q}(h_i^2-h_q^2)} \tag{6.62}$$

Specifically, for duplex reception from this formula we have

$$P=\frac{5(h_1^2+h_2^2)+3h_1^2h_2^2+8}{(2+h_1^2)^2(2+h_2^2)^2}. \tag{6.62a}$$

In the derivation of these formulas it was assumed that the spectral noise density in all branches is the same and the power of the signals differs.

A comparison of (6.62a) and (6.61) shows that when $h_1 \gg 1$ and $h_2 \gg 1$ both these formulas coincide asymptotically:

$$P\approx\frac{3}{h_1^2h_2^2}. \tag{6.62b}$$

Sometimes use is made of the method of diversity reception with coherent addition of signals and subsequent incoherent rectification [19]. With respect to resistance to interference it occupies an intermediate position between optimal coherent diversity reception and quadrature addition. The probability of error in the general case when the average values of the ratio between signal power and spectral noise density h_q^2 in the different branches is not the same as shown in [19], is

$$P=\frac{2^{Q-1}}{\prod\limits_{q=1}^{Q}(2+h_q^2)}. \tag{6.63}$$

As already indicated, the probability of error in actual diversity reception systems is greater than a theoretical system mainly due to the dissimilar amplification factors in the diversity branches. In the case of Rayleigh fading and duplex reception using a quadrature addition system and also a maximal power selection system the dependence of the probability of error on the asymmetry of amplification factors is computed in [12]. It is interesting to note that the quadrature addition system is much less sensitive to asymmetry than the maximal power selection system.

6. Carrying capacity of a channel with spaced reception. The spaced reception allows larger quantities of information to be derived from the signal than individual reception. Therefore it is possible to say that the increased carrying capacity of the channel is due to the utilization of spaced reception.

In case of optimum coherent addition, it is possible to estimate the carrying capacity on the basis of the Bpennan theory [2], which says that the resultant ratio of the signal power to the noise power is equal to the sum of the corresponding ratios in all branches. If all branches are identical, then

$$h^2_{рез} = Q h^2_q = Q h^2 \qquad (6.6)$$

By [illegible] (reception on the spaced antenna) the ratio increases by a factor of Q. It should be mentioned that the same result could be obtained by using a single antenna whose area is equal to the sum of the areas of all spaced antennas. If fading is absent, then in this case it is possible to find the spacing capacity by substituting the quantity QP_c for P_c in Shannons formula (3.84), with P_c understood to be the power of the signal in one branch of the receiver.

By the fading Rayleighs, if the latter is rigidly correlated in all branchos, it is possible to find the spacing capacity by way of substituting P_c for QP_c in formula (5.85). If, however, fading is not correlated in the branches, then, as it was noted in footnote 1, apart from the structure the power has in a reduced space and dispersed quantities $h_{рез}$, and with great significance the [illegible] conditions of reception approach a condition in the channel without fading as in the carrying capacity [illegible] (3.84) by substituting QP_c for P_c.

By [illegible] =1 from (6.6) it follows that resulting value h is not increased, and by [illegible] 1 it diminishes with the increase [illegible]. In these conditions the spaced reception permits the carrying capacity to increase only to account for the decreased dispersion h . Then with a conversion on the same resulting power of the signal the spacing capacity is approached with the increase of [illegible] from (5.85 to 3.84). As we already covered, the maximum difference between values in these terms cannot exceed 17%

In [16] the author computed the carrying capacity in the case of diversity reception in the absence of fading but allowing for the correlation of interference in the diversity branches. In [17] the author determines the carrying capacity of a channel formed by selecting the branch with the maximum transmission coefficient in the case of Rayleigh fading. Such a system is not physically realizable since the existence of interference foes not permit measuring exactly the coefficients of transmission buc it may serve approximately as a mathematical model of a system for selecting based on maximum power. For the carrying capacity the following expression is obtained:

$$C = F\sum_{k=1}^{Q} (-1)^{k} C_Q^{k} \exp\left(k\frac{P_n}{P_{c1}}\right) Ei\left(-k\frac{P_n}{P_{c1}}\right)\left[\frac{\text{natural electrons}}{\text{sec}}\right] ,$$

where P_{c1} is the average signal power in one branch. When Q = 1 this expression coincides with (5.8) and with an increase in Q it approaches the carrying capacity of a channel without fading with a signal power of P_{c1}.

The carrying capacity of diversity reception in the case of Rayleigh and gaussian* fading without reference to an addition system was calculated in work [15] as the upper limit of rate of transmission of the information contained in an aggregate of received signals in all branches with respect to a transmitted signal with variations of all possible transmitted signals with a given average power.

---

* Gaussian fading is an approximation of fading following a generalized Rayleigh distribution when the regular component is relatively large. In this case the interference was considered to be normal but not necessarily with a uniform spectrum. The correlation of transmission coefficients and also interference in the diversity branches was considered.

BIBLIOGRAPHY

1. Grisdale, G. L., J. G. Morris and D. S. Palmer, "Fading of Long-Distance Radio Signals and a Comparison of Space and Polarization Diversity Reception in the 6-18 Mc Range," PIEE, Pt. B, No. 13, 1957.

2. Brennan, D. G., "On the Maximum Signal-To-Noise Ratio Realizable from Several Noisy Signals," PIRE, No. 10, 1956.

3. Middleton, D., [illegible] [Introduction to Statistical Communication Theory], Vol. II, "Sovetskoye radio" Press, 1962.

4. Andronov, I. S. and L. M. Fink, [illegible] [Transmission of Discrete Messages Over Parallel Channels], "Sovetskoye radio" Press (in print).

5. Pierce, "Theoretical Limitations on Frequency and Time Diversity for Fading Binary Transmissions," IRE Trans., CS-9, No. 2, 1961.

6. Pierce, J. N. and S. Stein, "Multiple Diversity with Nonindependent Fading," PIRE, No. 1, 1960.

7. Levin, B. R., [illegible] [Theoretical Fundamentals of Statistical Radio Technology], Vol. I, "Sovetskoye radio" Press, 1966.

8. Turin, G. L., "Some Computation of Error Rates for Selectively Fading Multipath Channels," Proc. of Nat. Electron. Conf., Vol. XV, 1959.

9. [illegible], B. I., [illegible] [Transmission of Discrete Messages Over Radio Channels], "Svyaz'" Press, 1969.

10. [illegible], I. B., [illegible] [Potential Resistance to Interference in the Case of Diversity Reception of Discrete Information], Radiotekhnika, No. 5, 1961.

11. Turin, G. L., "On Optimal Diversity Reception," IRE Trans., IT-7, No. 3, 1962, CS-10, No. 1, 1962.

12. Fink, L. M. and I. S. Andronov, [illegible] [Resistance to Interference in One Method of Diversity Reception], Radiotekhnika, No. 8, 1966.

13. Lindsey, W. C., "Asymptotic Performance for the Adaptive Coherent Multireceiver and Incoherent Multireceiver Operating Through Rician Fading Multichannel," Trans. IEEE, Comm. Syst., No. 1, 1964.

14. Voelcer, H. B., "Phase-Shift Keying in Fading Channels," PIEE, Pt. B, No. 31, 1960.

15. Khvorostenko, N. P., [illegible] [Statistical Theory of Demodulation of Discrete Signals], "Svyaz'" Press, 1968.

16. Ovseyevich, I. A. and M. S. Pinsker, *Skorost' peredachi informatsii, propusknaya sposobnost' mnogoputevoy sistemy i priyem po metodu lineyno-operatornogo preobrazovaniya* [Rate of Transmission of Information, Carrying Capacity of Multipath System, and Reception by the Linear-Operator Conversion Method], Radiotekhnika, No. 3, 1959.

17. Nemirovskiy, A. S., *Propusknaya sposobnost' mnogoluchevogo kanala pri raznesennom priyeme s avtovyborom* [Carrying Capacity of Multibeam Channel in the Case of Diversity Reception with Automatic Selection], Radiotekhnika, No. 9, 1961.

18. Ovseyevich, I. A., *Propusknaya sposobnost' mnogoputevoy sistemy* [Carrying Capacity of Multipath System], In the collection: Problems of Information Transmission, Issue No. 14, 1963.

19. Andronov, I. S., *Potentsial'naya pomekhoustoychivost' odnogo iz metodov raznesennogo priyema* [Potential Resistance to Interference of One Method of Diversity Reception], Radiotekhnika, No. 6, 1964.

20. Andronov, I. S. and L. M. Fink, *K voprosu ob otsenke effektivnosti sistem raznesennogo priyema* [On Evaluating the Effectiveness of Diversity Reception Systems], Radiotekhnika. No. 4, 1968.

CHAPTER VII

CHANNELS WITH PARAMETERS DEPENDING ON FREQUENCY AND WITH FAST FADING

7.1. General Description of a Linear Communication Channel

The preceding chapters have discussed the conditions for transmitting signals in channels whose parameters do not depend on frequency. In all these channels noise resistance, as we have seen, does not depend on the fine structure of the signals, in particular on their base 2FT and other spectral characteristics, but is determined only by the signal power and the pairwise "scalar products" of the signals. For example, for all active-interval systems orthogonal in the intensified sense the probabilities of error are identical if the code base and signal element power are the same. In particular, the rate of information transmission in these channels may be made arbitrarily great at a given error probability by decreasing the element length and proportionally increasing signal power because this retains the ratio of signal element power to spectral noise density.

This and the succeeding chapters will study channels in which additive noise or distortion depends on frequency. The reception conditions in such channels depend both on the general power characteristics of the signal and on its spectrum.

A channel with selective fading is, as has already been noted in Chapter V, characterized by the signal coming to the receiving unit over several paths with travel differentials of Δt_i commensurate with the value 1/F or exceeding it, where F is the nominal signal frequency band. If, furthermore, the values of Δt_i are commensurate with signal element length T, then multipath propagation causes both selective fading and imposition of adjacent signals on each other (echoing).

If 2FT, the base of the system, is no more than several units, the values of 1/F and T are of the same order. Therefore in these signals the values of Δt_i are commensurate with 1/F only when they are close to T and the phenomenon of selective fading is always observed together with the echo-phenomenon. When base 2FT is large cases are possible where Δt_i is commensurate with 1/F, but substantially less than T. In these cases selective fading occurs without perceptible imposition of adjacent signal elements.

The interference of the components of the incoming beams and the imposition of the adjacent signal elements hinder reception. On the other hand, each arriving beam carries information about the message which is being transmitted, which, generally speaking, should increase the possibility of reliably distinguishing signals, as compared with the conditions in a single-path channel.

Up until relatively recently selective fading and echoing were regarded only as factors impeding communications and reducing carrying capacity and reception fidelity. In designing communication systems for multipath channels every effort was bent toward overcoming the effect of the arriving beams, except the first (or the most powerful) and toward approximating reception conditions in a "single-beam" channel (i.e., channel with general fading). Not a few clever ideas were proposed for this purpose, many of which have not lost their practical value even today. The chief ones will be described below.

Of late years (beginning about 1957) the approach to multibeam channels has drastically changed. Instead of combatting the multibeam nature of the signal efforts are made to utilize to some degree the information carried by each beam and to secure greater fidelity (or carrying capacity) in multibeam than in the single-beam channel.

It should not be thought that multibeam propagation is found only in radio channels. Multiple reflections, although less clearly expressed, are also found in wire channels, for example, as a consequence of nonuniform cable. Apparently they will also be inevitable in future waveguide channels.

Channel Models

The most general description of passage of a signal through a linear channel is provided by a random impulse transfer function[1] $H(t,\tau)$ which expresses the value of reaction at a channel output at instant t if at instant $t-\tau$ a single impulse (delta-function) is delivered to the channel input (Figure 7.1).

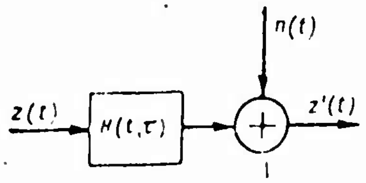


Figure 7.1. General Diagram of a Channel with Frequency-Dependent Variable Parameters.

On the basis of physical practicability any realization of a transfer function satisfies the condition

$$H(t,\tau)=0 \text{ when } \tau<0, \tag{7.1}$$

since the reaction at the output cannot occur any sooner than the action at the input. If a signal $z(t)$ arrives at a channel input, the signal at the output (without considering additive interference) is equal to

---

[1] These are other forms for expressing a channel impulse transfer function [1-3]. The expression used here is the most convenient.

$$z'(t) = \frac{1}{2}\int_0^{\infty} H(t, \tau)\, z(t-\tau)\, d\tau. \tag{7.2}$$

Inasmuch as $H(t,\tau)$ is a random function, then $z'(t)$, with an unchanging realization of $z(t)$, will also be a random function, even in the absence of additive interference. Therefore, the probability of error in such a channel with a reduction in interference spectral density, generally speaking, does not approach zero.

For any realization of $H(t - \tau)$ it is possible with an unchanging value of t to define the instantaneous transfer function as a Fourier transform with respect to τ:

$$Y(j\omega, t) = \int_0^{\infty} H(t, \tau) \exp(-j\omega\tau)\, d\tau \tag{7.3}$$

Here the lower limit of integration is zero in accordance with (7.1).

We will note that $Y(j\omega,t)$ does not have such a simple physical meaning as the transfer function $Y(j\omega)$ of a circuit with constant parameters which amounts to a ratio between complex amplitudes constituting, with an angular frequency of ω at the output and input circuits in a steady state. In a system with variable parameters a steady state, strictly speaking, generally does not occur. Therefore, it cannot be considered that the spectrum of a signal at the output of the channel is equal to $S_r(j\omega)Y(j\omega,t)$, where $S_r(j\omega)$ is the spectrum of realization if signal $z_r(t)$ at channel input.

Nevertheless, $Y(j\omega,t)$ can be considered as a complex signal at the channel output when an analytic monochromatic signal at a frequency of ω and a single amplitude of $z(t) = \exp(j\omega t)$ is delivered to the input. Indeed, substituting this value of $z(t)$ in (7.2) we obtain

$$\begin{aligned} z'(t) &= \int_0^{\infty} H(t, \tau) \exp(j\omega t - j\omega\tau)\, d\tau = \\ &= \exp(j\omega t)\int_0^{\infty} H(t, \tau)\exp(-j\omega\tau)\, d\tau = z(t)\, Y(j\omega, t). \end{aligned} \tag{7.4}$$

For an actual monochromatic input signal $z(t) = \cos(\omega t + \varphi)$ this means that the output signal is equal to

$$\begin{aligned} z'(t) &= \cos(\omega t + \varphi) \operatorname{Re} Y(j\omega, t) + \sin(\omega t + \varphi) \operatorname{Im} Y(j\omega, t) = \\ &= |Y(j\omega, t)| \cos[\omega t + \varphi - \psi(t)], \end{aligned}$$

where

$$\psi(t) = \operatorname{arctg} \frac{\operatorname{Im} Y(j\omega, t)}{\operatorname{Re} Y(j\omega, t)}.$$

Here it is important to note that the output signal will not be monochromatic inasmuch as it is modulated in amplitude and phase.

For the purpose of obtaining surveyable results, we will have to limit somewhat the generality of our observations by introducing certain conditions which satisfy, for all practical purposes, all communication channels in actual use. First of all we will consider H(t,τ), as function t (with a fixed τ), a stationary process. Then Y(jω,t) (with a fixed ω) is also a stationary process and its correlation function with respect to t

$$R_Y(j\omega,\ t_1,\ t_2) = \overline{Y(j\omega,\ t_1)Y^*(j\omega,\ t_2)} = R_Y(j\omega,\ \theta)$$

depends (not considering ω) only on the difference $t_2 - t_1 = \theta$.

The Fourier transform of $R_Y(j\omega,\theta)$ with respect to the variable θ is

$$\Gamma(j\omega,\ j\Omega) = \int_{-\infty}^{\infty} R_Y(j\omega,\ \theta)\exp(-j\Omega\theta)\,d\theta \qquad (7.5)$$

defines the spectral power density of fluctuations in the transfer function for components of the signal at frequency ω.

We will further assume that a channel has a limited memory, i.e., that there is an interval of time L during which the transfer function is almost completely damped, or, in other words, for any value of t when $\tau \geq L$

$$H(t,\ \tau) \approx 0. \qquad (7.6)$$

We will expand H(t,τ) with respect to the variable τ into a Fourier series over the interval $0 \leq \tau \leq L$:

$$H(t,\ \tau) = \mu_0(t) + \sum_{k=1}^{\infty}[\mu_{ck}(t)\cos k\Omega_1\tau + \mu_{ck}(t)\sin k\Omega_1\tau],\quad 0 \leq \tau \leq L, \qquad (7.7)$$

where

$$\mu_0(t) = \frac{1}{L}\int_0^L H(t,\ \tau)\,d\tau;$$

$$\mu_{ck}(t) - j\mu_{sk}(t) = \frac{2}{L}\int_0^L H(t,\ \tau)\exp(-jk\Omega_1\tau)\,d\tau = \frac{2}{L}Y(jk\Omega_1,\ t),\quad \Omega_1 = \frac{2\pi}{L}.$$

Substituting (7.7) in (7.2) and considering the limits of change of τ, we obtain the following expression for a signal at a channel output:

$$z'(t) = \mu_0(t)\int_0^L z(t-\tau)\,d\tau + \sum_{k=1}^{\infty}\Big[\mu_{ck}(t)\int_0^L z(t-\tau)\cos k\Omega_1\tau\,d\tau + \mu_{sk}(t)\int_0^L z(t-\tau)\sin k\Omega_1\tau\,d\tau\Big]. \qquad (7.8)$$

$$x(t) = \mu_0(t)\int_{t-T}^{t} z(\tau)\,d\tau + \sum_{k=1}^{\infty}\Big\{\mu_k(t)\int_{t-T}^{t} z(\tau)\cos k\Omega_1(t-\tau)\,d\tau +$$

$$+ \tilde{\mu}_k(t)\int_{t-T}^{t} z(\tau)\sin k\Omega_1(t-\tau)\,d\tau\Big\}$$

It can easily be seen that the integrals obtained represent the result of passage of signal z(t) through filters with the impulse responses:

$$\left.\begin{array}{l} g_0(t) = 1 \\ g_k(t) = \cos k\Omega_1 t \\ \tilde{g}_k(t) = \sin k\Omega_1 t \end{array}\right\} \text{when } 0 \leq t \leq T, \qquad (7.9)$$

$$g_0(t) = g_k(t) = \tilde{g}_k(t) = 0 \text{ when } t < 0 \text{ and } t > T.$$

This result permits us to construct a model of the channel shown in Figure 7.2. Signal z(t) is filtered by filters with constant parameters and impulse responses (7.9) and then each component is multiplied by its coefficient of transmission μ which is a random function of time. Such a channel will be called a selective fading model. The number of filters in this model is infinite but for practical purposes can always be delimited by a finite number, considering that the power of the input signal is outside a certain finite frequency band which is vanishingly small. It is easy to see that the spectral density of the power of the complex transmission coefficient $\mu_k = \mu_{ck} - i\mu_{sk}$ coincides with $H(jk\Omega_1, t)$. Coefficients μ_k with different subscripts are intercorrelated. They would be uncorrelated only if process H(t, τ) represented with respect to variable τ white noise and this does not occur in an actual channel. However, in many cases mutual correlation between μ_{k1} and μ_{k2} decreases rapidly with an increase in the difference $k_2 - k_1$.

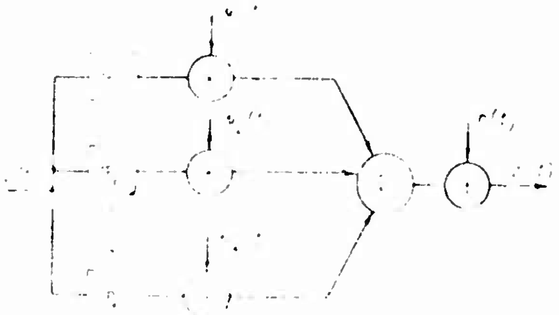


Figure 7.2. Selective Fading Model.

Additive interference which will, as formerly, be considered as gaussian white noise (or at least gaussian noise with a uniform spectrum in the frequency band exceeding the width of the output signal spectrum) is added to the signal at the channel output.

The advantage of the scheme in Figure 7.2 in comparison with the general diagram (Figure 7.1) is that here elements depending on time (multiplicative interference) and inertia elements determining the constant frequency distortion of the signal are separated.

To construct another model of the same channel we will introduce the additional assumption that a channel has, for all practical purposes, a limited passband, i.e., there is a magnitude Ω_2 such that for any t and $|\omega| > \Omega_2$

$$Y(\omega, t) = 0 \tag{7.10}$$

Of course, conditions (7.10) and (7.9) contradict one another inasmuch as two functions linked by a Fourier transform cannot both be finite. Furthermore, expression (7.10) contradicts the condition of physical realizability expressed in (7.1). Therefore, we introduce (7.6) and (7.10) as approximate equalities, assuming that they can be met with any desired accuracy if sufficiently large L and Ω_2 are selected. The latter holds for all actual channels inasmuch as they have losses as a consequence of which the transfer function becomes damped with an increase in τ, and inertness, as a consequence of which, with sufficiently large ω, the modulus of the transfer function $Y(j\omega,t)$ becomes as small as desired.

Of course, the channel models obtained in this way will also be approximate. However, they can be made precise by a limiting transition, assuming that L and Ω_2 approach infinity.

And so, assuming that condition (7.10) is met, it is possible to represent transfer function $H(t,\tau)$ in the form of a Kotelnikov series with respect to variable τ (see, for example, [4]):

$$H(t, \tau) = \sum_{k=0}^{\infty} \mu^{(k)}(t) \frac{\sin \Omega_2(\tau - k\tau_0)}{\Omega_2(\tau - k\tau_0)} \tag{7.11}$$

where $\mu^{(k)}(t) = H(t,k\tau_0)$; and $\tau_0 = \pi/\Omega_2$.

If condition (7.6) is also met, the upper limit in the sum may be set equal to L/τ_0.

Each term of the series represents a random function of time $\mu^{(k)}(t)$ multiplied by the transfer function of an ideal (physically unrealizable) pi-response filter of lower frequencies with a limiting angular frequency of Ω_2 shifted in time by $k\tau_0$. This permits formal representation of the channel circuit in the form of a delay line by line L which passes frequencies $|\omega| < \Omega_2$ with taps every τ_0. The voltages taken from a tap are multiplied by $\mu^{(k)}(t)$ and are then summated and the additive interference added (Figure 7.2). Thus, a channel model is obtained in accordance with which the signal passes from channel input to output along various paths ("beams") having different time-dependent coefficients of transmission $\mu^{(k)}(t)$. We will call such a model a multibeam propagation model. Its advantage in comparison with the general circuit (Figure 7.1) is that in each separate beam the coefficient of transmission depends only on time and not on frequency. The frequency dependence occurs only in the result of interference incurred in summating the beams.

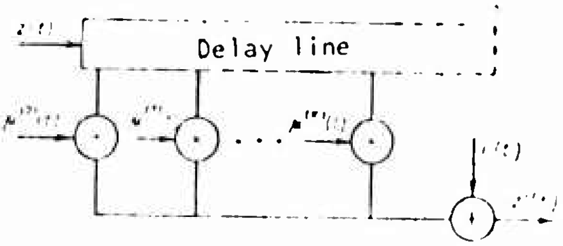


Figure 7.3. Model of Multibeam Propagation.

Both of these models represent the same channel and therefore can be, with equal right, used for analysis. Both of them describe the processes in a channel with the same approximation (increasing with an increase in L and Ω_2). In some cases it is more convenient to use one model and in other cases the other and this is determined mainly by the nature of the signal used. In deriving these models we base ourselves only on a phenomenological description of a channel using transfer function $H(t, \omega)$ without drawing on any physical ideas about actual processes occurring in a channel. In other words, the channel is considered a "black box" which we may, with equal right, consider met by the diagram in Figure 7.2 or Figure 7.3.

If we were to speak of the physical essence of the passage of a signal in a channel, it would be very little like any of the models obtained.

It is easy to see that in meeting conditions (7.6) and (7.10) each of the models has $2L(\Omega_2)/2\pi = 2(\Omega_2)/\Omega_1$ branches. With an improvement in the approximation (by increasing the calculated values of L and Ω_2) this number grows and this makes analysis difficult. In some cases it is possible to greatly decrease the number of branches. One of these cases is that when we may consider that a channel passes frequencies only within limits from

$$\Omega_1 = \omega_{av} - \frac{\Delta\Omega}{2} \text{ to } \Omega_2 = \omega_{av} + \frac{\Delta\Omega}{2},$$

where ω_{av} is a certain average frequency divisible by Ω_1. Obviously, it is possible in the model shown in Figure 7.2 (if only in the first approximation) to exclude branches with frequencies less than $\omega_{av} - \Delta\Omega/2$ and the total number of branches proves to be equal to

$$2L\frac{\Delta\Omega}{2\pi} = 2\frac{\Delta\Omega}{\Omega_1},$$

which yields a great reduction if $\Delta\Omega \ll \Omega_2$.

Similarly, it is possible to reduce the number of branches in the model of Figure 7.3. For this purpose we represent the transfer function by its envelope and the instantaneous phase by the variable φ:

$$\begin{aligned} H(t,\tau) &= H_0(t,\tau)\cos[\varphi_1(t,\tau)] = \\ &= H_0(t,\tau)\cos[\omega_{av}\tau - \theta(t,\tau)] = \\ &= H_0(t,\tau)\cos\theta(t,\tau)\cos\omega_{av}\tau + H_0(t,\tau)\sin\theta(t,\tau)\sin\omega_{av}\tau = \\ &= H_1(t,\tau)\cos\omega_{av}\tau + H_2(t,\tau)\sin\omega_{av}\tau \end{aligned} \quad (7.12)$$

Here, $H_0(t,\tau)$ is the envelope of the transfer function; $\varphi_1(t,\tau)$ is its initial phase; and

$$\theta(t,\tau) = \omega_{av}\tau - \varphi_1(t,\tau);$$
$$H_1(t,\tau) = H_0(t,\tau)\cos\theta(t,\tau);$$
$$H_2(t,\tau) = H_0(t,\tau)\sin\theta(t,\tau)$$

It can be shown that $H_1(t,\tau)$ and $H_2(t,\tau)$ are linked by a Gilbert transform and their spectrum is concentrated in a band from 0 to $\Delta\Omega/2$.

We will represent $H_1(t,\tau)$ and $H_2(t,\tau)$ in the form of a Kotelnikov series and obtain

$$\begin{aligned} H(t,\tau) = \sum_{k=0}^{J} \Big[& \mu_c^{(k)}(t)\frac{\sin\Delta\Omega(\tau - k\tau_1)}{\Delta\Omega(\tau - k\tau_1)}\cos\omega_{av}\tau + \\ & + \mu_s^{(k)}(t)\frac{\sin\Delta\Omega(\tau - k\tau_1)}{\Delta\Omega(\tau - k\tau_1)}\sin\omega_{av}\tau \Big], \end{aligned} \quad (7.13)$$

where

$$\mu_c^{(k)}(t) = H_1(t, k\tau_1);$$
$$\mu_s^{(k)}(t) = H_2(t, k\tau_1);$$
$$\tau_1 = \frac{2\pi}{\Delta\Omega}.$$

We will note that

$$\frac{\sin\Delta\Omega(\tau - k\tau_1)}{\Delta\Omega(\tau - k\tau_1)}\cos\omega_{av}\tau \approx \frac{\sin\Delta\Omega(\tau - k\tau_1)}{\Delta\Omega(\tau - k\tau_1)}\cos\omega_{av}(\tau - k\tau_1)$$

represents a transfer function of an ideal band filter with an average frequency of ω_{av} and a passband of $\Delta\Omega$ shifted by interval $k\tau_1$. The function conjugate with it which is obtained by shifting the phases of all spectral components through 90° (the Gilbert transform) will be

$$\frac{\sin\Delta\Omega(\tau - k\tau_1)}{\Delta\Omega(\tau - k\tau_1)}\sin\omega_{av}\tau.$$

In light of the possibility of rearranging the linear elements of the circuit we obtain the channel model shown in Figure 7.4. In this model the number of branches (degrees of freedom) is also equal to .

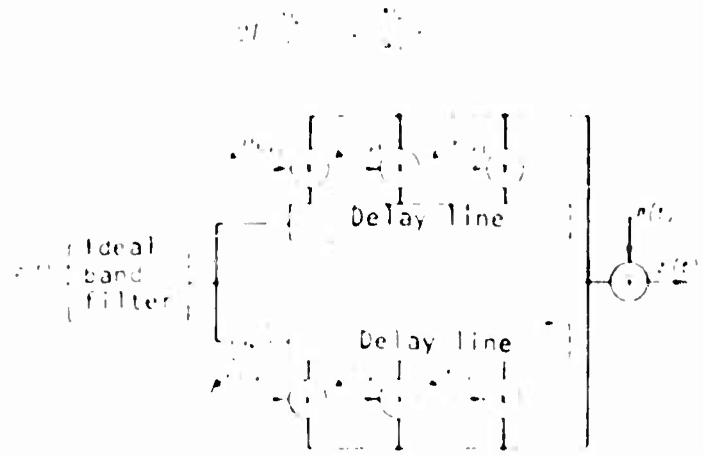


Figure 7.4. Model of Multibeam Propagation When the Signal Has a Limited Spectrum.

Rate of Fading

Variable coefficients of transmission ${}_{ck}$ and ${}_{sk}$ in the selective fading model and ${}^{(k)}$ and ${}^{(k)}$ or ${}^{(k)}$ in the multibeam model figure in all the channel models considered. Conditions for receiving signals depend in large measure on how rapidly these coefficients of transmission change. As in preceding chapters, we will call fading slow if the coefficients of transmission do not change noticeably over the length of a signal element T and rapid otherwise. However, in a channel with frequency dependent parameters there is some point in comparing the rate of fading not only with the length of signal element T but also with the "memory" of a channel T.

Let's assume that the power spectrum of processes ${}^{(k)}$ lie entirely in the range of angular frequencies from 0 to Ω_g and condition (7.6) is met exactly. Channels which satisfy the condition

$$T\Omega_g < 2\pi \qquad (7.14)$$

are usually called Category I channels and all others Category II channels.

In actual channels processes (t) do not have a strictly limited spectrum.[1] Consequently, all channels based on this definition should be relegated to Category II, especially since the memory of a channel is not always strictly limited. However, under these conditions we will assign a channel to Category I if the time of correlation of (t) processes is much greater than the length of channel memory L which is determined by any reasonable method (for example, as the interval over which 99% of the power of a transfer function is concentrated). We will relegate to Category II those channels in which the time of correlation of $\mu(t)$ processes is less than T. Understandably, such a definition is not precise since there may be intermediate cases encompassing a larger or smaller number of channels depending on what meaning is given to the work "significant." Nevertheless, for our analysis a no more precise subdivision is required.

---

[1] Processes with a strictly limited spectrum are determined (singular), i.e., they may be extrapolated with as much accuracy as desired over as small a sector as desired. Based on physical considerations it is clear that fading is always undetermined.

We will call the memory of a channel "short" if it is much less than the length of a signal element T and long if it is commensurate with T or greater.

We will note that depending on the ratio between the length of signal element T and the memory of a channel L and the time of correlation of transmission coefficients τ_k, there may be six typical cases:

a) Slow fading in a Category I channel with a short memory

$$(\tau_k \gg T \gg L).$$

b) Slow fading in a Category I channel with a long memory

$$(\tau_k \gg T \gtrsim L)^1.$$

c) Fast fading in a Category II channel with a long memory

$$(T \gtrsim \tau_k \gg L).$$

d) Slow fading in a Category II channel with a long memory

$$(L \gtrsim \tau_k \gg T).$$

e) Fast fading in a Category II channel with a long memory

$$(L \gtrsim T \gtrsim \tau_k).$$

f) Fast fading in a Category II channel with a short memory

$$(T \gg L \gtrsim \tau_k)$$

In practice all channels which are now in use belong to Category I. Thus, for cable channels (taking intermediate amplifiers into consideration) $L \approx 10^{-3}$ sec, $\Omega_3 \approx 10^{-4}$ -- 10^{-3}, so that $L\Omega_3 \approx 10^{-7}$ -- 10^{-6}. For shortwave radio channels variable L is determined by multiple reflections of radio waves from various layers of the ionosphere and reaches 10^{-3} -- 10^{-2} sec while Ω_3, characterizing the rate of fading under ordinary conditions, does not exceed 10 rad/sec, and therefore $L\Omega_3 \approx 10^{-2}$ -- 10^{-1}. In radio channels with tropospheric scattering $L\Omega_3 \approx 10^{-4}$ and in channels with ionospheric scattering $L\Omega_3 \approx 10^{-3}$. Thus, all listed channels belong to Category I. Incidentally, under conditions of magnetic storms the rate of fading in shortwave radio channels increases greatly

---

[1] The symbol $\gtrsim$ means "of the same order or greater."

and the product LF_s approaches a critical value of 2. In some cases Category II channels are hydroacoustic ultrasound channels.

As will be seen from what follows for convenience in obtaining the simplest decision principles possible the length of a signal element should be so selected that a channel with slow fading and a short memory is formed. Obviously, in Category II channels this is impossible. This circumstance imposes important limitations on the use of Category II channels. As V. I. Siforov [5] showed, the carrying capacity of a Category I channel approaches infinity if the power of additive interference approaches zero while the carrying capacity of a Category II channel under these conditions remains finite.

In what follows we will consider mainly Category I channels.

Several Ideas About an Optimal Decision Principle

We will assume that all distributions of probabilities of $\mu_k(t)$ processes in a selective fading model (or $\mu^{(k)}(t)$ in a multibeam model) are known. Then, in principle, it is possible to apply the criterion of maximal likelihood so that, based on arriving signal $z'(t)$ a decision is reached as to which of the possible realizations of signal $z(t)$ was transmitted. However, because of the finite memory L of the channel it is impossible, generally speaking, to limit ourselves here to analysis of an arriving signal over interval T, inasmuch as each element of a transmitted signal creates a reaction of duration T + L. Therefore, to extract complete information about one element of a signal, it is necessary to carry out an analysis over at least that same interval. Furthermore, it must be borne in mind that in the composition of a received signal there is simultaneously a reaction over several elements.

It would be possible to extract the most complete information by analyzing a received signal immediately over a long interval of time and by reaching a decision about the entire sequence of symbols transmitted over this period of time. However, such a method even in the simplest cases is exceedingly difficult and therefore preference is given to element-by-element (sequential) reception which can be described in general terms as follows [3]. A segment of a received signal of duration T + L is analyzed beginning from the instant of arrival of a new element. For all expected realizations of this element of $z_r(t)$ $(r = 1,\ldots,m)$ the likelihood function

$$w\left(z'(t)_{0<t<T+L} \mid z_r(t)_{0<t<T}\right)$$

is calculated in light of the distribution of probabilities of multiplicative and additive interference [$\mu_r(t)$ and $n(t)$] and also in light of previously received decisions about symbols preceding the one in question. The latter is important inasmuch as $z'(t)$ contains, along with a distorted signal element of $z_r(t)$ and additive interference, components caused by N preceding elements of the signal where N is the least whole number which is greater than or equal to L/T.

A decision is reached depending on which of the realizations of the signal element has the greatest likelihood function, i.e., $z_l(t)$ is considered received if

$$w[z'(t)|z_l(t)] \quad \max_r w[z'(t)|z_r(t)] \tag{7.15}$$

Such sequential reception is not optimal inasmuch as use is not made here of information about signal elements sent after completion of a given element of $z_r(t)$ and also forming components of the segment of $z'(t)$ being analyzed from T to T + L. This information could be obtained by analyzing a longer segment of the received signal.

Until now we have assumed that all distributions of probabilities of $\mu_r(t)$ are known. In that case when they are not known and only certain limitations imposed on $\mu_r(t)$ are known (for example, the frequency band in which the spectrum of multiplicative interference is lumped), it is possible to construct a principle based on a generalized criterion of maximal likelihood, i.e., to adopt a decision that signal element $z_l(t)$ was transmitted if

$$\begin{gathered}\max_{\mu_k(t)} [z'(t)_{T<t<T+L} | z_l(t)_{T<t<T+L}] > \\ > \max_{\mu_k(t)} [z'(t)_{T<t<T+L} | z_r(t)_{T<t<T+L}].\end{gathered} \tag{7.16}$$

With this criterion the decision principle is that first an evaluation of function $\mu_k(t)$ is based on received signal $z'(t)$ and then the likelihood functions are calculated. The same algorithm can be deducted from criterion (7.15) on the condition that $\mu_{ck}(t)$ and $\mu_{sk}(t)$ are gaussian processes [6].

In the general case such decision principles lead to exceedingly complex functional diagrams [5] which, however, are greatly simplified in certain cases for Category I channels if signals are properly selected. This selection must provide for the simplest possible extraction of information about functions $\mu_k(t)$ when a selective fading model is used or $\mu^{(k)}(t)$ when a multibeam model is used.

The problem of selecting signals can be explained from another point of view. A signal passing through a channel is subjected to various transformations which are both reversible and irreversible. In the case of irreversible transformations there is a partial loss of the information contained in the signal about the transmitted message. In the models presented multiplication by random functions $\mu(t)$, summation of the output voltages in branches, and the addition of additive noise are in the general case irreversible transformations. Nevertheless, in some particular cases it is possible to select signals so that some of these transformations are made reversible. This increases the amount of information obtained with the reception of a signal and, consequently, increases communication fidelity. For example, if each realization of a signal consists of a small number of harmonic components separated by a large frequency interval, under certain conditions the received signal can be more or less accurately separated into components which have passed through separate branches in a selective fading model. Thereby summation of output voltages of branches becomes (at least partially) a reversible operation. Furthermore, with such a possibility of separation of branches, the amount of received information is greater than in the case when a signal passes only along one branch with multiplicative interference. Similarly, if a signal amounts to short impulses separated by long intervals of time, then a received signal can be separated into components arriving along various branches in a multibeam model.

In subsequent paragraphs we will consider various particular cases of channels and certain methods which permit constructing relatively simple decision principles.

7.2. Channel with Constant Frequency-Dependent Parameters

The simplest case of a frequency-dependent channel is the channel with constant parameters in which the transfer function $H(t,\tau)$ does not depend on t and therefore can be designated $H(\tau)$. Included in this case in the first approximation may be channels in which $H(t,\tau)$ changes very slowly with t so that over a communication period which begins at instant $t = t_0$ it can be assumed that $H(t,\tau) \approx H(t_0,\tau)$. Most electric wire uncommutable channels, also long-wave radio channels if a communication period is sufficiently short, and ultrashort wave radio channels between mobile correspondents when communication is conducted within the limits of direct visibility are all included in such channels.

If, furthermore, $H(\tau)$ leads to a delta-function $H(\tau) = \delta(\tau - t_p)$ (where t_p is the time of signal passage), the parameters of a channel are constant in time as well as in frequency. This case was considered in Chapter III. It corresponds to an approximation of actual channels over a limited period of time if the transfer function of the channel (the Fourier transform of $H(\tau)$) is practically constant in the frequency band in which signal power is lumped.

We will consider a more general case when $H(\tau)$ is not expressed even approximately as a delta-function. If signal $z_r(t)$ arrives at a channel input, at the channel output the signal received will be

$$z'(t) = \int_0^t z_r(\tau) H(t-\tau)\, d\tau + n(t) = z'_{0r}(t) + n(t) \tag{7.17}$$

As can easily be seen, the problem can be reduced to that considered in Chapter III if it is assumed that not signals $z_r(t)$ but changed signals

$$z'_{0r}(t) = \int_0^t z_r(\tau) H(t-\tau)\, d\tau \quad (r = 1, \ldots, m) \tag{7.18}$$

are sent to a channel with constant parameters not depending on frequency. It should only be borne in mind that signals $z'_{0r}(t)$ have a duration not of T but of $T + L$ where L is the response time of the channel which we will consider limited. This circumstance is usually ignored if $T \gg L$. Otherwise it is possible to construct a system so that signal elements of duration T are sent at intervals of time T + L, i.e., delays of duration L are introduced. Finally, if $T > L$, it is possible to send signals continuously but deliver to a decision circuit only segments of a signal of duration T - L during which there is no overlapping of adjacent elements. Such a method is used rather widely in practice and is called the protective gap method. Of course, it is not optimal since it entails a loss of information contained in the rejected segments of the signal. Incidentally, when $T \gg L$, these losses are negligible.

In principle the selection $T > L$ is always possible. In order under this condition to provide for the required rate of information transmission, it is essential to select a sufficiently large code base m. However, with a high level of interference and an increase in m, the probability of error grows, especially since in a channel with a limited passband it is not always possible to have these signals orthogonal.

We will consider what the possibilities are to reduce the response time L and to select an optimal shape of signal providing for the greatest possible resistance to interference. For this purpose we will use the method of Section 3.6, namely, we will introduce two quadrapoles Φ_1 and Φ_2 (Figure 7.5a) where Φ_1 has the transfer function modulus $|\Phi_1(j\omega)| = |Y^{-1}(j\omega)|$ and the transfer function modulus of quadrapols Φ_2 coincides with $|Y(j\omega)|$ where $Y(j\omega)$ is the channel transfer function. We will note that these quadrapoles are physically realizable inasmuch as we are considering a physically realizable channel. As can easily be seen, at point b there will be a sum of signal $z_b(t)$ and gaussian interference with a spectral power density of $|\Phi_1(j\omega)|^2 = |Y^{-1}(j\omega)|^2$ and at point b signal $z_b(t)$ with the same modulus of amplitude spectral density as at point a against a background of white noise.

Reasoning the same as in Section 3.6, we may show that the decision circuit DC connected to point v will be optimal in that case if that part of the circuit within the broken line is an optimal circuit for a signal at point b. The latter, as was shown, consists of a "whitening" filter which in this particular case is quadrapole Φ_2 and optimal decision circuit DC for signal $z_v(t)$ in the case of white noise.

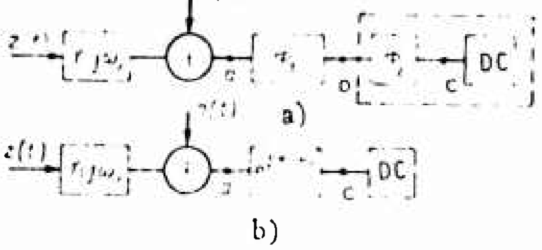


Figure 7.5. Pertaining to Conditions for Correction of a Channel with Frequency Dependent Parameters.

Signal $z_c(t)$, generally speaking, does not coincide with $z(t)$ inasmuch as only moduli of transfer functions are defined for quadrapoles. The sequential connection of these quadrapoles has the transfer function

$$\Phi(j\omega) = \Phi_1(j\omega)\Phi_2(j\omega) = |\Phi_1(j\omega)||\Phi_2(j\omega)| e^{-j\beta(\omega)} = e^{j\beta(\omega)} \tag{7.19}$$

where $\beta(\omega)$ is an arbitrary function satisfying the condition of physical realizability.

Thus, sequential connection of the two quadrapoles Φ_1 and Φ_2 amounts to a phase loop.

If it is desirable to reduce the length of signal element T to a minimum, it is advisable to select $\beta(\omega)$ so that the circuit formed by sequential connection of a channel and a phase loop with transfer function (7.19) has the least possible length of transfer function. It can be shown that for this purpose the phase-frequency characteristic of the resulting circuit must be linear over the entire range of frequencies in which the modulus of the channel transfer function $Y(j\omega)$ differs from zero.[1] Such a phase correction of a channel characteristic often occurs in practice. When this is so the diagram shown in Figure 7.5b is used.

Since there is normal white noise and signal $z_v(t)$ which amounts to the result of passage of the initial signal $z(\tau)$ through a circuit with a transfer function of $Y(j\omega)e^{j\beta(\omega)}$ at point c, the greatest resistance to interference with a given power of signal $z(t)$ will occur provided for when the power of signal $z(t)$ is maximal.

---

[1] This assertion is valid if the length L of the transfer function H(t) is understood in a mean-square sense:

$$L^2 = \frac{\int_0^T (\tau - \tau_1)^2 H^2(\tau)\,d\tau}{\int_0^T H^2(\tau)\,d\tau},$$

where

$$\tau_1 = \frac{\int_0^T \tau H^2(\tau)\,d\tau}{\int_0^T H^2(\tau)\,d\tau}$$

(See Note 5 at the end of the Chapter).

We will select any value T exceeding the length of impulse response H(t) of a corrected circuit. Then for any signal z(t) of length T

$$z_c(t) = \int_0^T H(t - \tau)z(\tau)d\tau \qquad (7.20)$$

We will consider the following Fredholm integral equation:

$$\int_0^T H(t - \tau)\varphi_k(\tau)d\tau = \lambda_k\varphi_k(t) \qquad (7.21)$$

It has solutions $\phi_k(t)$ which are called eigenfunctions for certain values of λ_k which are enumerated in decreasing order of magnitude: $\lambda_1 \geq \lambda_2 \geq \lambda_3 \ldots$ As is known, (see, for example, [7], Appendix II) functions $\phi_k(t)$ form a completely orthonormalized system over the interval $0 \leq t \leq T$. Therefore, any signal can be expanded into a series in accordance with these functions:

$$z(t) = \sum_{k=1}^{\infty} a_k\varphi_k(t), \qquad (7.22)$$

and

$$\int_0^T z^2(t)dt = \sum_{k=1}^{\infty} a_k^2$$

Substituting (7.22) in (7.20) and considering (7.21), we obtain

$$z_c(t) = \int_0^T H(t - \tau)\sum_{k=1}^{\infty} a_k\varphi_k(\tau)d\tau = \sum_{k=1}^{\infty} a_k\lambda_k\varphi_k(t). \qquad (7.23)$$

On the basis of orthonormalcy of the eigenfunctions

$$\int_0^T z_c^2(t)dt = \sum_{k=1}^{\infty} a_k^2\lambda_k^2\int_0^T \varphi_k^2(t)dt = \sum_{k=1}^{\infty} a_k^2\lambda_k^2. \qquad (7.24)$$

From this equality it is apparent that transformed signal $z_c(t)$ will have the greatest power over the interval (0 - T) in that case when all coefficients a_k are set equal to zero and, furthermore, when it corresponds to the maximal eigenvalue λ_1. Thus, the following is the optimal signal

$$z(t) = \pm a_1\varphi_1(t), \qquad (7.25)$$

where a_1 is defined by limitations imposed on signal power at channel input.

Using both signs in (7.25) we obtain an optimal binary system with opposed signals. After signals z(t) are selected, it is easy to calculate the probability of error which, in the case of coherent reception, is equal to

$$p = \frac{1}{2}\left[1 - \Phi(\sqrt{2}\lambda_1 h)\right]$$

If a code base of $m > 2$ is required to increase the rate of information transmission, it is possible to use several orthogonal forms of signal which coincide with the eigenfunctions of equation (7.21) corresponding to the greatest eigenvalues.

7.3. Fast General Fading

Another relatively simple particular case of the channel shown in Figure 7.1 occurs when transfer function $H(t, \tau)$ can be shown in the form of a product of two cofactors of which one depends only on t and the other only on τ:

$$H(t, \tau) = \mu(t) H_1(\tau) \tag{7.26}$$

The first cofactor designated $\mu(t)$, amounts to a variable transmission coefficient and the second cofactor $H_1(\tau)$ is a constant channel transfer function. The instantaneous transfer function (7.3) in this case may also be expressed by two cofactors:

$$Y(j\omega, t) = \mu(t) Y(j\omega), \tag{7.27}$$

where

$$Y(j\omega) = \int_0^\infty H_1(\tau) \exp(-j\omega\tau) d\tau.$$

If $|Y(j\omega)|$ can be considered constant in the frequency band in which the main part of signal energy is concentrated and the duration of a signal element T is much less than the interval of correlation of process $\mu(t)$, there is general slow fading as discussed in Chapters V and VI. In this section we will study the case when $\mu(t)$ changes greatly over the length of signal element T, i.e., the case of fast fading.

In communication channels actually used, this case is encountered rarely. However, it has many times been shown that with the development of space radio communication, fast fading must inevitably be considered. Inasmuch as the power sources for signal transmission are limited and fading over cosmic distances is great, the length of T must be increased to provide for a high ratio between power of signal element and spectral noise density. As a result T may exceed the time of fading correlation [8]. This may take place also in the case of "ground" shortwave communication when because of certain considerations it is necessary to transmit messages if only at a slow rate using a very low-power transmitter. If in these cases narrow-band signals are used, it is possible to ignore the selective nature of fading and also the frequency distortions in the channel.

In the case under consideration in the channel model shown in Figure 7.3 all functions of $\mu_{ck}(t)$ and $\mu_{sk}(t)$ will be respectively the same (with an accuracy to the constant coefficients determined by the transfer function of $Y(j\omega)$). Assuming the signal to be so narrow-band that within the limits of its spectrum $|Y(j\omega)|$ = const, it is possible to reduce this model to the simpler one shown in Figure 7.6.

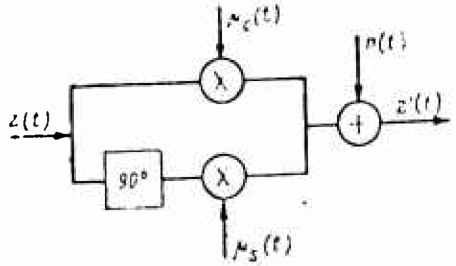


Figure 7.6. Model of a Channel With General Fading.

It can easily be seen that such a circuit can be obtained based on Figure 7.4 if $\mu_c^{(k)}$ and μ_s^k in all branches except one pair are equal to zero. Therefore, the concept of "general" fading and "fading in a single beam channel" coincide completely.

If the spectral density of additive interference is much less than the power of a signal over the interval of fading correlation then the function $\mu(t)$ can be evaluated with great accuracy based on the signal received by using the circuit suggested by V. I. Siforov in calculating the carrying capacity of a channel with general fading [9]. For this purpose the harmonic component, which does not depend on the information being transmitted (pilot signal), is included in the composition of the signai. Component $\mu(t)a \cos \omega_n t$, the spectrum of which is concentrated in band $_3$, determined by the rate of fluctuation in transfer coefficient $\mu(t)$, will be present in the signal received. If the spectrum of the signal components carrying the information lies mainly outside this band, it is possible to separate the pilot signal using a filter and, inasmuch as the transmitted pilot signal is known, obtain an evaluation for $\mu(t)$. After separating the signal received for $\mu(t)$, it is possible with sufficient accuracy to "eliminate" fading and use a decision principle for a completely known signal.

In essence, such a principle amounts to an idealization of the method used in practice of instantaneous automatic gain control (IAGC) based on a pilot signal. Errors occurring due to the presence of additive interference in the pilot signal channel and also because the spectrum of fluctuation of $\mu(t)$ is only approximately concentrated in frequency band $_3$ are considered in detail in [9].

When the interference spectral density is great it is possible to use the idea expressed in the work of John Kostas [8]. This idea, if the terminology used here is employed, amounts to the following. A signal element of length T is broken down into Q "subelements" of length $T_1 = T/Q$ and time-diversity reception is conducted over Q branches. The length of subelement T_1 is so selected that it is less than the interval of fading correlation so that reception of each subelement is done in the usual way under conditions of slow fading.

As a consequence of the small ratio between the power of a signal subelement and spectral interference density, the probability of error in the reception of a subelement will be great but can be decreased to any given value through incoherent accumulation, i.e., through time-diversity reception by selecting for a fixed value of T_1 a sufficiently large $T = QT_1$. Of course, with an increase in T the rate of information transmission decreases.

We will determine the probability of error in such a system. We will assume, as was done in work [8], that the system is binary and in the transmission of subelements relative phase modulation[1] is used and the fading is Rayleigh. Furthermore, we will assume that the fading in the subelements which are added can be considered independent.

According to formula (6.38), in light of Note 3 to Chapter VI, the probability of error is equal to

$$p = \sum_{k=0}^{Q-1} C^k_{Q+k-1} p_1^Q (1-p_1)^k,$$

where p_1 is the probability of incorrect reception of one subelement.

Considering that the length of one subelement may be a magnitude of the same order as the interval of fading correlation, probability p_1 can be determined from formula (5.76a) by substituting in it for h_0^2 the magnitude $h_Q^2 = 1/Qh_0^2$. As a result, after simple transformations we obtain

$$p = \sum_{k=0}^{Q-1} C^k_{Q+k-1} \left[\frac{1}{2}\left(1 - M\frac{h_0^2}{h_0^2+Q}\right)\right]^Q \times \times \left[\frac{1}{2}\left(1 + M\frac{h_0^2}{h_0^2+Q}\right)\right]^k, \tag{7.28}$$

where

$$M = \frac{4Q}{T}\int_0^{\frac{2T}{Q}}\left(1 - \frac{\tau Q}{2T}\right)R(\tau)\,d\tau;$$

and $R(\tau)$ is the coefficient of correlation of the process $\mu(t)$.

The dependence of M on Q hinders analysis of formula (7.28). If it is assumed that the length of a subelement is much less than the interval of correlation τ_k of fading, then $M \approx 1$. In this case, as was shown in Section 6.3, for a given h_0^2 there is an optimal value of Q providing for the least probability of error. Inasmuch as with an increase in Q, M increases, it is possible to assume that the optimal value of Q is somewhat greater than that calculated under the assumption that $M = 1$.

---

[1] This means that signal $z_1(t)$ amounts to a harmonic oscillation with a constant phase and in signal $z_2(t)$ the phase, with a shift from one subelement to another, changes by 180°.

In work [8] the problem is raised of determining the length of subelement T_1 which provides for a maximal transmission rate (i.e., a minimal T) with a given probability of error. It is shown that with a gaussian correlation function the optimal value of T_1 is equal to $0.6\tau_k$. In this case $M \approx 0.75$. Incidentally, this result is obtained insufficiently rigorously. Specifically, the decrease in probability of error with an increase in Q with a constant value of h_0^2 is not considered.

We will note that in actual channels widening of the frequency band due to the decrease in T_1 leads to a situation wherein selective fading begins to manifest itself. In other words, with small T_1 the magnitude L of channel memory cannot be ignored.

With a known correlation function of the quadrature components $\mu_c(t)$ and $\mu_s(t)$, it is possible to deduce an optimal decision principle for general fading at an arbitrary rate [10]. Let the received signal have the form

$$z'(t) = \mu_c(t)\,z_r(t) + \mu_s(t)\,\tilde{z}_r(t) + n(t), \quad 0 \leq t \leq T. \tag{7.29}$$

We will limit ourselves to the case of Rayleigh fading, i.e., we will assume that $\mu_c(t)$ and $\mu_s(t)$ are independent random processes with a zero average and the same correlation coefficient.

$$R(t_1, t_2) = \frac{\overline{\mu_c(t_1)\,\mu_c(t_2)}}{\overline{\mu_c^2(t)}} = \frac{\overline{\mu_s(t_1)\,\mu_s(t_2)}}{\overline{\mu_s^2(t)}}. \tag{7.30}$$

The likelihood function for signal $z_r(t)$ with known $\mu_c(t)$ and $\mu_s(t)$ is equal to

$$\begin{aligned} w(z'/z_r, \mu_c(t), \mu_s(t)) = N \exp\Big\{ &-\frac{1}{\nu^2}\int_0^T z'^2(t)\,dt + \\ &+ \frac{2}{\nu^2}\int_0^T \mu_c(t)\,z'(t)\,z_r(t)\,dt + \frac{2}{\nu^2}\int_0^T \mu_s(t)\,z'(t)\,\tilde{z}_r(t)\,dt - \\ &- \frac{1}{\nu^2}\int_0^T \mu_c^2(t)\,z_r^2(t)\,dt - \frac{1}{\nu^2}\int_0^T \mu_s^2(t)\,\tilde{z}_r^2(t)\,dt - \\ &- \frac{2}{\nu^2}\int_0^T \mu_c(t)\,\mu_s(t)\,z_r(t)\,\tilde{z}_r(t)\,dt \Big\}, \end{aligned} \tag{7.31}$$

where N is a constant not dependent on r.

We will represent realizations of random processes $\mu_c(t)$ and $\mu_s(t)$ over the interval (0, T) by the canonic expansion [?]

$$\mu_c(t) = \sum_{n=1}^{r} \mu_{cn}\varphi_n(t), \quad \mu_s(t) = \sum_{n=1}^{r} \mu_{sn}\varphi_n(t), \tag{7.31}$$

where μ_{cn} and μ_{sn} are independent random variables and $\varphi_n(t)$ are eigenfunctions which are solutions of the Fredholm integral equation

$$\int_0^T R(t_1, t_2)\varphi(t_2)\,dt_2 = \lambda\varphi(t_1) \quad (0 \le t_1 \le T) \tag{7.32}$$

and which are enumerated in decreasing order of eigenvalues λ_n. Then

$$\begin{aligned} w(z'/z_r, \mu_c, \mu_s) &= w(z'/z_r, \mu_{cn}, \mu_{sn}, \varphi_n) = N\exp\Big\{-\frac{1}{\nu^2}\int_0^T z'^2(t)\,dt + \\ &+ \frac{2}{\nu^2}\sum_{n=1}^{r}\mu_{cn}\int_0^T z'(t)\,z_r(t)\,\varphi_n\,dt + \\ &+ \frac{2}{\nu^2}\sum_{n=1}^{r}\mu_{sn}\int_0^T z'(t)\,\tilde{z}_r(t)\,\varphi_n(t)\,dt - \\ &- \frac{1}{\nu^2}\int_0^T\Big(\sum_{n=1}^{r}\mu_{cn}\varphi_n(t)\Big)^2 z_r^2(t)\,dt - \\ &- \frac{1}{\nu^2}\int_0^T\Big(\sum_{n=1}^{r}\mu_{sn}\varphi_n(t)\Big)^2 \tilde{z}_r^2(t)\,dt - \\ &- \frac{2}{\nu^2}\sum_{n=1}^{r}\sum_{i=1}^{r}\mu_{cn}\mu_{si}\int_0^T \varphi_n(t)\,\varphi_i(t)\,z_r(t)\,\tilde{z}_r(t)\,dt\Big\}. \end{aligned}$$

We will note the units of functions $\varphi_n(t)$ are $sec^{-1/2}$, of the eigenvalues λ_n sec, and of μ_{cn} $sec^{1/2}$.

By assuming that the signal spectra do not overlap with the fluctuation spectrum of $\mu(t)$ and, consequently, with the spectrum of coordinate functions (which is practically always met in regular channels) we can set

$$\int_0^T\Big(\sum_{n=1}^{r}\mu_{cn}\varphi_n(t)\Big)^2 z_r^2(t)\,dt \approx \frac{1}{T}\int_0^T z_r^2(t)\,dt\int_0^T\Big(\sum_{n=1}^{r}\mu_{cn}\varphi_n(t)\Big)^2 dt \tag{7.33a}$$

and make similar simplifications in the remaining integrals.

Considering the orthonormalcy of functions $\varphi_n(t)$, we find

$$w(z'/z_r, \mu_{cn}, \mu_{sn}, \varphi_n) = k\exp\Big\{ [illegible] \int_0^T z'^2(t)\,dt$$

$$+ [illegible] \sum \mu_{cn} \int_0^T z'(t)\, z_r(t)\, \varphi_n(t)\, dt \qquad (7.54)$$

$$+ [illegible] \sum \mu_{sn} \int_0^T z'(t)\, \tilde{z}_r(t)\, \varphi_n(t)\, dt - [illegible] \sum [illegible] \Big\}$$

where $P_r = \frac{1}{T}\int_0^T z_r^2(t)\,dt$ is the power of arriving signal $z_r(t)$.

In what follows we will assume that an active-interval system is used, i.e., $P_r = P_s$. We will use the generalized criterion of maximal likelihood. For this we will find the values of μ_{cn} and μ_{sn} which maximize the likelihood function (7.54). These values, $\tilde{\mu}_{cnr}$ and $\tilde{\mu}_{snr}$, are solutions of the equations

$$\begin{aligned} \frac{\partial}{\partial \mu_{cn}} \ln w(z'/z_r, \mu_{cn}, \mu_{sn}, \varphi_n) &= 0, \\ \frac{\partial}{\partial \mu_{sn}} \ln w(z'/z_r, \mu_{cn}, \mu_{sn}, \varphi_n) &= 0. \end{aligned} \qquad (7.55)$$

Substituting in (7.55) the conditional probability (7.54), we find

$$\begin{aligned} \tilde{\mu}_{cnr} &= \frac{\mu^2}{P_r} \int_0^T z'(t)\, z_r(t)\, \varphi_n(t)\, dt = \frac{\mu^2}{P_r} X_{rn}, \\ \tilde{\mu}_{snr} &= \frac{\mu^2}{P_r} \int_0^T z'(t)\, \tilde{z}_r(t)\, \varphi_n(t)\, dt = \frac{\mu^2}{P_r} Y_{rn}, \end{aligned} \qquad (7.56)$$

where

$$\begin{aligned} X_{rn} &= \int_0^T z'(t)\, z_r(t)\, \varphi_n(t)\, dt, \\ Y_{rn} &= \int_0^T z'(t)\, \tilde{z}_r(t)\, \varphi_n(t)\, dt. \end{aligned} \qquad (7.57)$$

The decision principle based on the generalized criterion of maximal likelihood amounts to recording symbol Y_l if

$$w(z'/z_l, \tilde{\mu}_{cnl}, \tilde{\mu}_{snl}, \varphi_n) > w(z'/z_r, \tilde{\mu}_{cnr}, \tilde{\mu}_{snr}, \varphi_n) \quad \text{when } r \neq l,$$

or

$$\sum_{n=1}^{\infty} (X_{ln}^2 + Y_{ln}^2) > \sum_{n=1}^{\infty} (X_{rn}^2 + Y_{rn}^2), \qquad (7.58)$$

for all $r \neq l$.

On the basis of the principle obtained we may construct the decision principle expressed in Figure 7.7. It consists of m branches corresponding to signals $z_r(t)$. In each branch the result of multiplication of arriving signal by $z_r(t)$ applies to an infinite number of multipliers where multiplication by functions $\varphi_n(t)$ is performed. The products obtained are integrated, as a result of which, X_{rn} and Y_{rn} are formed. The subsequent operations are clear from (7.38).

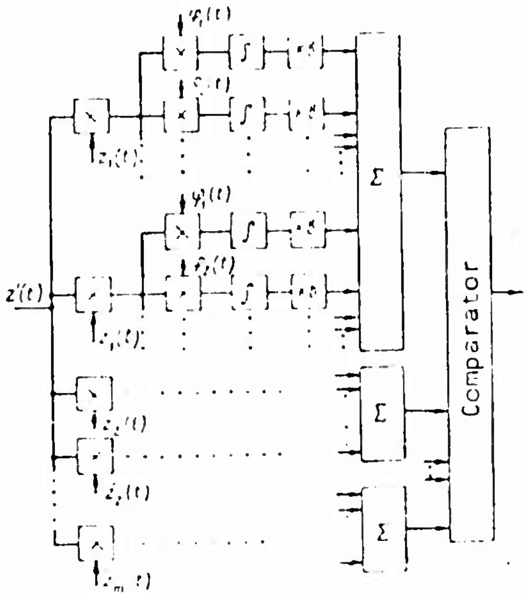


Figure 7.7. Decision Principle for a Channel with Fast General Fading.

Of course, such a principle is unrealizable because of the infinite number of terms[1] in (7.38). However, it can be shown that the mean-square values of variables X_{rn} and Y_{rn} with an increase in n rapidly approach zero and therefore a principle using a finite number of eigenfunctions is close to optimal.

Jumping somewhat ahead, we will show that if only N functions $\varphi_n(t)$ with the greatest eigenvalues are used, where N is a magnitude on the order of T/τ_k (depending on h_0^2), with further increase in N the probability of error decreases insignificantly.

The diagram shown in Figure 7.7 can be transformed by replacing multiplication of the signal and regular functions by passage of it through an appropriate filter. Figure 7.8 shows several variations of circuits for one of m branches in which the variable $X_{rn}^2 + Y_{rn}^2$ is formed using filters matched with $\varphi_n(t)$ (Figure 7.8a) or with $z_r(t)$ (Figure 7.8b). If, as we have supposed, the spectra of functions $\varphi_n(t)$ occupy a frequency band lower than the spectra of $z_r(t)$, we may show that $z_r(t)\varphi_n(t)$ is a signal linked with the product $z_r(t)\varphi_n(t)$. This permits representing the decision principle in the form of Figure 7.8c not containing multipliers in which filters are matched with $z_r(t)\varphi_n(t)$. Many other variations are possible, specifically, a circuit having synchronous heterodyning and filtering using an intermediate frequency.

---

[1] By transforming the decision principle (7.38) it is possible to obtain a circuit which does not have an infinite number of branches but contains filters with variable parameters [6, 10].

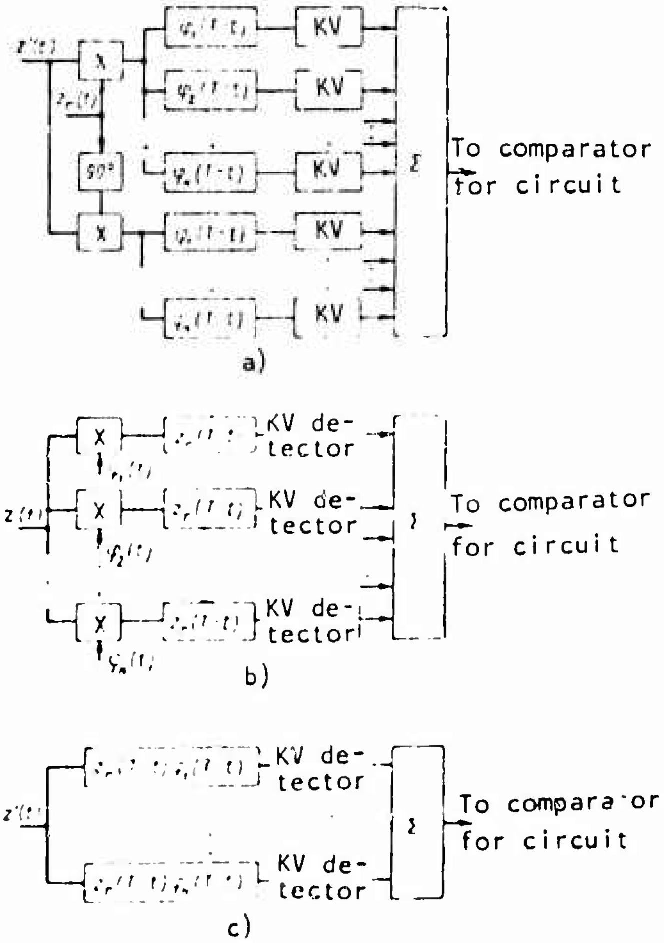

z'(t)
X
z_r(t)
90°
X
φ₁(T-t)
φ₂(T-t)
φ_n(T-t)
φ_n(T-t)
φ_m(T-t)
KV
KV
KV
KV
KV
Σ
To comparator for circuit
a)
z(t)
X
X
X
φ₁(t)
φ₂(t)
φ_m(t)
z_r(T-t)
z_r(T-t)
z_r(T-t)
KV detector
KV detector
KV detector
Σ
To comparator for circuit
b)
z'(t)
z_r(T-t)φ₁(T-t)
z_r(T-t)φ_m(T-t)
KV detector
KV detector
Σ
To comparator for circuit
c)

Figure 7.8. Several Variations of a Branch of a Decision Circuit for a Channel with Fast General Fading.

In the general case all these circuits are very complex. However, in two particular cases when $T \ll \tau_k$ and when $T \gg \tau_k$, they become greatly simplified.

In the first case, when fading is relatively slow, simplification is obvious inasmuch as we may limit ourselves to a consideration of one eigenfunction, $\varphi_1(t)$. It can easily be seen that in this case the decision circuit shown in Figure 7.8c is the same as in the case of slow fading (or in the case of absence of fading), the only difference being that the filters are matched not with $z_r(t)$ but with the product $z_r(t)\varphi_1(t)$.

In the case when $T/\tau_k \to 0$, as might be expected, $\varphi_1(t) \to 1$.

In the second case, with orthogonal signals having nonoverlapping spectra, by using transformations of the decision principle and making a few allowances, it is possible to represent the decision principle in the form shown in Figure 7.9 (see Note 6). Here the band filters $\varphi_1 - \varphi_m$ have a passband on the order of the width of the spectrum of the fluctuations in the transmission coefficient. They effect coherent accumulation of a signal being received during time on the order of τ_k after which incoherent quadratic accumulation (addition) occurs in an integrator. It can easily be seen that this circuit coincides with the circuit for integration following a detector as considered in Chapter IV. At the same time the idea of John Kostas as described above is embodied in it [8].

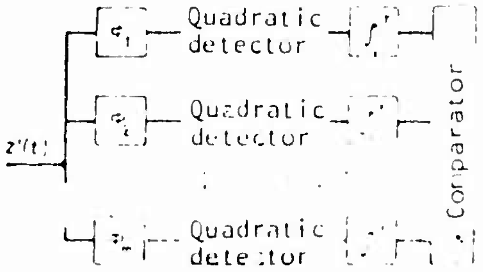


Figure 7.9. Simplified Decision Principle for Signals with Non-Overlapping Spectra in the Case When $T \gg \tau_k$.

In order to evaluate the effect of rate of fading on probability of error we will consider the simplest case of a binary system with an active interval. We will assume that the signals are orthogonal in the intensified sense and orthogonality is retained following passage through a channel with fast fading. Such an assumption is valid, for example, for an FM system with a large frequency spread and also for signals considered in the work of John Kostas [8]. If the length of a subelement is much less than τ_k. The decision principle of (7.38) for a binary system has the form

$$\xi = \sum_{n=1}^{N}(X_{1n} + Y_{1n} - X_{2n} - Y_{2n}) \gtrless 0 \qquad (7.39)$$

With transmission of signal $z_1(t)$

$$z'(t) = \sum_{n=1}^{N} \mu_n x_n(t) z_1(t) + \sum_{n=1}^{N} \mu_n y_n(t) \tilde{z}_1(t) + n(t). \qquad (7.40)$$

We will find elements of the correlation matrix of random variables entering into quadratic formula (7.39). Considering (7.55a):

$$\overline{X_{1n}^2} = \overline{Y_{1n}^2} = \overline{\left[\frac{\mu_n}{\mu_0} P_s + \int_0^T n(t) z_1(t) x_n(t) dt\right]^2}$$

$$= \frac{\nu^2}{2\mu_0^2} P \left[\frac{\lambda_n}{T} h_0^2 + 1\right], \qquad (7.41)$$

$$\overline{X_{2n}^2} = \overline{Y_{2n}^2} = \frac{\nu^2}{2\mu_0^2} P.$$

$$\overline{X_{1n} Y_{1p}} = \overline{X_{1n} X_{2p}} = \overline{Y_{1n} Y_{2p}} = 0 \text{ for all n and p.}$$

where $h_0^2 = P_s T/\nu^2$. Normalizing to $\nu^2/2\mu_0^2 P_s$, we find

$$\mathbf{K} = \left|\begin{array}{cccc:c} \frac{\lambda_1}{T} h_0^2 + 1 & 0 & 0 & \dots & \\ 0 & \frac{\lambda_2}{T} h_0^2 + 1 & 0 & & 0 \\ 0 & 0 & \frac{\lambda_3}{T} h_0^2 + 1 & \dots & \\ \hdashline & & & & \begin{matrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \dots & 1 \end{matrix} \\ & & 0 & & \end{array}\right|$$

and the square matrix is

$$\mathbf{A} = \left|\begin{array}{c:c} \mathbf{I} & 0 \\ \hdashline 0 & \mathbf{I} \end{array}\right|$$

Here I is a singular matrix of the N order.

We will find the roots of the equation

$$|\mathbf{KA} - \varkappa \mathbf{I}| = 0, \qquad (7.42)$$

$$\varkappa_n = \begin{cases} \frac{\lambda_n}{T} h_0^2 + 1 & (n \leqslant N), \\ -1 & (N < n \leqslant 2N) \end{cases} \qquad (7.43)$$

By using the method described in Note 4 to Chapter V it is possible to find the characteristic function and then the density of distribution of ξ (7.39). The probability of error, i.e., the probability that $\xi < 0$ is equal to

$$p = 1 - \sum_{n=1}^{N} \frac{\left(\frac{\lambda_n}{T} h_0^2 + 1\right)^{N-1}}{\left(\frac{\lambda_n}{T} h_0^2 + 2\right)^{N} \prod\limits_{\substack{p=1 \\ p \neq n}}^{N} \left(\frac{\lambda_n}{T} - \frac{\lambda_p}{T} h_0^2\right)} . \qquad (7.44)$$

With an increase in N this probability of error decreases and approaches a magnitude determining the potential resistance to interference. For small N if condition $\lambda_n h_0^2 \gg T$ is met for all $n \leq N$, it is possible to obtain from (7.44) the convenient approximate formula:

$$p \approx \frac{C_{2N-1}^{N}}{\prod\limits_{n=1}^{N} \left(\frac{\lambda_n}{T} h_0^2\right)} . \qquad (7.44a)$$

In the particular case when the function of fading correlation is exponential:

$$R(t_1, t_2) = \mu_0^2 \exp\left(-\frac{|t_1 - t_2|}{\tau_k}\right),$$

the eigenfunctions $\varphi_n(t)$ are equal to (see [17] on page 270):

$$\varphi_n(t) = \sqrt{\frac{2}{T + \lambda_n}} \cos\left[\omega_n\left(t - \frac{T}{2}\right) + (n-1)\frac{\pi}{2}\right]. \qquad (7.45)$$

where ω_n are positive roots of the equation

$$\tan \omega T = \frac{2\omega\tau_k}{\omega^2\tau_k^2 - 1}, \qquad (7.46)$$

$$\lambda_n = \frac{2\tau_k}{\omega_n^2\tau_k^2 + 1}. \qquad (7.47)$$

Figure 7.10 shows the dependence of probability of error on N in the case for $h_0^2 = 100$ and various T/τ_k ratios. These results confirm the fact that in practice it is possible to limit oneself to a number of eigenfunctions N on the order of T/τ_k.

Figure 7.11 shows the dependence of probability of error on h_0^2 calculated for a rather large N. As can be seen from these curves, potential resistance to interference increases with an increase in the rate of fading and approaches the potential resistance to interference of a channel without fading (broken line in Figure 7.11). There is nothing surprising about this since the greater the rate of fading, the less is the probability that over the length of signal element T unfavorable ratios between instantaneous signal power and interference will be retained. However, it should be kept in mind that with a selected system of signals, with an increase in the rate of fading sooner or later the condition of orthogonality will be disrupted for signals which pass through the channel and the formulas used to calculate the curves presented will no longer hold.

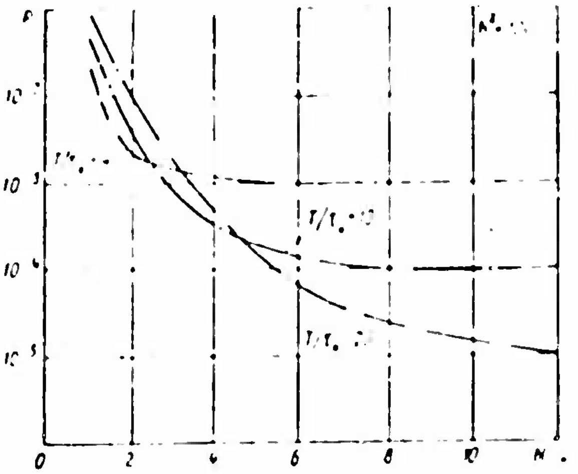


Figure 7.10. Dependence on Probability of Error on the Number of Subchannels Used in a Quasi-Optimal Receiver.

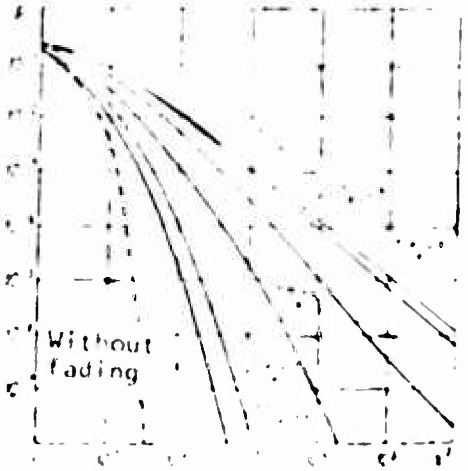


Figure 7.11. Probability of Error in the Case of Optimal Reception of Binary Orthogonal Signals with an Active Interval in a Channel with Fast Fading

[illegible]

In conclusion, we will note that with sufficiently fast fading, errors become practically independent inasmuch as fading in adjacent elements can be considered uncorrelated.

7.4. Slow Selection Fading

In this section we will consider that case when the length of signal element T is much greater than channel memory L, but much less than the interval of fading correlation $_k$. In this case it is convenient to use the model of selective fading (Figure 7.2).

If a signal occupies a relatively narrow band of frequencies, by doubling the signal in several bands separated so much that processes $_{nc}(t)$ and correspondingly $_{ns}(t)$ are weakly correlated for them, it is possible to effect frequency-diversity reception, greatly reducing the probability of error. We will not discuss this inasmuch as problems of diversity reception were considered in Chapter VI.

However, it is possible to obtain approximately the same gain as in diversity reception if separate signal realizations occupy non-overlapping frequency bands with weakly correlated fading [11]. We will consider a simple [illegible] binary system with signals

$$s_1(t) = \mu_1 c \cos \omega_1 t$$
$$s_2(t) = \mu_2 c \cos \omega_2 t \qquad (0 < t < T),$$

assuming that fading at frequencies ω_1 and ω_2 is weakly correlated.

To evaluate the potentialities of this reception method, let us assume that fading occurs sufficiently slowly so that the values of the transmission factors μ_1 and μ_2 and of the phases [illegible] φ_1 and φ_2 of the received signal element can be predicted with the required accuracy [illegible]. Then

$$w_1[x(t)|s_1] = [illegible]$$

In this case coherent element-by-element reception is possible. It must be noted here that in selective fading, when $\mu_1 \neq \mu_2$, the expected signals during reception of each [illegible] are no longer in antipodal form [illegible] because signal power is different for the different transmitted [illegible]. Therefore, expression [illegible] and the error probability [illegible] expression [illegible], where

$$[illegible]$$

[illegible] the receiver [illegible]

$$h_0 = [illegible]$$

and we consider the mean square of the transmission factor $\overline{\mu^2}$ identical for both signals.

The error probability in coherent reception of a given signal element is

$$p_{\text{e.e.}} = \frac{1}{2}\left[1 - [illegible]\left(1 [illegible]\right)\right] \qquad [illegible]$$

and the unconditional probability of error may be derived by averaging this expression with respect to μ_1 and μ_2 in conformity with their joint probability distribution. If the fading is Rayleigh and so selective that the [illegible] between [illegible] and [illegible] may be disregarded, the average error probability is

[illegible]

where [illegible] is the [illegible] probability [illegible] approximately

[illegible]

[illegible] error probability during reception [illegible] channel with [illegible] [illegible] [illegible] [illegible] of the first degree as at the second [illegible] [illegible] differ at [illegible] of general [illegible].

Figure [illegible] shows the [illegible] of the error probabilities [illegible] [illegible] [illegible] [illegible] [illegible] the probability of error.

[illegible] correlation exists between [illegible] reception, the power gain [illegible] separate [illegible] decreases [illegible] average error probability when [illegible] are correlated [illegible] designate

[illegible]

The probability density of u is can easily be shown, after [illegible], [illegible], as

[illegible]

where r is the correlation factor between [illegible]$_{c1}$ and [illegible]$_{c2}$ (or [illegible]$_{s1}$ and [illegible]$_{s2}$).

By averaging expression (7.42) we find

$$[illegible] \qquad (7.51)$$

The [illegible] of p and [illegible] when r = 0.5 is also shown in Figure 7.12.

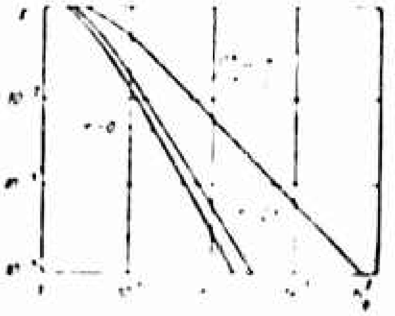


Figure 7.12. Error Probability in Coherent Reception of Binary FM Signals Under Conditions of Selective and General Fading.

[illegible] beam Propagation

[illegible] multipath propagation (Figures 7.3 and 7.4) is convenient in the case when the length of a signal element T is less than the memory of the channel, or is not much greater than it. It is possible to arrive at such [illegible] that [illegible] a channel with selective fading T is decreased for the purpose of increasing the rate of transmission. If we limit ourselves to consideration of a [illegible] channel, when T [illegible], then from [illegible] it follows that T [illegible], i.e., that fading is slow. In other words, it is possible to consider that transfer function [illegible] does not depend on t over the length of several packets.

Let transfer function [illegible] for a certain segment of time t be known. Then, in principle, it is possible to effect optimal reception by forming expected signals at a receiver as a sequence of realizations of transmitted signal [illegible] with [illegible] and applying a coherent decision principle, or a quadrature incoherent principle if [illegible] is known with an accuracy to an initial phase. The difficulty here is, first, in measuring the transfer function and, second, in that the received signal must be analyzed over a rather long segment of time inasmuch as the expected signal elements passing through the channel are mutually overlapping. The second difficulty is overcome if suboptimal sequential reception [3] is used as discussed in Section 7.1. As far as measurement of [illegible] is concerned, it can be effected by periodically sending a test pulse of known form into the channel and recording the pulse received. This idea is used in the [illegible] system which employs a test pulse [82].

In another variation [13] a linear quadrapole formed by a delay line with taps in such a manner that all arriving beams except the first are compensated for is controlled based on a test pulse in the receiver.

The test pulse must be sent sufficiently often so that the transfer function $H(t,\tau)$ over the period of time between two impulses does not succeed in changing noticeably. A sufficient amount of time not less than the length of the channel memory T (the difference in movement between the first and subsequent beams) must be allocated for reception of the test pulse. Naturally, in this process a great deal of the time allocated for information transmission is wasted.

It is easy to draw an analogy between this method and V. I. Siforov's method which was described in the preceding section. Indeed, the test pulse in a time domain acts in a way similar to that of the pilot signal in a frequency domain. In one case unchangeability in the coefficient of transmission with respect to frequency (gèneral fading) is required and in the other case relative unchangeability with respect to time (slow fading). The pilot signal is continuously emitted but occupies only part of the assigned frequency band while the test pulse must occupy the entire frequency band of the channel but only part of the time used for transmission. Both methods are intended for a channel with a relatively low level of interference.

The probability of error for systems with a test pulse and also for several other systems in a multibeam channel are computed in works [14, 15].

As a consequence of the complexity in constructing decision circuits under conditions of multibeam propagation many designers prefer to increase the length of a signal element in order to knowingly provide for inequality $T \geq (2-3)T$ and use the protective interval method described in Section 7.2, thereby making a channel a single-beam channel with selective fading. In order to provide for the required high rate of information transmission a code with a base $m > 2$ is used. Most frequently this is done by methods of multiplexing channels [16] and therefore such systems will be considered in chapter IX.

In shortwave radio channels and also in several other communication channels a model of multibeam propagation is not simply a convenient method of representing a channel but reflects the physical essence of passage of a signal with multiple reflections. For such channels the transfer function may be sufficiently well approximated by a sum of delta-functions:

$$H(t,\tau) = \sum_i \mu_i(t)\,\delta(\tau - \tau_i) \qquad (7.12)$$

where τ_i is the delay time of the i-th beam, and $\mu_i(t)$ is its intensity which changes slowly with t.

This formula can be obtained from (7.11) if ℓ approaches infinity by setting $\mu^{(k)}(t)\,\ell = \mu_i(t)$ and considering that several of the coefficients $\mu_i(t)$

(corresponding to instants of arrival of actual beams) in the limiting case remain finite.

From the physical model, in light of the fact that functions are determined by processes in various layers of areas of the ionosphere (or other reflectors), it follows that $\mu_i(t)$ can be considered independent for different beams. Below we consider several ways of receiving signals in such a channel.

Methods of Single-Beam Discrimination

A significant number of methods proposed to receive signals under multipath propagation conditions are based on discrimination of one of the incoming beams. Only a very few of these methods have found practical application, mainly because of the complexity of their instrumentation. The discriminated beam (the first or most powerful) is also usually subject to fading, but this fading is now general, not selective, and does not impede transmission at very high speeds. Furthermore, the fading in the separate beam is most often quasi-Rayleigh with a relatively large regular component, for which reason reception fidelity of an isolated beam may be substantially greater than that of all the interfering beams.

As early as the 1930's in the United States a method based on use of a receiving antenna with a narrow and automatically controlled directional pattern in the vertical plans (MUSA--multiple unit steerable antenna) was proposed and put into operation in several long-range shortwave radio communication lines. Since the different beams arrive at the receiving antenna from different angles relative to the horizon this antenna can discriminate one of them. The automatic tracking device permits the selection of the most "powerful" of the incoming beams.

In the same fashion the use of narrowly directional transmitting and receiving antennas in communications lines utilizing ionospheric and tropospheric scatter permit a considerable degree of elimination of the multipath nature of propagation.

Many other methods have been proposed to discriminate a single beam with respect to the time of its arrival. The simplest in concept is the method which uses short pulsed signals all of whose power is concentrated in a small fraction of time T reserved for element transmission. Here the pulse duration τ must be less than the amount t_{min} of relative lag of two adjacent beams, while element length T must be longer than the amount t_{max} of relative lag of the next beam. At receiver input the incoming beams form mutually nonoverlapping pulses (Figure 7.15 showing the envelopes of the radio pulses radiated by the transmitter and arriving at the receiver). The receiver opens for a period not much longer than τ at the frequency at which the signal elements are following each other, and in so doing, the moment of opening synchronizes with the arrival of the strongest beam (in Figure 7.15 this is the second beam). The signals matching the various symbols are distinguished from one another in phase, frequency, or amplitude.

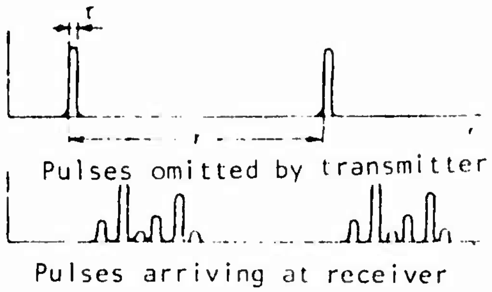


Figure 7.13. Multipath Propagation in Short Pulse Emission.

We would note that this signal system is not simple because when expanding an impulse of length τ into a Fourier series over interval T a large (in principle, an infinite) number of non-zero terms results. The effective frequency band occupied by such a signal is considerably broader than in simple systems.

The method described possesses essential defects which have prevented it, as far as we know, from being realized in practice, at least in the shortwave range where multipath propagation manifests itself most strongly. The greatest obstacle in the way of using this method is the difficulty of producing sufficiently short pulses of the power necessary to assure the requisite fidelity of reception.

It is not, however, at all obligatory that the signal be of an impulse nature for the receiver to be afforded the possibility of discriminating one of the arriving beams. It suffices for this that the signals have a base of $2FT \gg 1$ and an approximately uniform spectral density in frequency band F [17]. Such wide-band signals are often called noise-like, although in reality they are completely regular and knowledge of their structure permits them to be discriminated from noise in the receiver.*

For the sake of definiteness we will assume that signals $z_r(t)$ represent realizations of a gaussian process. They can be obtained, for example, by a random independent selection of 2FT of the coefficients of a Fourier series from a general aggregate with a normal probability distribution, zero mathematical expectation, and a given dispersion of [illegible] P_s/[illegible]. Inasmuch as slow fading is under consideration, we will consider the transfer coefficient μ_i in each beam to be constant random magnitudes over the length of one signal element.

---

* Let us remark that nowhere in all the preceding chapters, except in isolated, especially stipulated examples, was the size of the signal base limited. Therefore, all the earlier derived results remain true also for wide-band signals.

Let us demonstrate how one of the arriving beams is discriminated in coherent reception of a wide-band signal. Let the incoming signal

$$z'(t) = \sum_{i=1}^{N} \mu_i z_r(t - \Delta t_i) + n(t) \tag{7.53}$$

(where N is the number of arriving beams) go to a multiplier (Figure 7.14) in which it is multiplied by the signal coming from a local oscillator, equalling $z_r(t)$ to a constant factor of accuracy, and synchronized with one of the incoming beams. For convenience we will in expression (7.53) read out the lag time from the moment that the local signal begins, so that the values of Δt_i may be both negative and positive. After multiplication and integration the comparator system will at readout moment t = T receive voltage

$$X_r = \int_0^T z'(t) z_r(t)\,dt = \sum_{i=1}^{N} \int_0^T \mu_i z_r(t) z_r(t - \Delta t_i)\,dt + \int_0^T n(t) z_r(t)\,dt. \tag{7.54}$$

In contrast to the case of the single-path channel examined in Chapter III expression (7.59) contains terms which express the result of integrating the product of the local signal and the signals $\mu_i z_r(t - \Delta t_i)$ which arrived by different paths and with a lag (or a lead).

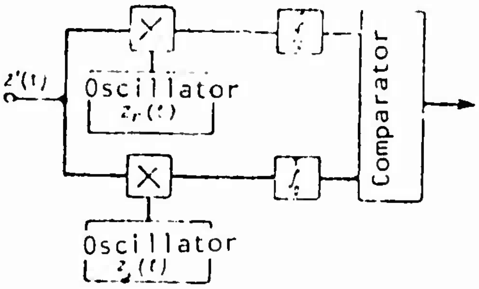


Figure 7.14. Coherent Reception of Wide-Band Signal.

In similar fashion the voltage

$$X_l = \sum_{i=1}^{N} \int_0^T \mu_i z_l(t) z_r(t - \Delta t_i)\,dt + \int_0^T n(t) z_l(t)\,dt. \tag{7.55}$$

at the readout moment will enter the comparator system, coming from the second multiplier circuit which has received signal z_l from the second local oscillator.

Let us calculate the degree of correction introduced by the additional beams into the multiplication and integration results, as compared to a single-path channel. For this purpose let us examine one of the integrals

$$J_i = \frac{1}{T}\int_0^T \mu_i z_r(t - \Delta t_i)\, z_r(t)\, dt. \tag{7.56}$$

entering into expression (7.54).[1]

Representing $z_r(t)$ as a Fourier series over the interval (0,T) we may write this integral as

$$J_i = \frac{\mu_i}{T}\int_0^T \sum_{k_1}^{k_2}(a_{rk}\cos k\omega_0 t + b_{rk}\sin k\omega_0 t)\sum_{k_1}^{k_2}[a_{rk}\cos k\omega_0 (t - \Delta t_i) + b_{rk}\sin k\omega_0 (t - \Delta t_i)]\, dt = \frac{\mu_i}{2}\sum_{k_1}^{k_2}(a_{rk}^2 + b_{rk}^2)\cos k\omega_0 \Delta t_i. \tag{7.57}$$

Let us now take into account that signal base 2FT = 2(k_2 - k_1 + 1) is large and that the value of $a_k^2 + b_k^2$ is about the same for all subscripts k (i.e., the signal has an adequately uniform spectral density in band F). Here, obviously,

$$a_k^2 + b_k^2 = \frac{2P_{si}}{\mu_i FT}, \tag{7.58}$$

where P_{si} is the power of the signal arriving over the i-th path.

Hence it follows that

$$J_i = \frac{P_{si}}{\mu_i}\,\frac{1}{FT}\sum_{k_1}^{k_2}\cos k\omega_0 \Delta t_i. \tag{7.59}$$

In particular, $\Delta t_i = 0$ for the beam with which the local oscillator signal is precisely synchronized and

$$J_i = J_{rec} = \frac{P_{s\,rec}}{\mu_{rec}}. \tag{7.60}$$

where the subscript rec denotes the beam received.

In the general case where $\Delta t_i \neq 0$, let us transform expression (7.59) by designating m = k - k_1 + k_2/2 = k - k_1 - FT - 1/2 (or with even FT, the nearest whole number). Then

---

[1] Strictly speaking, expression (7.54) is true in the case where the preceding (when $\Delta t_i > 0$) or the following (when $\Delta t_i < 0$) signal element is also $z_r(t)$. Otherwise the integral must be broken into two parts, from 0 to Δt_i and from Δt_i to T (when $\Delta t_i > 0$), where in the first of these $z_1(t - \Delta t_i)$ is to be substituted instead of $z_r(t - \Delta t_i)$ if the preceding element was $z_1(t)$. This refinement does not affect the qualitative result which will be obtained below.

$$J_i = \frac{P_i}{\mu_i}\frac{1}{H}\sum_{m=-\frac{H-1}{2}}^{\frac{H-1}{2}}\cos\left(m + k_1 + \frac{H}{2}\right)\omega_\beta\Delta t_i =$$

$$= \frac{P_i}{\mu_i}\frac{1}{H}\left[\cos\left(k_1 + \frac{H}{2}\right)\omega_\beta\Delta t_i \sum_{m=-\frac{H-1}{2}}^{\frac{H-1}{2}}\cos m\omega_\beta\Delta t_i - \right. \quad (7.61)$$

$$\left. - \sin\left(k_1 + \frac{H}{2}\right)\omega_\beta\Delta t_i \sum_{m=-\frac{H-1}{2}}^{\frac{H-1}{2}}\sin m\omega_\beta\Delta t_i\right] =$$

$$= \frac{P_i}{\mu_i}\cos\omega_{av}\Delta t_i\frac{1}{H}\left(1 + 2\sum_{m=1}^{\frac{H-1}{2}}\cos m\omega_\beta\Delta t_i\right),$$

where $\omega_{av} = \left(k_1 + \frac{H}{2}\right)\omega_\beta = \frac{k_2 + k_1 + 1}{2}\omega_\beta$ is the average signal frequency.

When deducing expression (7.61) we took into account that the cosine is an even function and the sine, an odd.

Thus, the variable J_i, regarded as a function of Δt_i, is an oscillatory variable of frequency ω_{av} and its envelope equals

$$\frac{P_i}{\mu_i}\frac{1}{H}\left|1 + 2\sum_{m=1}^{\frac{H-1}{2}}\cos m\omega_\beta\Delta t_i\right| \quad (7.62)$$

The doubled cosine sum which appears here has a maximum of H-1 when $\Delta t_i = 0$, while as Δt_i increases, the sum rapidly diminishes and oscillates around zero.

Figure [illegible] gives the graphs of function

$$\frac{1}{H}\left|1 + 2\sum_{m=1}^{\frac{H-1}{2}}\cos m\omega_\beta\Delta t_i\right|$$

at different values of H, from which it is evident how, with an increase in the base, the peak width at $\Delta t_i = 0$ decreases and the level of the "background" drops at $\Delta t_i \neq 0$. As this is an even function, it is shown only for the positive values of $\omega_\beta\Delta t_i$.

By using a well-known formula for the sum of cosines, it may be shown that

$$\frac{2}{H}\sum_{m=1}^{\frac{H-1}{2}}\cos m\omega_\beta\Delta t_i \quad (7.63)$$

$$y = \frac{1}{H}\left|1 + 2\frac{\cos\frac{H+1}{4}\omega_0\Delta t_1 \sin\frac{H-1}{4}\omega_0\Delta t_1}{\sin\frac{1}{2}\omega_0\Delta t_1}\right| = \frac{1}{H}\left|\frac{\sin\frac{H}{2}\omega_0\Delta t_1}{\sin\frac{1}{2}\omega_0\Delta t_1}\right|$$

Envelope (7.63) has, besides its principal maximum at $\Delta t_1 = 0$, maximums near Δt_1 values determined by condition $\frac{H}{2}\omega_0\Delta t_1 = \left(k + \frac{1}{2}\right)\pi$, where k is a whole number.

In fact, when $H \gg 1$ the denominator of expression (7.63) changes much more slowly than the numerator, and therefore the positions of the maximum of the quotient almost coincide with those of the maximum of the numerator. The greatest of these maximums corresponds to k = 1, i.e.,

$$\Delta t_1 = \frac{T}{2H}\cdot 3 = \frac{3}{2}\cdot\frac{1}{F},$$

and has the value (when $H \gg 1$)

$$\frac{1}{H}\cdot\frac{1}{\sin\frac{3\pi}{2H}} \approx \frac{2}{3\pi} \approx 0.2,$$

i.e., substantially less than the principal maximum which is unity. The other maximums have even smaller values.

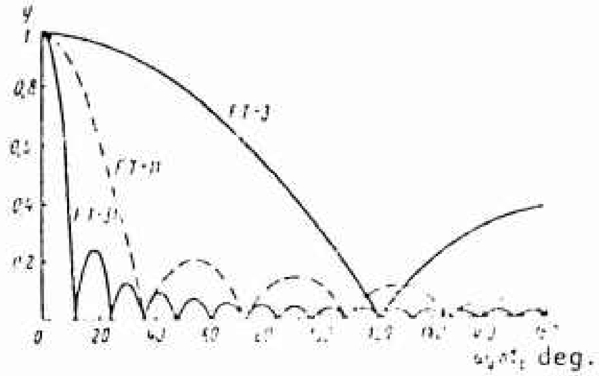


Figure 7.15. Graph of Function $y = \frac{1}{H}\left|1 + 2\sum_{m=1}^{\frac{H-1}{2}} \cos m\omega_0\Delta t_1\right|$

Half of the peak width at $\Delta t_1 = 0$ may be calculated by finding the first value of Δt_1 at which expression (7.63) becomes zero. This occurs when

$$\frac{H+2}{4}\omega_0\Delta t_1 = \frac{\pi}{2}, \text{ whence } \Delta t_1 = \frac{T}{H+2} \approx \frac{1}{F}.$$

Therefore, all the beams leading or lagging the fundamental beam by more than 1/F create only very small potentials at the output of the multiplier circuit.

In the second multiplier all the beams create a voltage which is expressed by the first term of formula (7.55). Substituting in it $z_u(t - \Delta t_i)$ and $z_i(t)$ in the form of a Fourier series, we may show that for the i-th beam this voltage is

$$J_i' = \frac{\mu}{2}\Big[\sum_{k} \text{[illegible]} + \sum_{k} \text{[illegible]}\Big] \qquad (7.64)$$

When $\Delta t_i = 0$ (i.e., for the fundamental beam) this expression equals zero for signals which are orthogonal in the intensified sense. For the other beams it differs from zero and with a random value of Δt_i it represents the sum of a large number of random terms. This enables us to regard the probability distribution of J_i' as approximately normal. In other words, all the beams, except the one being received, act upon the receiver in about the same way as normal fluctuation noise of uniform spectrum in frequency band F.

Let us calculate the power of this extra noise. It is obvious that it is always advisable to discriminate the most powerful incoming beam. Let us, however, examine the worst case where N beams of approximately equal intensity arrive. Then the extra noise power is about $(N - 1)P_s$.

At first blush it may seem that this noise, several times greater than signal power, would completely disrupt reception. This would, in fact, have happened with small-base signals, but when $FT \gg 1$ and with optimum or close to optimum methods of reception this noise has little effect on reception fidelity because its power is evenly distributed in frequency band F and the complete spectral noise density rises by only $((N - 1)P_s)/F$.

If, leaving multipath propagation out of account, the received beam permits a signal power to spectral noise density ratio of $h^2 = P_s/T\nu_0^2$, the harmful effect of the remaining beams will decrease this ratio to

$$h'^2 = \frac{P_s T}{\nu_0^2 + \frac{N-1}{F}P_s} = \frac{h^2}{1 + \frac{(N-1)h^2}{FT}}, \qquad (7.65)$$

equalling a power loss by a factor of $1 + \frac{(N-1)h^2}{FT}$ times.

The additional noise arising in a certain communication system is usually called system noise.

The number of beams in power commensurate to the fundamental, ordinarily does not exceed three or four. Then, so that the power loss be no more than by a factor of 3-4, it suffices to provide a signal base on the order of the requisite value of h^2, which is ordinarily no more than several hundreds under quasi-Rayleigh fading conditions (characteristic of the case where one beam is received). This loss proves in most cases to be less, inasmuch as the total power of all the interfering beams rarely exceeds the power of the fundamental beam.

It is not at all necessary to employ coherent reception to discriminate a beam in wide-band signals. It is easy to show that the same result may be obtained, for instance, by a quadrature system of incoherent reception, which provides, as was shown in preceding chapters, almost the same fidelity as coherent reception. Moreover, this makes the conditions for synchronizing the local signal with the incoming beam. Whereas in coherent reception this synchronization must be accomplished with accuracy to a small fraction of the average signal frequency period (which is technically unrealizable in many cases), in quadrature reception this exactitude is limited only to the width of the basic peak in expression (7.63), i.e., need be on the order of a fraction of 1/F. As signal band width increases, the requisite synchronization accuracy grows larger. Another variety of incoherent reception of wide-band signals, which is based on the synchronous heterodyne method, is used in the "Rake" system. As concerns the results obtained, this system is in no way different from the quadrature method and demands the same precise synchronization.

There are possible methods of incoherent reception of wide-band signals with single-beam discrimination which solve the synchronization problems more simply. Among these we find the variant of optimum incoherent reception based on linear filters matched with the signals.

Since the matched filters are linear the principle of superposition is applicable to them and the effect of each beam in isolation may be examined. Because the pulse response of the filter matched to signal $z_r(t)$ is $g(t) = kz_r(T - t)$, where k is an arbitrary proportionality factor, the filter's response at moment t to signal $z_r(t - \Delta t_i)$ delivered at moment Δt_i may be precisely found to a constant factor by Duhamel's integral

$$\begin{aligned} u(t) &= \int_{\Delta t_i}^{t} z_r(x - \Delta t_i)\, g(t - x)\, dx = \\ &= k \int_{\Delta t_i}^{t} z_r(x - \Delta t_i)\, z_r(T - t + x)\, dx \quad (t \geq \Delta t_i). \end{aligned} \tag{7.66}$$

Let us exchange the variable, designate $T - t + x = y$, and also introduce the notation $\Delta t' = T - t + \Delta t_i$. Then

$$u(t) = k \int_{\Delta t'}^{T} z_r(y - \Delta t')\, z_r(y)\, dy \quad (\Delta t' \leq T). \tag{7.67}$$

which to a constant factor of accuracy coincides with expression (7.56) if t' is substituted for t_1. When t traverses the range from t_1 to T + t_1 the value of t' changes from T to zero. Therefore, the dependence of a(t) may be obtained by means of the corresponding substitution in expression (7.61) [illegible]

$$[illegible] \left| \sum [illegible] \right| \qquad (7.6[illegible])$$

where [illegible]$_{av}$(t' - T).

The graph of the envelope of this function agrees with the curves in Figure 7.15 if $t + t_1 - T$ is laid off along the x-axis instead of t_1. This envelope hence has a sharp peak on the order of 2/F wide at moment $t = T + t_1$. Each incoming beam forms its own peak at the proper moment and if the travel distance between adjacent beams is no less than 2/F these peaks are not superimposed on each other (Figure 7.16) and therefore do not interfere as in the case of narrow-band signals. By combining the moment of readout with the greatest peak, single-beam reception may be accomplished.

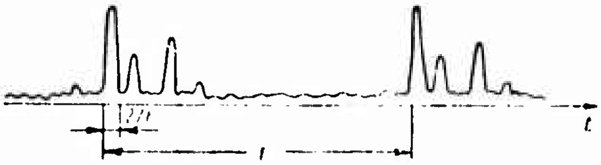


Figure 7.16. Voltage Envelope at Output of Filter Matched to Wide-Band Signal Being Received in Multipath Propagation.

The power ratios in a circuit with matched filters are the same as in coherent reception inasmuch as all the incoming beams set upon filters matched to other signals, about like fluctuation noise.

Methods of Utilizing Several Beams

The described methods of single-beam discrimination enable us, at least in principle, to get rid of interference between incoming beams, i.e., of heavy (Rayleigh) selective fading and of echoing; but the fidelity of reception by these methods is determined only by the power of the single discriminated beam. The question naturally arises whether it is not possible to use our knowledge of the structure of the multipath signal to separate the incoming beams and add them in such a way as to get the whole power of the entire signal and thus increase reception fidelity.

The addition of incoming beams may also employ wide-band signals which permit the incoming beams to be separated [21, 28]. This concept has been put into practice in the already mentioned Rake shortwave system [18]. Figure 7.17 presents a simplified block diagram of the Rake receiving unit. As already mentioned, this system uses wide-band signals, to be precise, signals of frequency band F = 10 kc and element length T = 22 msec, hence, signal base 2FT = 440. To realize optimum incoherent reception by the method of synchronous heterodyning the receiving unit contains two local signal generators which reproduce the transmitted signals $z_1(t)$ and $z_2(t)$, but with the frequencies of all components shifted by some frequency F_1.

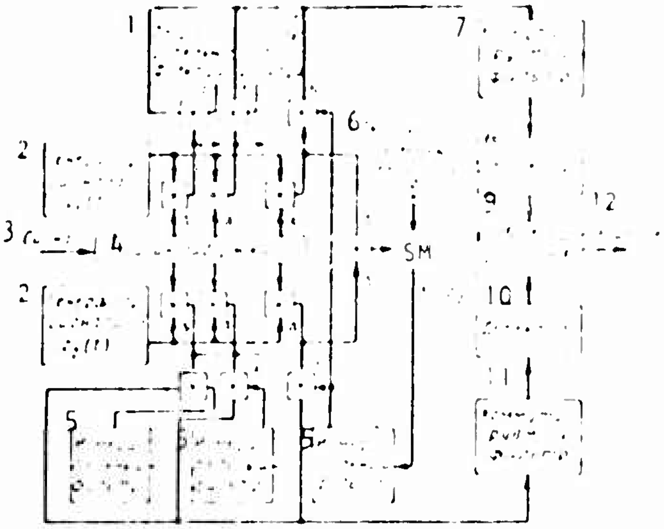


Figure 7.17. Simplified Block Diagram of Receiving Unit in the "Rake" System. Key: 1, From measuring filter B; 2, Signal generator $z_1(t)$; 3, Signal; 4, Delay line; 5, Measuring filter; 6, Auxiliary generator F_2; 7, Kinematic filter; 8, Detector; 9, Comparing device; 10, Detector; 11, Kinematic filter; 12, Decision.

The received signal goes to a delay line whose transit time must be not less than maximum relative time lag Δt_{max} of the beams under consideration, and in the given system is 3 msec. This line has fixed taps on both sides at intervals of 1/F, i.e., 100 microsec. Thus, the total number of taps on both sides is $\Delta t_{max} F = 30$. The taps are connected to the multipliers M. To the multipliers of the top taps is fed a voltage from local signal generator $z_1(t)$, and to those of the bottom taps, from the local generator signal $z_2(t)$.

The local signal generators are so synchronized that the beginning of their element coincides with the moment the beginning of the element of the received signal which has traversed the shortest path appears at the last tap of the delay line. Then at the last pair of multipliers there appear voltages which contain frequency component F_1 (See Note 7 to Chapter IV) and with an amplitude proportional to values of V_r (4.11) and corresponding to the maximum of the curve in Figure 7.15. The remaining beams, provided they lag the first by more than 1/F, will generate no perceptible voltage of frequency F_1 in the last pair of multipliers, because for them the abscissa of the curve in Figure 7.15 lies outside the principal maximum. But each lagging beam is at some one tap synchronized with the local signal generator with an accuracy of no worse then 1/I and generates voltages of frequency $_1$ at its corresponding multipliers and with amplitudes proportional to the values of V_r for that beam.

Thus, the series of multipliers can obtain distinct effects from the incoming beams. But, on the other hand, most of the multipliers do not satisfy the stipulation of synchronism and voltages are segregated on them which are generated only by noise.

The voltage of frequency $_1$ from the output of each multiplier go to a second multiplier (mixer) B, to the second input of which is fed a voltage of frequency $_1 - _2$ from a measuring filter. This voltage also has an amplitude approximately proportional to the mean-square value U_i of the voltage generated by the corresponding beam. There is a measuring filter of this sort in the circuit of each pair of taps. Figure 7.17 shows the connection of this filter only for the last pair of taps. It consists of the following: the voltages of frequency $_1$ obtained at the output of the multipliers of a given tap are added and fed to mixer SM; there they are mixed with the oscillations of the auxiliary generator of sinusoidal waves of frequency $_2$ (common to all the taps), and the emitted oscillations of frequency $_1 - _2$ are fed to a narrow-band measuring filter whose passband is of the order of 1 cps (matching the fluctuation rate of the state of the ionosphere). Therefore the voltage at the output of the filter is almost independent of noise and is determined by the power of the corresponding beam. In particular, at the output of the filter of the tap in which not one beam was synchronized with the signals of the local generators this voltage is practically zero, hence there is likewise no voltage at the output of the multipliers B of this tap. As for the circuits of the "active" taps in which a certain beam is separated, then at the output of the multipliers B there are voltage of frequency $_2$ and amplitude approximately proportional to the produce $V_r U_i$.

The initial phase of the voltages of frequency $_2$ at the outputs of the multipliers B at all the taps is the same and agrees with the initial phase of the auxiliary generator. We easily satisfy ourselves of this by tracing all the frequency conversions through the circuit and taking into account that the initial phases are transformed in the mixers (multipliers) just as are the frequencies. Therefore, by feeding all these voltages from the upper taps to one common bus and all the voltages from the lower taps to another common bus we can arithmetically add the amplitudes of the currents resulting from the processing of each beam. Therefore we are here practically effectuating almost coherent addition of the individual beams (accurate to the noise at the output of the measuring filter).

The thus added voltages from the z_1 and z_2 signal buses go to kinematic filters tuned to frequency Δ_2 and screening out the other harmonic components, are then detected, and are compared in a comparator. In this detection the information about signal phase is lost. Therefore the Rake receiver performs incoherent detection of the signal obtained from coherent addition of the beams.

Discrete generators of binary sequences in shift registers with feedback are used in the Rake system to shape wide-band noise-like signals (see, for example [20]). Such a generator emits a sequence of pulses of positive and negative polarity with a length of 0.1 microsec every 8.5 microsec. The period of a sequence contains 1023 pulses and lasts 8.525 msec. This exceeds the maximum propagation difference observed in the shortwave band.

The pulses received are delivered to a filter with a passband of 10 kc at the output of which the noise-like voltage is separated. To decrease the peak factor this voltage is limited with respect to maximum and then goes to another filter also having a passband of 10 kc. Inasmuch as the generators of the pulse sequences, filters, and limiters in the transmitting and receiving devices are the same, it is possible to obtain practically identical noise-like voltages.

By sequential transformations the frequencies form a signal in the working band of frequencies. The same noise-like voltage is used to obtain signal pair $z_1(t)$ and $z_2(t)$ but the average frequencies differ by $\Delta f = 181.8$ cps. This permits modulation in the frequency transforming circuit of the transmitting device.

The length of element $T = 22$ msec $= 4/\Delta f$ is not equal to and not a multiple of the length of the period of the generator of the noise-like signal. However, this does not alter the conditions of optimal incoherent reception inasmuch as with synchronized functioning of the generators in the transmitting and receiving devices reception of each element amounts to adopting a decision as to which of two signals was transmitted and the shapes of the expected signals are known (with the degree of synchronism attainable in practice with an accuracy to the initial phase) although they are different for different elements.

If the number of received beams and the ratios of signal power to spectral noise density are known in each beam, the probability of error may be calculated from formula (6.63).

Inasmuch as the period of a noise-like signal is 8.525 msec more than the memory of channel T (which in a shortwave radio channel coincides with the propagation difference between the first and subsequent beams and usually does not exceed 5-6 msec), the signals arriving by different paths to the receiver are almost orthogonal regardless of the length of element T. This permits in principle increasing the rated transmission speed in a multibeam channel when there is appropriate widening of the frequency band F to provide a sufficiently large base. Incidentally, it should be borne in mind that with a frequency band greater than approximately 100 KH the transfer function of a wide-band radio channel may not be approximated by formula (7.52) since the diffuse nature of each reflected beam begins to take effect and the delta-function must be replaced by transfer functions of finite length. In such a situation the signal must be subjected to more complex processing than in the Rake system [3].

In the receiving device the signal is amplified and its spectrum shifts to the average frequency f = 455 kc. The remaining frequencies shown in Figure 7.17 are f_1 = 20 MHz, f_2 = 9 MHz. The measuring filters are tuned to frequency $f_1 - f_2$ = 11 MHz and have a passband on the order of 1 cps. All multipliers (mixers) use 6Zh6 tubes and provide for linearity within the limits of a dynamic range of 100 db.

A system for receiving wide-band signals could be designed which would perform incoherent addition of the incoming beams by means of matched filters. As the preceding section demonstrated, the voltage envelope at the output of a filter matched to transmitted signal $s_r(t)$ has peaks proportional to the value of V_r for every beam, and they do not mutually overlap if the difference in course of the adjacent beams exceeds 1/W. By detecting this envelope with a quadratic detector and feeding the detection result to a capacitor we can under certain conditions obtain a voltage proportional to the sum of the squares of these values for all the incoming beams.

Figure 7.18 shows such a system for receiving wide-band signals when $T \gg L$. Here the capacitor charges directly by the difference between the two detectors. It is connected to the detector circuit at moment T of the ending of the element in the first of the arriving beams and remains connected for time L. At moment T + L the voltage on the capacitor is read out and a decision is reached as to the signal transmitted based on its sign. After this the capacitor is disconnected from the detector circuit and at moment T + L + τ (where $0 < \tau \ll T$) discharges in readiness to receive the next element. The curves show the voltages at different points in the circuit in reception of series of signals s_1, s_2, s_3.

At first blush this system of reception is simpler than the Rake method. Specifically, here the requirements for synchronization of transmitting and receiving devices are less rigid. However, it is necessary to consider that a filter matched with a complex signal is far from being simple [19].

The probability of error in any system with beam addition depends on the number of beams and on their intensity. If the number of beams is known and the ratios between signal power and spectral noise density in each beam are known, with quadratic beam addition (for example, in the diagram shown in Figure 7.16) the probability of error can be calculated from formula (6.62). It is somewhat greater than in a Rake system (formula (6.63)).

Fast Fading in a Multibeam Channel

All systems previously named which are intended for multibeam channels presuppose rather slow fading in each of the beams. With fast fading when $F_k T \approx 1$ a multibeam channel with $L \approx T$ is a Category II channel and this greatly hinders processing of the received signal. The only system known to the author which retains working capacity under these conditions was first suggested in 1959 and was named AME (Anti-Multipath Equipment)--a device for protecting against multibeam propagation.

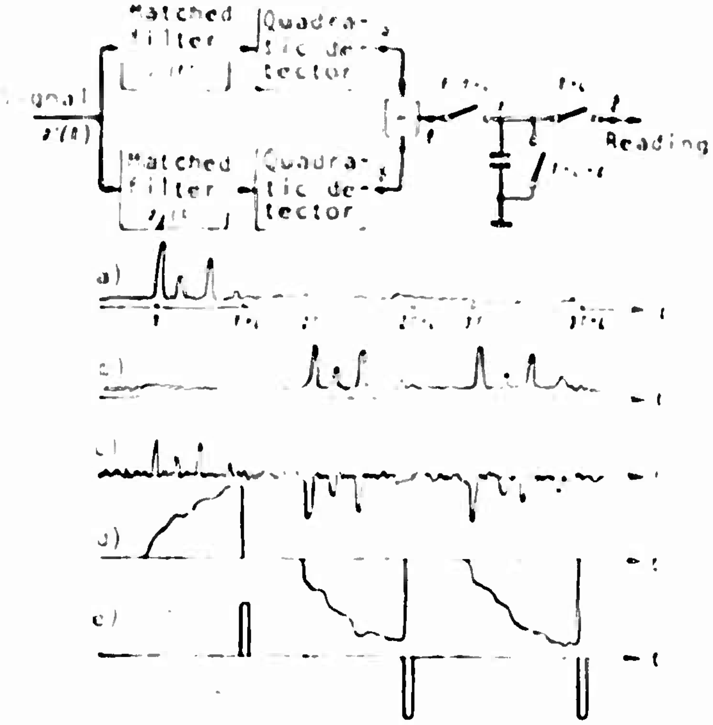


Figure 7.18. Matched-Filter System for Receiving Wide-Band Signals and Adding Beam Powers.

In this system for transmission of binary symbols y_1 and y_2 in accordance with a certain program use is made of n pairs of different simple signals:

$$\left.\begin{array}{l} z_{1,1}(t) = a\cos\omega_{1,1}t \\ z_{1,2}(t) = a\cos\omega_{1,2}t \\ \dots\dots\dots \\ z_{1,n}(t) = a\cos\omega_{1,n}t \end{array}\right\}\text{for symbol } y_1.$$

$$\left.\begin{array}{l} z_{2,1}(t) = a\cos\omega_{2,1}t \\ z_{2,2}(t) = a\cos\omega_{2,2}t \\ \dots\dots\dots \\ z_{2,n}(t) = a\cos\omega_{2,n}t \end{array}\right\}\text{for symbol } y_2.$$

In transmission of the first element signal $z_{1,1}(t)$ is emitted if symbol y_1 should be transmitted or $z_{2,1}(t)$ if symbol y_2 should be transmitted. In the second element use is made of signals $z_{1,2}(t)$ of $z_{2,2}(t)$ respectively, etc., until transmission in the n th element of signal $z_{1,n}(t)$ or $z_{2,n}(t)$. In the (n + 1)-th element again use is made of signals $z_{1,2}(t)$ and $z_{2,1}(t)$ and then the cycle repeats itself. The sequence of frequencies is shown in Figure 7.19. The spread between adjacent frequencies must be sufficiently great so that signal spectra, with account taken of their widening due to fading, do not overlap.

The number of pairs of frequencies n is selected based on the condition nl [illegible] L., therefore after reception of a certain signal $z_{1,1}(t)$ this same signal or a paired signal $z_{\nu+1}(t)$ may arrive at an input only after time lapse nI when the arrival of all beams from a previously transmitted signal ends. This permits discriminating with a receiver only that band of frequencies in which at a given time the arrival of the main beam is expected, as shown for the particular receiver shown in Figure 7.20. The beams corresponding to the transmitted signals have frequencies not falling in the passband of the receiver.

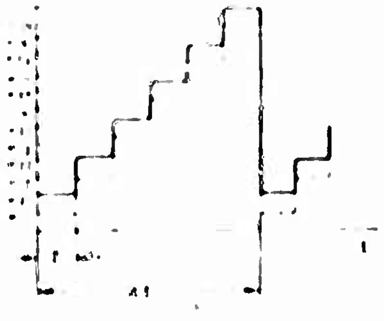


Figure 7.19. Distribution of Frequencies in an AME System. Solid line, frequencies of z_2. Broken line, frequencies of z_1.

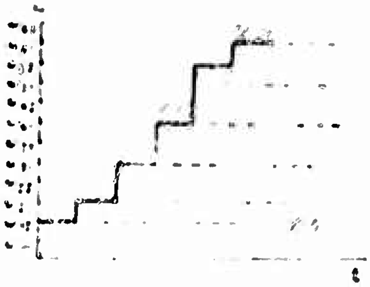


Figure 7.20. Transmission of Sequence of Symbols $y_2, y_1, y_1, y_1, y_2, y_1$ (Shaded Areas Show Receiver Passbands). Solid line, first beam. Broken line, subsequent beams.

Thus, this system permits eliminating beams which are delayed with respect to the main beam by more than I. The beams with the less delays act partially on the receiver and create additional interference fading but have no effect on reception of subsequent beams. The less I is, the better are interfering beams eliminated. From this point of view it is desirable to decrease I and, if the power of a signal is insufficient to provide the required fidelity, to use frequency-time diversity reception, transmitting several sequential elements for one code symbol. However, as can easily be seen, shortening I leads to abrupt widening of the frequency band occupied due to the increase in n and widening of the spectrum of each signal element.

By making the receiving device more complex it is possible to use the power of subsequent beams similar to the way shown in Figure 7.16. The difference in frequencies permits determining exactly to what signal element a given received oscillation should be relegated, even if it arrives later than oscillations corresponding to subsequent elements.

Summing up what has been said in this chapter, we note that in principle channels with parameters which change rapidly and depend on frequency are entirely suitable for transmitting discrete messages. Furthermore, with the selected signals in the proper form it is possible to get a higher degree of fidelity in such channels than in channels with slow general fading. However, decision principles in many cases are complex and difficult to implement, especially when it is necessary to transmit information at a great rate, exceeding 1/I.

Notes

1. (See Section 7.1) The construction of models of a channel with frequency-dependent variable parameters is described basically in accordance with works [1, 2, 3]. Several departures are due to an effort to correct the inexactness found in [1, 2] where the authors make formal use of a Fourier transform which they know not to exist for squared unintegratable functions and they also, without any qualification, use singular processes having a limited spectrum.

2. (See Section 7.1) Representation of functions having a spectrum which is not strictly limited in the form of a Kotelnikov series (7.11) or (7.13) should be considered as approximate. The average mean-square error in it can be defined by a fraction of the power of the function being expanded which lies beyond the limits of the "boundary" frequency. The higher the frequency selected the more rapidly the spectrum is damped beyond its limits and the more accurate is this representation (see, for example [22, 23]. We will note that if a signal spectrum diminishes with an increase in more rapidly than transfer function Y(j,t), then an approximate "boundary" signal frequency can be used for in formula (7.11). The same pertains to in formula (7.13).

3. (See Section 7.1) The channel models shown in Figures 7.2 and 7.3 also encompass those cases when a signal (or its separate components) acquire a doppler frequency shift. It is easy to see, for example, that if $_c(t)$ = a cos t and $_s(t)$ = b sin t, then the angular frequencies of an arriving signal will be shifted by . Generally speaking, if signal z(t) = Re (t) passes through a fading channel, the signal at channel output is equal to

Re{[illegible]} Re{[illegible] exp{[illegible]}}

where (t) defines a change in envelope and $\phi(t)$ [illegible] is the change in phase. The derivative d/dt represents a doppler shift in frequency. In a selective fading model or a multibeam model this pertains to separate branches in a channel. We will note that it is the doppler shift in frequency in separate beams (caused, for example, by a shift in reflecting areas) which for the most part determines interference fading.

4. (See Section 7.1) The definition given here for Category I and Category II channels is in accordance with work [6] and also with a report given by F. Green at the All-Union Scientific Session of the Scientific and Technical Society of Radio Engineering and Telecommunications named for A. S. Popov in 1962. A somewhat different definition was suggested by V. I. Siforov [5] who relegates to Category I those channels in which the passband is wider than the total width of the fluctuation spectrum of transfer coefficients for all beams. These two definitions in essence coincide if it is considered that in a channel a correction in the phase-frequency characteristic takes place, since in this case the length of response can be considered inversely proportional to the passband.

5. (See Section 7.2) We will present a proof to show that among constant linear circuits with a given amplitude-frequency characteristic $K(j\omega)$ a circuit with a linear phase-frequency characteristic has the least mean-square length of response. Let $K(j\omega) = C(\omega)\exp[-j\varphi(\omega)]$. The circuit impulse transfer function of $H(t)$ is a Fourier transform of $K(j\omega)$ and for physically realizable circuits $H(t)$ equals zero when $t < 0$. With a physically realizable stable circuit $K(j\omega)$ when $H(t)$ are integratable when squared. The derivative $d\varphi(\omega)/d\omega$ defines the phase lag of a signal in the circuit.

We will introduce the following definitions. We will call the following the average phase lag

$$\theta_1 = \frac{\int_{-\infty}^{\infty} \theta(\omega) C^2(\omega)\, d\omega}{\int_{-\infty}^{\infty} C^2(\omega)\, d\omega}, \tag{7.69}$$

and the following the mean-square of the phase lag

$$\theta_2^2 = \frac{\int_{-\infty}^{\infty} \theta^2(\omega) C^2(\omega)\, d\omega}{\int_{-\infty}^{\infty} C^2(\omega)\, d\omega}. \tag{7.70}$$

Similarly, we will call the following the average group lag

$$\tau_1 = \frac{\int_0^{\infty} t H^2(t)\, dt}{\int_0^{\infty} H^2(t)\, dt} \tag{7.71}$$

and the following the mean-square of the group lag

$$\tau_2^2 = \frac{\int_0^{\infty} t^2 H^2(t)\, dt}{\int_0^{\infty} H^2(t)\, dt}. \tag{7.72}$$

We will first show that $\theta_1 = \tau_1$. For this purpose we will designate $\mathrm{Re}\,K(j\omega) = a(\omega)$ and $\mathrm{Im}\,K(j\omega) = -b(\omega)$. Then $C^2(\omega) = a^2(\omega) + b^2(\omega)$ and $\varphi(\omega) = \arctan\frac{b(\omega)}{a(\omega)}$. **Whence,**

$$\theta(\omega) = \frac{d}{d\omega}\left[\arctan\frac{b(\omega)}{a(\omega)}\right] = \frac{b'(\omega)a(\omega) - a'(\omega)b(\omega)}{C^2(\omega)}, \tag{7.73}$$

where the primes indicate derives with respect to ω.

Substituting (7.73) in (7.69) we obtain

$$\theta_1 = \frac{\int\limits_{-\infty}^{\infty} [b'(\omega)a(\omega) - a'(\omega)b(\omega)]\,d\omega}{\int\limits_{-\infty}^{\infty} G^2(\omega)\,d\omega}. \qquad (7.74)$$

But

$$a(\omega) = \int\limits_0^{\infty} H(\tau)\cos\omega\tau\,d\tau,$$

$$b(\omega) = \int\limits_0^{\infty} H(\tau)\sin\omega\tau\,d\tau,$$

$$a'(\omega) = -\int\limits_0^{\infty} \tau H(\tau)\sin\omega\tau\,d\tau,$$

$$b'(\omega) = \int\limits_0^{\infty} \tau H(\tau)\cos\omega\tau\,d\tau.$$

Therefore, considering that a(.) is an even and b(.) is an odd faction, we obtain

$$\int\limits_{-\infty}^{\infty} [b'(\omega)a(\omega) - a'(\omega)b(\omega)]\,d\omega$$

$$= \int\limits_{-\infty}^{\infty} \Big[a(\omega)\int\limits_0^{\infty} \tau H(\tau)\cos\omega\tau\,d\tau + b(\omega)\int\limits_0^{\infty} \tau H(\tau)\sin\omega\tau\,d\tau\Big]\,d\omega$$

$$= \int\limits_{-\infty}^{\infty}\int\limits_0^{\infty} \tau H(\tau)\,[a(\omega) - jb(\omega)]\exp(j\omega\tau)\,d\tau\,d\omega.$$

Substituting this result in (7.74) and changing the order of integration (which is easily seen to be permissible) and also considering that according to the Plansherel theorem

$$\int\limits_0^{\infty} H^2(\tau)\,d\tau = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} G^2(\omega)\,d\omega, \qquad (7.75)$$

we obtain

$$\theta_1 = \frac{2\pi\int\limits_0^{\infty} \tau H^2(\tau)\,d\tau}{\int\limits_{-\infty}^{\infty} G^2(\omega)\,d\omega} = \frac{\int\limits_0^{\infty} \tau H^2(\tau)\,d\tau}{\int\limits_0^{\infty} H^2(\tau)\,d\tau} = \tau_1. \qquad (7.76)$$

Further, we will find the dependence between θ_2 and τ_2. Inasmuch as $\tau H(\tau)$ is the Fourier transform of $d\Phi(j\omega)/d\omega$, then

$$\int\limits_0^{\infty} \tau^2 H^2(\tau)\,d\tau = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\Big|\frac{d\Phi(j\omega)}{d\omega}\Big|^2 d\omega = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} |[G'(\omega) -$$

$$- j\varphi'(\omega)G(\omega)]\exp[-j\varphi(\omega)]|^2\,d\omega = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} [G'^2(\omega) + G^2(\omega)\varphi'^2(\omega)]\,d\omega. \qquad (7.77)$$

Substituting this expression in (7.72) and considering (7.75) we obtain

$$\tau_{11}^2 = \theta_{11}^2 + \frac{\int_{-\infty}^{\infty} C'^2(\omega)\,d\omega}{\int_{-\infty}^{\infty} C^2(\omega)\,d\omega}. \qquad (7.78)$$

We define the mean-square length of response as

$$L^2 = \frac{\int_0^{\infty} (\tau - \tau_1)^2 H^2(\tau)\,d\tau}{\int_0^{\infty} H^2(\tau)\,d\tau}.$$

It can easily be seen that $L^2 = \tau_{11}^2 - \tau_1^2$ or, considering (7.76) and (7.78):

$$L^2 = \theta_{11}^2 - \theta_1^2 + \frac{\int_{-\infty}^{\infty} C'^2(\omega)\,d\omega}{\int_{-\infty}^{\infty} C^2(\omega)\,d\omega} = \frac{\int_{-\infty}^{\infty} [\theta(\omega) - \theta_1]^2 C^2(\omega)\,d\omega}{\int_{-\infty}^{\infty} C^2(\omega)\,d\omega} + \qquad (7.79)$$

$$+ \frac{\int_{-\infty}^{\infty} C'^2(\omega)\,d\omega}{\int_{-\infty}^{\infty} C^2(\omega)\,d\omega}.$$

The second term on the right side is determined completely by the given amplitude-frequency characteristic of the circuit and the first term is not negative. It follows that L^2 reaches a minimal magnitude when the first term is equal to zero. For this purpose it is essential that $\theta(\omega) = \theta_1$ = const., i.e., that the circuit phase-frequency characteristic $\varphi(\omega)$ be linear.

Similar relationships between instantaneous signal frequency and its spectrum are obtained in work [24] from which it follows that with a given envelope a signal with a constant instantaneous frequency has the least spectrum width.

6. (See Section 7.3) We will consider the case of very rapid fading ($\tau_k \ll T$) under the assumption that the power spectrum of fluctuations is strictly limited and uniform in a band from 0 to Ω_3. In this case the correlation function is

$$R(t_1, t_2) = \frac{\mu_0^2}{2}\,\frac{\sin \Omega_1 (t_2 - t_1)}{\Omega_1 (t_2 - t_1)}.$$

Inasmuch as $T \gg \tau_k$ in equation (7.33) the upper limit of the integral can be set equal to infinity. Its solution will be

$$\varphi_n(t) = \frac{1}{\pi}\,\frac{\sin[\Omega_1 t - (n-1)\pi]}{[\Omega_1 t - (n-1)\pi]}. \qquad (7.80)$$

Let signal $z_r(t) = a\cos\omega_r t$ where $\omega_r = n\Omega_э = 2\pi/T$. Then the filters in Figure 7.8c have transfer functions equal with an accuracy to a constant coefficient of

$$\frac{\sin[\Omega_э t-(n-1)\pi]}{[\Omega_э t-(n-1)\pi]}\cos\omega_r t, \quad (0<t<T)$$

The transfer function of such a filter, to a great degree of accuracy, amounts to pi-shaped function with a passband of $2\Omega_э$, an average frequency of ω_r, and a phase lag equal to $(n-1)\pi/\Omega_э$. It can easily be seen that in this circuit at the summator output at instant of readout will be the sum of the squares of the values of the envelope of the received signal which has passed through a band filter taken at a Kotelnikov interval of $\pi/\Omega_э$. Such a circuit can be replaced with one band filter with a following quadratic detector by summing the value of the detected voltage using an integrator as shown in Figure 7.9. Apparently such a circuit will be close to optimal in the case of fast fading with a varying fluctuation spectrum but the filters must be in a certain sense matched with this spectrum.

7. (See Section 7.5) In evaluating system noise in a wide-band receiver providing for discrimination of one beam or addition of beams, we assumed that the leading and lagging beams act on a coherent detector (multiplier) or on a matched filter as gaussian noise. Undoubtedly this is justified if randomly selected realizations of a normal process in a given frequency band are used as signals. A question arises as to whether it is not possible to especially select an ensemble of signals so as to decrease system noise.

If we are speaking of selection of one realization of a signal (for example, for a system with a passive interval or with opposed signals), then for complete elimination of system noise it should be required that the signal be orthogonal to its copy which is shifted by any time segment. Inasmuch as this is impossible, usually we limit ourselves to the requirement that in the case of a shift by any time segment above a certain minimal value (on the order of 1/F) the condition of orthogonality be met at least approximately. In this case the leading and lagging beams will act on the decision circuit (matched in time with the beam being selected) much less than normal noise with the same power. Therefore, the requirement is met by signals which are modulated in phase by so-called Barker codes (see, for example, [25]) or a pseudo-random sequence of pulses also called a sequence of maximum length [20].

We can construct a signal whose autocorrelation function represents a single peak of length 1/F, i.e., provides for exact orthogonality of signal and its copy shifted by any interval not divisible by the period and not exceeding 1/F. For this purpose a signal is modulated in a balanced modulator by a special sequence of pulses with a variable amplitude [26]. It can easily be seen that the noise-like signal obtained in this process is modulated in amplitude as well as in phase.

Additional information about optimal reception of signals in a channel with randomly changing parameters can be found in works [2, 3, 27].

BIBLIOGRAPHY

1. Sherman, G., "Fast Multiplicative Fluctuations," In the book: Lectures on the Theory of Communication Systems, edited by Ye. Dzh. Bagdad. Translated from the English, edited by B. R. Levin, "Mir" Press, 1964.

2. Keylos, T., "Channels with Parameters Changing Over Time," Op. cit.

3. Vozenkraft, Dzh. M., "Sequential Reception in Communication Over a Channel with Parameters Changing Over Time," Op. cit.

4. Kharkevich, A. A., "Ocherki obshchey teorii svyazi" [Essays on General Communication Theory], Gostekhizdat, 1955.

5. Siforov, V. I., "Conditions for Obtaining a Higher Carrying Capacity in Communication Channels with Randomly Changing Parameters," Elektrosvyaz', No. 1, 1958.

6. Kailath, T., "Correlation Detection of Signals Perturbed by a Random Channel," Trans. IRE, IT-6, No. 3, 1960.

7. Pugachev, V. S., "Theory of Random Functions and Its Application to Problems in Automatic Control," Second edition, Fizmatgiz, 1960.

8. Kostas, Dzh. P., "Carrying Capacity of Fading Channels Under Conditions of Strong Interference," TIIER, No. 3, 1965.

9. Siforov, V. I., "Carrying Capacity of Communication Channels with Random Changes in Absorption," Radiotekhnika, No. 2, 1958.

10. Finkel'shteyn, Ye. Z., "Optimal Reception of Signals in Channels with Rapid Fading," Works of the Scientific Technical Conference of the LEIS, No. 1, 1967.

11. Law, H. B., "The Detectability of Fading Radiotelegraph Signals in Noise," PIEE, Pt. B, No. 13, 1957.

12. Klovskiy, D. D., "System of Optimal Reception in a Channel Containing Echo Signals," Works of Communications Educational Institutions, No. 19, 1964.

13. Hulst, R., "Inverse Ionosphere," IRE Nat. Conv. Rec., Pt. 1, 1959.

14. Klovskiy, D. D., "Potential Resistance to Interference in Channels with Echo-Signals," Radiotekhnika, No. 12, 1964.

15. Klovskiy, D. D., "Problems in Potential Resistance to Interference in Channels with Echo Signals," Radiotekhnika, No. 10, 1966.

16. Mosier, R. R., Clabaugh, R. G., "Kineplex, a Bandwidth-Efficient Binary Transmission System," Commun. and Electronics, No. 34, 1958.

18. Price, R. and P. E. Green, "A Communication Technique for Multipath Channels," PIRE, No. 3, 1958.

19. Turin, G. L., "An Introduction to Matched Filter," IRE Trans. IT-6, No. 3, 1960.

20. Kaffmen, D. A., "[illegible]" [Synthesis of Linear Multibeat Coding Circuits], In the collection: Theory of Message Transmission, Foreign Literature Press, 1957.

21. Okunev, Yu. B. and L. A. Yakovlev, "[illegible]" [Wide-Band Communication Systems with Component Signals], "Svyaz'" Press, 1968.

22. Kharkevich, A. A., "Concerning the Kotel'nikov Theorem," Radiotekhnika, No. 8, 1958.

23. Turbovich, I. T., "Applying the Kotel'nikov Theorem to Time Functions with an Unlimited Spectrum," Radiotekhnika, No. 8, 1958.

24. Fink, L. M., "Relationships Between Spectrum and Instantaneous Signal Frequency," Problems of Information Transmission, No. 4, 1966.

25. Lezin, Yu. S., "[illegible]" [Optimal Filters and Impulse Signal Storage Circuits], Second edition, Sovetskoye Radio Press, 1969.

26. Kaffmen, D. A., "Investigation of Signals Equivalent to a Pulse," Radiotekhnika, No. 8, 1964.

27. Smol'yaninov, V. M. and L. I. Filippov, "Analysis of Optimal Reception of Signals Passing Through a Channel with Random Parameters," Radio-Tekhnika, No. 2, 1967.

28. Petrovich, N. T. and M. K. Razmikhnin, "[illegible]" [Systems of Communication with Noise-Like Signals], Sovetskoye Radio Press, 1969.

CHAPTER VIII

CHANNEL WITH LUMPED AND IMPULSE INTERFERENCE

8.1. Definition and Basic Characteristics of Lumped and Impulse Interference

In communication channels, along with fluctuation interference which is well approximated by gaussian noise, additive interference of another kind is often encountered. In radio channels, especially medium and short wave, a dominating role is played by lumped and impulse interference, to which this chapter is devoted.

Additive interference in which the main part of the power is concentrated in separate frequency bands less than or comparable to 1/T, where T is the length of a signal element, is called lumped noise. It occurs most frequently in radio channels as a result of the action on the receiver of signals belonging to extraneous communication channels. In this case lumped noise is also called mutual.

Additive interference which differ from zero only in separate time intervals differing greatly from T and separated by much longer intervals free from interference is called impulse interference. Impulse interference is a regular or random sequence of interfering impulses. The sources of impulse interference in radio channels are extremely varied. Included in them are atmospheric discharges, industrial installations, ignition systems in internal combustion engines, medical and household instruments and appliances, etc.

Lumped and impulse interference are found in other communication channels, e.g., in cable, hydroacoustic, etc. The sum of monochromatic interference with random (but not changing in time) amplitudes, frequencies, and phases is an idealized limiting case of lumped noise. The overall frequency band occupied by such interference has a measure of zero and therefore does not reduce the carrying capacity of a channel. From this it follows that in principle there should be methods of signal reception with which idealized lumped noise can be completely suppressed, i.e., will not cause errors.

A sequence of delta-functions occurring at random instants of time with random intensities is an idealized limiting case of impulse interference. The power spectrum of such interference is unlimited but its total time of existence has the measure zero. Consequently, it also does not reduce the carrying capacity of a channel and methods of reception must be used which permit complete suppression of the idealized impulse interference.

Actual components of lumped noise are not precisely monochromatic just as actual impulses are not delta-functions. Therefore, complete suppression of such interference is impossible although it can be partially suppressed. We will explain what we mean.

Let there be at a channel output a signal and a certain additive, non-gaussian interference which we will describe by the average spectral power density. We will assume that the decision circuit is selected so as to be optimal for gaussian interference. The probability of error in this case will be, depending on the nature of the interference, either greater or smaller than in the case of gaussian interference with the same spectral density. However, in many cases, as will be shown below, it differs little from the probability of error in the case of gaussian interference. If it is possible to select signals and construct a decision circuit such that the probability of incorrect reception of a signal element of finite length T with given interference will be as small as desired, we will say that the interference is completely suppressed. If the probability of error remains finite but much less than in the case of gaussian interference with the same spectral density, we will say that the interference is partially suppressed.

Unfortunately, for all practical purposes there does not exist at the present time a general theory for optimal reception in the case of non-gaussian interference. Arriving at such a theory entails great difficulty inasmuch as such interference cannot be completely described by the first two moments. Furthermore, even a unidimensional probability distribution of non-gaussian interference is not invariant with respect to linear conversions.

The problems involved in selecting an optimal signal system and decision circuit for a given interference structure have been considered theoretically only for a few particular cases in which additional simplifying assumptions are made. Included in such particular cases are idealized lumped or impulse interference. Investigating channels in which lumped, impulse, and fluctuation interference is simultaneously present entails tremendous difficulty.

In many works (for example [1, 2]), the effects of lumped and impulse interference on a decision circuit optimal for the case of gaussian interference have been studied in detail and the probabilities of occurring errors have been calculated. The solutions to such problems are rather trivial and their practical importance is limited to those channels in which the intensity of impulse or lumped interference is small in comparison with the intensity of fluctuation interference. In the interest of economy of space, we will not discuss these problems (see Note 1 to this chapter).

Many methods of suppressing lumped and impulse interference have been developed based on intuitive ideas. At the present time the theory behind this problem at best explains the essence of the methods used and serves to explain several details.

We will note that methods permitting good suppression of lumped interference usually worsen conditions for suppressing impulse interference and vice versa. This will be shown below based on several examples. Furthermore, all methods lead to a situation wherein the decision circuit is not optimal for fluctuation interference.

Radio channels in the long, medium, and short wave ranges are always subject to the action of a great deal of lumped noise. This is the consequence of the radio wave propagation conditions in these ranges which result in the generation of a perceptible potential field at considerable distance from the transmitter at every emission. The majority of this noise is of relatively low strength, but the addition of them all together forms the general background noise which is little different in its characteristics from normal white noise. Of this sort of noise, all the findings of the preceding chapters from the study of fluctuation noise are true, but this noise is not the subject of the present chapter. We will here be interested in the individual lumped noise which stands out against the common background noise and is commensurate in power with the useful signal. This noise is encountered in all ranges, and during the planning of systems and equipment for radio communication the possibility of it is always taken into consideration.

With respect to the mechanism of its action on signal reception lumped noise may be divided into three types:

a) noise whose spectrum is concentrated in a frequency band completely or partially coinciding with that occupied by the signal;

b) noise whose spectrum lies outside of the signal frequency band (often called "adjacent-channel" interference); and

c) noise which at the input of the receiving unit has a spectrum which lies outside the signal frequency band, but which, because of nonlinear conversion in the receiving device, form components which fall in the same frequency band as the signal.

Reducing the chances for such nonlinear noise effect is one of the basic problems in developing radio receiving devices. It is studied in detail in all manuals on radio receiving equipment; we will therefore touch upon it only to the degree that it is associated with introducing special nonlinear elements into the receiver circuit to protect against other types of noise.

The rapid development of radio communications, as well as of other applications of radio-electronics, has brought about a situation wherein radio frequency bands are overloaded with different emissions. The result is that mutual interference in many cases predominates over all other factors in restricting the real carrying capacity of radio channels.

The fundamental methods of protection against lumped noise which have been applied from the very beginning of radio communication right up to the present time are based on frequency selectivity. Spatial selectivity which is also widely used is provided by directional transmitting and receiving antennas. Although it was shown as early as the 1930's that frequency and spatial selectivity was by no means the sole method for discriminating a useful signal from noise [3]. The first attempts to put other methods into practice are found only very recently.

Frequency selectivity for the purpose of eliminating mutual interference presupposes a certain regulation of the frequencies set aside for various communication channels. In the "ideal" case the problem of mutual interference could, we should think, be completely solved if each communication channel were assigned a certain frequency band and all other emissions were forbidden in this band. For a number of reasons, however, this solution proves to be impossible. The chief obstacles to ideal regulation [illegible]

1. Existing international agreements regulating the use of [illegible] frequency range are often, and for different reasons, [illegible]

2. Strict regulation in which each radio channel is assigned its own frequency band free from other emissions [illegible] because the number of existing channels in many frequency ranges [illegible] bands which could be set aside [illegible] the best systems [illegible] sible spatial selectivity. Furthermore, the [illegible] tinually increasing, and the [illegible] also rising. This usually [illegible] bands.

3. The frequency band in which the [illegible] essentially [illegible] must be taken into consideration [illegible] certain reserve, "protective" [illegible] to transmit information. The [illegible] because it possible [illegible]

4. In many cases radiation [illegible] emissions of the transmitters, [illegible] the harmonics or combined frequencies [illegible]

In actual fact, [illegible] are [illegible] and transmitters [illegible] interference [illegible] field strength [illegible] limits, however, [illegible] not rise [illegible] tolerable. [illegible] in free space [illegible] more reliable, [illegible]

In cases where natural interference [illegible] the frequency band is often restricted, [illegible] other operating channels.

Various problems of importance for the further development of communications engineering arise in connection with [illegible] noted the problem of decreasing the effect of [illegible] of radio frequency regulation and of correctly [illegible] volves questions of decision system design. [illegible] also be referred the problems associated with [illegible] and spurious receiving channels, as well as the use of spatial selectivity. These problems are not examined here.

A second range of problems consists in studying the possibilities of reducing mutual interference when designing new systems of communication.

Many of these problems have at present not yet been satisfactorily solved. A great obstacle to the development of this field of theory is the lack of adequate statistical data on lumped noise. These data are difficult to obtain because the nature of lumped noise differs in the different radio frequency ranges, at different times of the day, and even in the different seasons of the year. [illegible] that the distribution of lumped noise also depends on the regulation of [illegible] frequency use and the degree to which this is observed. We therefore have to employ rough approximate concepts which do not always lead to reliable conclusions.

8.2. Optimal and Suboptimal Reception in the Case of Lumped Interference

In this section we will limit ourselves to the case of reception of completely known signals [illegible] in a channel lumped noise in the form of harmonic oscillations with varying amplitudes, frequencies, and phases created by [illegible] remaining unchanged at least over a certain time interval [illegible] where [illegible] is the length of a signal element. Without repeating, [illegible]

We [illegible] represent [illegible] signal [illegible] the signals transmitted [illegible] in the [illegible] interval [illegible]

$$[illegible]$$

where [illegible]

The number [illegible] signal realizations m depends on n. If the system is binary and any sequence of symbols is possible, then $m = 2^n$.

---

[*] We will remind the reader that we are speaking not about a signal element but about a sequence consisting of n elements. We will consider all sequences equiprobable.

For an ensemble of interference realizations [illegible]$_k$ and [illegible]$_k$ are random values. Assuming that the frequencies of jumped noise are removed from one another on an average by a magnitude significantly greater than 1/nT, we may consider that [illegible]$_k$ are mutually independent. This assumption is closer to reality, the greater is n. Furthermore, it is natural to assume that all [illegible]$_k$ are evenly distributed over the interval (0, [illegible]) and are independent of one another and also of [illegible]$_k$. The distribution of probabilities of [illegible]$_k$ depends on actual conditions in the channel and will not impose any limitations on it. It can best be characterized by the probability distribution of [illegible]$_k$ which we will designate [illegible]$_k$([illegible]). We will assume this function to be continuous and to have at least a first derivative. It can easily be seen that it is possible, without violating the assumptions made, to include in n(t) also fluctuation interference in the form of white noise by suitably changing function [illegible]$_k$.

The likelihood function for a certain realization of a signal [illegible] consisting of n elements [illegible]$_r$ [illegible] is equal to

$$[illegible]$$

and its logarithm

$$[illegible]$$

where

$$[illegible]$$

The decision system which is optimal in accordance with the criterion of maximal likelihood should select that realization of signal [illegible] for which ln [illegible] > ln [illegible]$_r$ for all r ≠ [illegible]. Figure 5.1 shows the functional diagram needed to make this selection. From the received signal in each of the m branches a realization of the transmitted signal is subtracted and the difference obtained is delivered to an array of filters matched with a segment of cosinusoid cos [illegible]$_j$t of length nT.

The voltage at the output of each filter at instant of readout is equal to [illegible]. After squaring, the voltages obtained go to a nonlinear

---

[1] For generality we will assume that [illegible] can vary for different k. This actually happens in many radio channels inasmuch as in connection with regulation of working frequencies the probability of suppression of powerful interference at certain frequencies is greater than at others.

inertialess quadrapoles with characteristics $U_{out} = f_k(U_{in})$. After adding the outputs of these quadrapoles of each of the m branches we obtain logarithms of the likelihood functions (8.3) from which the comparator selects the greatest.

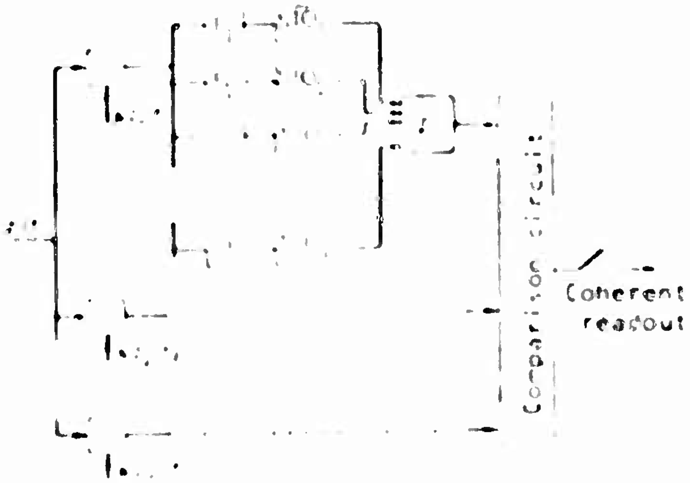


Figure 8.1. Optimal Decision System for a Channel with Lumped Noise. F_k, Filters matched with segments of cosinusoid of length nT, NI_k, Nonlinear inertialess quadrapoles with characteristics $y = \ln f_k(x)$.

A typical peculiarity of this circuit is that the received signal is analyzed immediately over a rather long segment of time nT and a decision is reached not about each code symbol one after the other but about a sequence consisting of n symbols. This is altogether natural inasmuch as the unchangeability of components over a long period of time is an important difference between the interference under consideration and fluctuation interference and only the use of this difference permits suppressing the interference partially. The magnitude of n predetermines the complexity of the system. As can easily be seen the number of elements in such a system is approximately proportional to $n2^n$. It is not possible to significantly simplify the system obtained and still retain its optimality in the general case.

A simpler system can be built which will be asymptotically optimal when the radio between the power of the interference at each of the frequency components approaches zero. With a finite signal power this system can be considered as suboptimal.

For this purpose, assuming in (8.3) that $a_{rk}^2 + b_{rk}^2 \ll A_k^2 + B_k^2$ we will expand the function f_k into a Taylor series and we will limit ourselves to the first two terms:

$$[illegible] \qquad (8.4)$$

The first term of (8.4) does not depend on the subscript r. Therefore, an approximate decision principle can be written in the following form

$$[illegible] \qquad (8.5)$$

for all r ≠ [illegible].

We will set [illegible] and rewrite (8.5) in the following form

$$[illegible] \qquad (8.6)$$

According to the Parseval theorem

$$[illegible] \qquad (8.7)$$

where

$$[illegible] \qquad (8.7a)$$

This permits us to write the decision principle as follows

$$[illegible] \qquad (8.8)$$

If any combinations of signal elements are possible, inequality (8.8) is equivalent to n inequalities of the type

$$\int_{\theta}^{\theta+T} \xi(t) z_k(t)\,dt > \int_{\theta}^{\theta+T} \xi(t) z_r(t)\,dt. \qquad (8.9)$$

where $z_r(t)$ is one signal element. Thus, if we assume that $_k$ are known, decision principle (8.8) is realized by element-by-element reception. In actuality it is necessary to analyze an arriving signal over the interval nT in order to determine $_k$. Therefore, the suboptimal decision system consists of two parts. In the first part the received signal z'(t) is analyzed over length of time nT, values of $_k$ are determined, and a "corrected" signal (t) is formed. In the second part principle (8.9) is realized and the received code symbols are determined in turn. This part coincides completely with the decision circuit used for fluctuation interference, the only difference being that instead of received signal z'(t), (t) is delivered to the input (Figure 8.2).

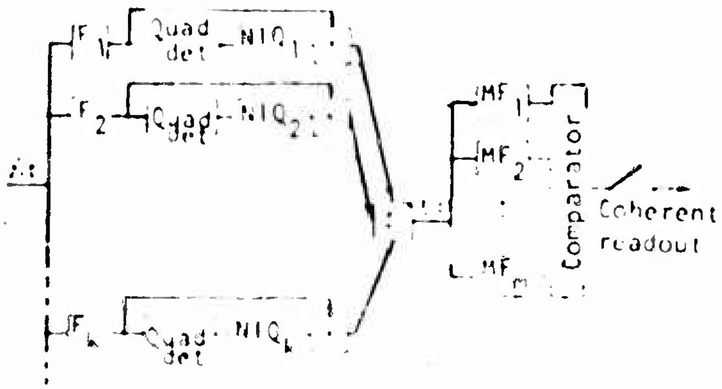


Figure 8.2. Decision System for a Channel with Lumped Noise in the Case of a Weak Signal. F_k, Filters matched with segments of cosinusoid cos k $_1$t of length nT; NIQ_k, Nonlinear inertialess quadrapoles with the characteristic y = $-f'_k(x)$, MF_r, Filter matched with $z_r(t)$.

In that particular case when interference is made up of normal white noise, $w_k(x) = N\exp\left(-\frac{x^2}{2\sigma^2}\right)$, where N is constant and is the interference dispersion. Then,

$$f_k(x) = \ln N - \frac{x^2}{2\sigma^2} \quad \text{and} \quad f'_k(x) = -\frac{1}{2\sigma^2} = \text{const}$$

Thus, $_k$ is constant and the decision system coincides with that obtained in Chapter III. In the case of gaussian interference with a nonuniform spectrum, the magnitudes of $_k$ do not depend on x but vary for different subscripts k. If it is considered that the circuit shown in Figure 8.2 is for weak signals, when $A_k \approx {}_k$, $B_k \approx {}_k$, it can easily be seen that it amounts to the circuit for the "whitening" filter shown in Chapter III (3.71).

In the general case the components of lumped noise have a distribution differing from normal and therefore the coefficients of γ_k depend on $C_k^2 = A_k^2 + B_k^2$. In many radio channels, according to observations, the distribution of probabilities of the square of the amplitude C^2 of lumped noise is close to normal-logarithmic:

$$w_k(x) = \frac{1}{\sqrt{2\pi}\beta} \exp\left[-\frac{1}{2\beta^2} \ln^2\left(\frac{x}{a_k}\right)\right] \qquad (8.10)$$

where a_k is the median value of C_k^2 and β is the mean-square deviation of ln C_k^2. The dimensionless magnitude β, depending on the load of the band and other actual conditions, lies within limits from 2 to 5. In this case,

$$\gamma_k = \frac{1}{C_k^2}\left[1 + \frac{1}{\beta^2} \ln\left(\frac{C_k^2}{a_k}\right)\right]. \qquad (8.11)$$

Considering that ln C_k^2/a_k has a normal distribution with a zero mathematical expectation and a dispersion of β^2, it can be asserted that, with a probability of the order of 0.7,

$$\left|\frac{1}{\beta^2} \ln\left(\frac{C_k^2}{a_k}\right)\right| < \frac{1}{\beta}.$$

Therefore, with sufficiently large β in the first approximation

$$\gamma_k = \frac{1}{C_k^2}. \qquad (8.11a)$$

In this case γ_k can be found within a priori knowledge of the median values of a_k. The signal $\zeta(t)$ formed in this case has an amplitude of spectral components inversely proportional to the amplitudes of corresponding components of signals z'(t).

We will not try to compute the probability of error in the circuits considered inasmuch as this entails a great deal of difficulty. By way of indirect evaluation of the degree of suppression of lumped noise, we will determine the gain in the ratio between signal power and average spectral interference density when a'(t) becomes $\zeta(t)$. This gain is sufficiently well characterized by the increase in resistance to interference in changing from a Kotelnikov receiver circuit which is optimal in the case of white gaussian noise to a circuit which is optimal or suboptimal in the case of lumped noise. We will limit ourselves to the suboptimal circuit shown in Figure 8.2.

The strength of useful signal contained in z'(t) is equal to

$$\frac{1}{2}\sum_k (a_{rk}^2 + b_{rk}^2) = \frac{1}{2}\sum_k c_k^2.$$

The strength of the noise in z'(t) is equal to $\sum_k \gamma_k^2$. The number of terms in these sums is equal to nFT where F is the conditional frequency band occupied by the signal. The ratio between the power of signal element and the average spectral density of the interference is equal to

$$h^2 = \frac{\sum_k c_k^2 / T}{\sum_k \gamma_k^2}. \tag{8.12}$$

After transforming the signal received into ζ(t) in accordance with (8.7a), the strength of the signal proves equal to $\frac{1}{2}\sum_k \psi_k^2 c_k^2$ and interference strength is equal to $\frac{1}{2}\sum_k \psi_k^2 \gamma_k^2$. Therefore the ratio between signal element strength and average spectral interference density in ζ(t) will be equal to

$$h^{*2} = \frac{\sum_k \psi_k^2 c_k^2 / T}{\sum_k \psi_k^2 \gamma_k^2}. \tag{8.13}$$

The gain in such a transformation is equal to

$$G = \frac{h^{*2}}{h^2} = \frac{\sum_k \psi_k^2 c_k^2 \sum_k \gamma_k^2}{\sum_k c_k^2 \sum_k \psi_k^2 \gamma_k^2}. \tag{8.14}$$

If nFT = 1 (signal ΔT or FT in analysis of interference over only one signal element), then

$$G = \frac{\psi^2 c^2 \gamma^2}{c^2 \psi^2 \gamma_k^2} = 1,$$

i.e., there will be no suppression of interference.

We will consider another extreme case when nFT ≫ 1. Dividing the numerator and denominator of (8.14) by (nFT)^2 and using a wavy line to indicate averaging over all frequency components, we obtain

$$G = \frac{\widetilde{\psi_k^2 c_k^2} \cdot \widetilde{\gamma_k^2}}{\widetilde{c_k^2} \cdot \widetilde{\psi_k^2 \gamma_k^2}}. \tag{8.15}$$

With an increase in NFT the average values in the numerator and denominator of (8.15) approach in probability corresponding mathematical expectations so that when nFT ≫ 1

$$G \approx \frac{\overline{\psi_k^2 c_k^2}\;\overline{\gamma_k^2}}{\overline{c_k^2}\;\overline{\psi_k^2 \gamma_k^2}}.$$

Assuming, as formerly, that $C_k^2 \ll \gamma_k^2 \approx C_k^2$, it is possible to consider γ_k^2 and c_k^2 as independent variables, consequently,

$$G \approx \frac{\overline{\psi_k^2}\,\overline{c_k^2}\,\overline{\gamma_k^2}}{\overline{c_k^2}\,\overline{\psi_k^2}\,\overline{\gamma_k^2}} \cdot \frac{\overline{\psi_k^2}\,\overline{\gamma_k^2}}{\overline{\psi_k \gamma_k}^2}. \qquad (8.16)$$

In that case when γ_k^2 has a normal-logarithmic distribution (8.10) and all α_k are the same, and the values of ψ_k are determined from the approximate formula (8.11a), it is easy to calculate

$$\overline{\gamma_k^2} = \alpha_k e^{\beta^2/2},\ \overline{\psi_k^2} = \left(\overline{\frac{1}{C_k^2}}\right) \approx \left(\overline{\frac{1}{\gamma^2}}\right) = \frac{1}{\alpha_k} e^{\beta^2};$$
$$\overline{\psi_k^2}\,\overline{\gamma_k^2} \approx \left(\overline{\frac{1}{\gamma^2}}\right) = \frac{1}{\alpha_k} e^{\beta^2/2},$$

and, finally,

$$G = e^{\beta^2}. \qquad (8.17)$$

With values of β ranging from 2 to 4 the gain G fluctuates within limits $G \approx 3 \cdot 10^3 - 8 \cdot 10^{12}$. This means that even a suboptimal system suppresses lumped noise almost completely if nFT is sufficiently large.

The magnitude of n is determined by the interval of time over which the strengths of the components of lumped noise can be considered constant and is usually small. Therefore, to realize the possibility of sufficiently effective suppression of lumped noise, wide-band dense signals with a large base FT should be used. These can be impulse signals differing from zero only for a small part of the length of element T or the noise-like signals [5] which were mentioned in Chapter VII. Sometimes it is convenient to shape a wide-band signal from a simple narrow-band signal using frequency diversity.

In the latter case another method of reception based on the selection of that diversity branch in which the lumped noise has the least strength is possible. It is simpler although further removed from optimal. This means that in reception principle (8.6) the values of ψ are not determined from formulas (8.11) but are considered equal to unity for that component for which the lumped noise is minimal and zero for all remaining components.

In work [4] this method is considered for that case when the signal and the lumped noise are subjected to Rayleigh fading and this describes the situation in shortwave radio channels rather well. This means that the amplitude of γ_k of the lumped noise component has a conditional distribution density of

$$w(\gamma/\sigma_0^2) = \frac{\gamma}{\sigma_0^2}\exp\left(-\frac{\gamma}{2\sigma_0^2}\right),$$

where σ_0^2 is a random variable conforming to the distribution shown in (8.10). It can be shown that even in this situation it is possible to obtain quite significant suppression of lumped noise.

8.3. Mutual Interference and Frequency Band Occupied by a Signal

In the preceding section we showed that in a channel with narrow-band lumped noise it is advisable to use wide-band signals. As long as wide-band signals are used only in exceptional cases, they indeed permit obtaining a large power gain with respect to lumped noise. If wide-band radio communication systems come into wide use, the picture will change greatly inasmuch as mutual interference will no longer be narrow-band and the statistics will vary greatly from the picture presented above.

We will assume that the load of a band, i.e., the number of transmitters operating simultaneously, does not change. Then in the transition from narrow-band signals to wide-band signals the average number of sources determining the interference component in (8.1) will increase by as many times as did the base of the signals. With an increase in the base the probability density of instantaneous values of the components of mutual interference will approach normal. Therefore, the optimal decision system will approach a Kotelnikov system and there will no suppression of interference. A question arises as to whether it is possible to decrease mutual interference by proper selection of the signal base.

For the more than half-century of existence of radio communication the conviction has reigned that the only way to reduce mutual interference lies in reducing the frequency band assigned to each radio channel. Indeed, the narrower this frequency band, the more such bands can be designated in a given range for various channels, thereby providing for absence of mutual interference in the passband. Even in the case of incomplete regulation of frequencies, the reduction in assigned band in reducing a receiver's passband reduces the probability of random interference entering this band.

The development of communication equipment has been guided in large measure by this idea. In the area of transmitting discrete messages over radio channels (radiotelegraphy) the transition from spark transmitters which emitted damping oscillations with a wide power spectrum to arc transmitters and then to tube-type transmitters of "non-damping" oscillations permitted a large increase in the number of "operating frequencies" in a given range. Until approximately the 1950's the frequency band assigned to a radio channel was determined mainly by frequency instability. The shortage of free frequency bands which grew worse with every passing year ("crowded ether") was the principal stimulus in developing equipment with greater frequency precision and stability and this, unquestionably, was a progressive trend. Frequency precision increases by approximately one order over a period of ten years. At the present time in most cases the required minimal receiver passband (at least in the range of radio frequencies to 20-25 mc and with $T \leq 20$-25 msec) is determined not by frequency imprecision but by the width of the signal power spectrum which, in turn, depends on the base.

The effort to reduce the receiver passband has led to a tendency to decrease the signal base. The single-channel systems of voice-frequency telegraphy in which signals were segments of a modulated harmonic oscillation, the base of which was much greater than unity, formerly in wide use have almost completely disappeared. In FK systems the difference in signal frequencies has gradually been reduced and at the present time in some systems has reached 1/T. In a binary system this corresponds to FT = 2. Finally, during the past few years RPM systems with FT = 1 have been introduced.

However, as has been stated many times, the conditional frequency band F and the width of the power spectrum have different meanings. A receiver passband cannot be reduced to F since the transient processes in a filter cause an impermissible increase in the probability of error. In the effective width of a power spectrum of a signal and, consequently, the required receiver passband is much wider than the band of frequencies F, dividing a range among radio channels proves to be uneconomical.

For a more economical use of a frequency range, it is desirable to use those signals the power of which is concentrated as well as possible within the limits of frequency band F. But for this the signal base must be great (see Note 4 to Chapter III).

For purposes of explanation we will assume that in a certain band F, N transmitters are operating and FT » 1. For simplification we will assume that messages are encoded in a binary code and each of the transmitters uses a pair of orthogonal or opposed signals but that these pairs are selected randomly for each channel.[1] A useful signal for which the decision system of a receiver is calculated arrives at the input of any of the channels being considered along with interference which is the sum of N - 1 signals of the remaining channels using the given frequency band.[2] With a sufficiently large N this interference can be considered normal fluctuating interference, especially in that case when wide-band signals have a uniform spectrum in band F, i.e., are noise-like.

---

[1] If all signals of the channels using the frequency band were selected from one set of mutually orthogonal signals, each of them could be separated with a filter matched with it and mutual interference would be absent as in the case of an orthogonal separation multiplexing system (see Chapter IX). However, this is unrealizable if signals are emitted by different transmitters and arrive at the input of a transmitter with different delays since the orthogonality of the signals, generally speaking, is disrupted when one is shifted in time with respect to another.

[2] Since when FT » 1 the share of signal power outside the band is small, the receiver passband can be only a little wider than F and this permits ignoring the interference lying outside the frequency band. It is possible to also ignore interference caused by spurious emissions in comparison with the interference created by the principal emission of the transmitters, especially if N is large.

The spectral density of lumped noise in this case is

$$N_\Sigma = \frac{1}{F}\sum_{i=1}^{N-1} P_i \approx \frac{(N-1)P_{av}}{F}, \tag{8.18}$$

where P_i is the power of the i-th interfering signal and P_{av} is the average signal power.

The ratio between the power of the useful signal and the total spectral density of interference which determines the probability of error is equal to

$$h^2 = \frac{P_s T}{N_\Sigma + N_0} = \frac{P_s T F}{N_0 F + (N-1)P_{av}}. \tag{8.19}$$

In "overloaded" radio frequency ranges the spectral density of fluctuation interference (mainly receiver set noise and some atmospheric noise) N_0 can be ignored in comparison with N_Σ. In this case

$$h^2 \approx \frac{P_s}{(N-1)P_{av}} FT. \tag{8.19a}$$

If h^2 is known based on obtaining a required level of fidelity, then

$$N \approx \frac{P_s}{P_{av}}\frac{FT}{h^2} + 1, \tag{8.20}$$

i.e., the permissible number of simultaneously operating independent channels in frequency band F is approximately proportional to the signal base. In this case the permissible range load density, i.e., the number of channels per unit of frequency band (when N ≫ 1) is

$$\frac{N}{F} \approx \frac{P_s T}{P_a h^2} \tag{8.20a}$$

does not depend on F in the first approximation.

Separating the required signal from a large number of interfering signals with commensurate power is done by using matched filters or other circuits similar to the way in which beams are separated in the reception of wide-band signals in a channel with multi-beam propagation (Chapter VII). Specifically, in a system with matched filters the useful signal creates sharp peaks in the envelope while all remaining signals with which the filter is not matched create only background noise.

We will note that the ratios obtained remain valid if among the N interfering signals there are narrow-band signals, as long as they are uniformly distributed in frequency band F. On the other hand, as already noted, wide-band signals do not represent for narrow-band systems significantly greater interference than do narrow-band signals for the same spectral density. This can easily be understood since only a very small share of the power of a wide-band signal enters the receiver passband of a narrow-band signal. Therefore, it can be asserted that wide-band and narrow-band communication systems cannot be considered incompatible.

We will present the following numerical example to demonstrate the formulas obtained.

Let $h^2 = 20$ (which provides for a probability of error less than 10^{-4} in the case of incoherent reception in the absence of fading or with quasi-Rayleigh fading if the share of the regular component is sufficiently great), $T = 10^{-2}$ sec, $F = 10^5$ cps, and $P_s = P_{av}$. In this case the signal base $B = 2FT =$ $= 2000$. From (8.20) it follows that in this frequency band signals from $N = 51$ transmitters can be transmitted simultaneously.

This number can be increased if power P_s of the useful signal at the input of the receiver is greater than the average power P_{av} of the interfering signal. This in practice is almost always the case if only because in the selection of frequencies conditions of radio wave propagation are considered and optima are obtained for a given disposition of transmitter and receiver. Furthermore, even when using antennas with a small directivity factor, it is always possible to achieve an advantage for useful signal in each channel. Assuming that $P_s = 2P_{av}$, we obtain $N \approx 100$ and this gives a density of range load on the order of one channel per kilocycle.

This coincides approximately with the load density when using narrow-band signals, for example, FM, in the case of regulated frequency distribution. Indeed, the minimal frequency band F occupied by a binary FM signal is equal to $2/T$, or when $T = 10^{-2}$ sec to $F = 200$ cps. But the receiver passband, as we have seen, must be much wider in order to avoid errors from transient processes and constitutes a minimum of 500-600 cps. If we consider the required "protective" band, which we discussed earlier, it appears that the range load density is greater than one channel by a kilocycle when $T = 10^{-2}$ sec is not provided with an FM system. Of course, it is possible to increase the load density in the case of a regulated distribution of narrow-band signals by using not FM but PM or RPM which cuts F in half. However, in the case of narrow-band signals there are additional reserves for increasing the range load density.

One such reserve is reduction of the permissible magnitude of h^2. This can be achieved by using diversity reception and in channels with multibeam propagation the method of adding beams described in Chapter VII. It is also possible to reduce the permissible value of h^2 by using correcting codes, especially in feedback systems (see Chapter XI).[1]

---

[1] We will not consider increasing the range load density by lengthening a signal element here because increasing T reduces the rate of information transmission. The number of channels in a given frequency band is proportional to T not only for wide-band but also for narrow-band signals.

The number of channels using a given frequency band can be greatly increased if it is considered that not all channels operate simultaneously. We will use α to indicate the probability that a certain channel is operating (for simplicity we will consider this probability the same for all channels). We will stipulate that h must be not less than a permissible value, say, a probability of 0.99. This means that with a probability of 0.99 the number of simultaneously operating channels must not exceed N in formula (8.20). By using n to denote the number of operating channels and N_0 the total number of channels using a given frequency band, it is possible to find the probability of obtaining a magnitude for n which is determined by the binomial distribution:

$$p(n) = C_{N_0}^{n} \alpha^{n} (1-\alpha)^{N_0-n},$$

and the probability that n will not exceed the permissible value of N:

$$P\{n \leq N\} = \sum_{n=1}^{N} C_{N_0}^{n} \alpha^{n} (1-\alpha)^{N_0-n}. \qquad (8.21)$$

For large N_0 it is possible to replace the binomial distribution with a normal distribution and consider as approximately correct

$$P\{n \leq N\} = \Phi\left[\left(\frac{N}{N_0} - \alpha\right)\sqrt{\frac{N_0}{\alpha(1-\alpha)}}\right] \qquad (8.22)$$

Assuming this probability to be equal to 0.99 and considering that $0.99 \approx \Phi(2.6)$, it is easy to obtain:

$$N = 2{,}6\sqrt{N_0\alpha(1-\alpha)} + \alpha N_0,$$

hence for given α and N it is easy to find N_0. For example, with $\alpha = 1/2$ and N = 100 the number of channels can be $N_0 = 170$. With an increase in N (for example, by increasing the base) the N_0/N ratio increases and tends toward $1/\alpha$.

In the case of a regulated distribution of frequency bands among narrow-band channels it is impossible to use non-simultaneous operation of them in order to increase the range load density. Indeed, if one frequency band is assigned to channels and their transmitters create powers of the same order at the input of the receivers, when they are operating simultaneously with a probability of α^2, communication will be disrupted in both channels.

In work [6] there is a comparison of narrow-band and wide-band signals from the point of view of mutual interference in the case of unregulated band distribution. The author of this work gives preference to wide-band signals. The essence of his argumentation amounts to the following. For a given range load and given signal powers the average spectral density of lumped noise is the same for narrow-band or wide-band signals. But in a frequency band of a narrow-band signal the spectral density of interference fluctuates within wide limits while in a frequency band of a wide-band signal it changes little. Therefore, in the case of wide-band signals a larger range load than in the case of narrow-band signals is permissible in obtaining a guaranteed high probability of satisfactory communication. In other words, a wide-band system provides for satisfactory communication with small fluctuations in the probability of error while in the case of a narrow-band system under the same conditions communication will sometimes be excellent and sometimes unsatisfactory.

It would hardly be correct to assume that in the case of wide-band systems it is possible to refrain completely from regulation of frequencies. However, regulation for wide-band signals under certain conditions is easier and need not be too rigid and deviation from regulated use of frequencies leads to consequences not as severe as in the case of narrow-band signals.

8.4. Mathematical Description of Impulse Interference

For analysis of conditions for signal reception in the case of impulse interference we will represent an interfering impulse $n_1(t)$ in the form of a Fourier series over an interval of time equal to the length of an element of a signal being received:

$$n(t)=\sum_{k=1}^{\infty}(a_k\cos k\omega t+b_k\sin k\omega t)\quad (0<t<T), \tag{8.23}$$

where

$$\omega=\frac{2\pi}{T}.$$

In essence series (8.23) represents not a single impulse $n_1(t)$ but a periodic sequence of such impulses with a period of T. But inasmuch as we are studying element-by-element reception, we are interested in the behavior of such a signal as well as the interference over the interval (0, T). It is also possible to view (8.23) as an expansion of a single impulse into a series with respect to segments of a sinusoid and a cosinusoid. If function $n_1(t)$ expresses the realization of a random impulse then a_k and b_k in (8.23) are realizations of certain random values.

For realization of an idealized impulse

$$n_1(t)=A\delta(t-t_1) \tag{8.24}$$

(where t_1 is the moment of occurrence of the impulse $(0 \le t_1 < T)$ and A is a coefficient the physical essence of which will be explained below) the values of coefficients a_k, b_k in (8.23) can easily be determined using the usual rule for expanding a function into a Fourier series:

$$\left.\begin{aligned} a_k &= \frac{2}{T}\int_0^T A\,\delta(t-t_1)\cos k\omega_1 t\,dt = \frac{2A}{T}\cos k\omega_1 t_1 \\ b_k &= \frac{2}{T}\int_0^T A\,\delta(t-t_1)\sin k\omega_1 t\,dt = \frac{2A}{T}\sin k\omega_1 t_1 \end{aligned}\right\} \quad (8.25)$$

We will consider the values of A and t_1 to be random and independent.

We will note that in distinction from the case of white noise the coefficients of the Fourier series a_k and b_k are not independent. Indeed, by knowing any two of these coefficients it is possible to unambiguously determine the magnitudes of A and t_1 and in accordance with them regenerate the values of all remaining coefficients. Nevertheless, all Fourier coefficients are pairwise not correlated as long as the magnitude of t_1 with equal probability adopts any value in the interval $0 \leq t_1 \leq T$. This can easily be seen by computing the mathematical expectation of the product of any pair of coefficients. For example, for a_{k1} and a_{k2} $(k_2 \neq k_1)$, we have

$$\begin{aligned} \overline{a_{k_1}a_{k_2}} &= \overline{\frac{4}{T^2}A^2\cos k_1\omega_1 t_1\cos k_2\omega_1 t_1} \\ &= \frac{2}{T^2}\overline{A^2}\,\overline{[\cos(k_1+k_2)\omega_1 t_1 + \cos(k_1-k_2)\omega_1 t_1]} \\ &= \frac{2}{T^2}\overline{A^2}\left[\int_0^T \frac{1}{T}\cos(k_1+k_2)\omega_1 t_1\,dt_1 + \right. \\ &\quad \left. + \int_0^T \frac{1}{T}\cos(k_1-k_2)\omega_1 t_1\,dt_1\right] = 0, \end{aligned}$$

whence it follows that a_{k1} and a_{k2} are not correlated. The same result can be obtained for any pair of coefficients and and also for any pair of coefficients with different subscripts.

The mathematical expectation of a_k and b_k with a uniform distribution of t_1 and any distribution of probabilities of A is equal to zero. The dispersion of each of the coefficients of a Fourier series is

$$\overline{a_k^2} = \overline{\frac{4A^2}{T^2}\cos^2 k\omega_1 t_1} = \overline{b_k^2} = \overline{\frac{4A^2}{T^2}\sin^2 k\omega_1 t_1} = \frac{2}{T^2}\overline{A^2} \qquad (8.26)$$

We will set $2(\overline{A^2})/T = {}_1$. By analogy with (3.16) it is possible to define the physical meaning of ${}_1$ as the spectral density of an interfering impulse. In distinction from the case of white noise the magnitude of ${}_1$ is not constant but assumes different values for different elements of the signal. Specifically, it is equal to zero if over the length of a given element interfering impulses do not arrive at the receiver input. When there is a sufficiently strong interfering impulse, as a rule, the magnitude of λ_1 proves to be much greater than the spectral density of fluctuation interference and impulse interference can for all practical purposes completely destroy the information contained in a given signal element.

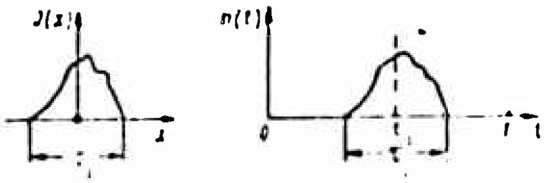


Figure 8.3. Interfering Impulse.

Actual interfering impulses have a finite length. Let an impulse be described by a certain function (Figure 8.3)

$$n_{u}'(t) = J(t - t_1),$$

where J(x) depicts the shape of the pulse and when $|x| > \tau/2$ $J(x) = 0$. Then we obtain an expression for the Fourier coefficients

$$\left.\begin{aligned} a_k &= \frac{2}{T}\int_0^T J(t - t_1)\cos k\omega_0 t\,dt, \\ b_k &= \frac{2}{T}\int_0^T J(t - t_1)\sin k\omega_0 t\,dt \end{aligned}\right\} \quad (8.27)$$

If $\tau \ll 2\pi/k\omega_0$, then, inasmuch as the integrand differs from zero only within limits from $t_1 - \tau/2$ to $t_1 + \tau/2$, it is possible to set approximately

$$\cos k\omega_0 t \approx \cos k\omega_0 t_1 \text{ and } \sin k\omega_0 t \approx \sin k\omega_0 t_1$$

Then

$$\left.\begin{aligned} a_k &\approx \frac{2}{T}\cos k\omega_0 t_1\int_0^T J(t - t_1)\,dt, \\ b_k &\approx \frac{2}{T}\sin k\omega_0 t_1\int_0^T J(t - t_1)\,dt, \end{aligned}\right\} \quad (8.28)$$

which coincides with (8.23) if we use λ to denote the "impulse area"

$\int_0^T J(t - t_1)\,dt \approx \int_{-\infty}^{\infty} J(x)\,dx$. Thereby the physical meaning of the coefficient λ is expressed in an idealized representation of an impulse (8.24).

If the condition $\tau \ll 2\pi/k\omega_0$ is not met, expression (8.28) will not be valid. However, in this case a rather rigid connection is maintained between the Fourier coefficients (8.27) when $\tau \ll T$.

Indeed, for example, let a_k and b_k be known. We will show that this permits finding coefficients a_{k+1} and b_{k+1} approximately:

$$\begin{aligned} a_{k+1} &= \frac{2}{T}\int_0^T J(t - t_i)\cos(\omega_0 + k\omega_0)t\,dt \\ &= \frac{2}{T}\int_0^T J(t - t_i)\cos k\omega_0 t\cos\omega_0 t\,dt \\ &- \frac{2}{T}\int_0^T J(t - t_i)\sin k\omega_0 t\sin\omega_0 t\,dt \approx \\ &\approx \frac{2}{T}\cos\omega_0 t_i\int_0^T J(t - t_i)\cos k\omega_0 t\,dt \\ &- \frac{2}{T}\sin\omega_0 t_i\int_0^T J(t - t_i)\sin k\omega_0 t\,dt \\ &= a_k\cos\omega_0 t_i - b_k\sin\omega_0 t_i \end{aligned} \tag{8.29}$$

Here there is an approximate equality because for all values of t for which $J(t - t_i)$ differs from zero, cos $\omega_0 t$ and sin $\omega_0 t$ change imperceptibly and they change less, the less is the τ/T ratio and they can be replaced by cos $\omega_0 t_i$ and sin $\omega_0 t_i$. Similarly,

$$b_{k+1} \approx a_k\sin\omega_0 t_i + b_k\cos\omega_0 t_i \tag{8.30}$$

From these equalities it follows that

$$a_{k+1}^2 + b_{k+1}^2 \approx a_k^2 + b_k^2 \tag{8.31}$$

By simple transformations from (8.29) and (8.30) we also obtain the approximate equation:

$$\left[\frac{a_k a_{k+1} + b_k b_{k+1}}{a_k^2 + b_k^2}\right]^2 + \left[\frac{a_k b_{k+1} - b_k a_{k+1}}{a_k^2 + b_k^2}\right]^2 \approx 1 \tag{8.32}$$

From equations (8.31) and (8.32), knowing a_k and b_k, it is possible to find approximately a_{k+1} and b_{k+1} in the case of an unknown impulse shape and instant of arrival t_i and the precision of the result will be greater, the less is the τ/T ratio.

We will examine the case when n independent random interfering impulses represented by a delta-function arrive at a receiver input during the length of a signal element.

Then the impulse interference is

$$n_i(t) = \sum_{r=1}^{n} A_r\delta(t - t_r) \tag{8.33}$$

where t_m is the instant of occurrence of the m-th impulse, and $2(\overline{\Lambda_m^2})/T$ is its spectral density.

In this case, as can easily be seen,

$$\left.\begin{aligned} \xi_k &= 2\sum_{m=1}^{n}\frac{\Lambda_m}{T}\cos k\omega_0 t_m, \\ \eta_k &= 2\sum_{m=1}^{n}\frac{\Lambda_m}{T}\sin k\omega_0 t_m, \end{aligned}\right\} \qquad (8.34)$$

$$\overline{\xi_k^2} = \overline{\eta_k^2} = \frac{2}{T^2}\sum_{m=1}^{n}\Lambda_m^2.$$

If n is sufficiently large and Λ_m are random values having a limited dispersion, then according to the central limit theorem ξ_k and η_k have an approximately normal probability distribution. Considering that they furthermore are not mutually correlated, it can be concluded that with a large number of impulses in interval T the impulse interference differs little from normal white noise as was discussed above. This same result can be generalized to apply to the case of impulses of finite length, the only difference being that the noise formed by such impulses is not white since its spectral density at high frequencies diminishes faster, the longer the length of the impulses.

To describe the Fourier coefficients of impulse interference formed by chaotically occurring impulses, it is further essential to note that ξ_k and η_k are not correlated with one another for different elements of a received signal. This obviously follows from the fact that various impulses which are not independent on one another take part in the formation of these coefficients.

8.5 Possibilities in Principle of Suppressing Impulse Interference

The rigid functional dependence between coefficients ξ_k and η_k of impulse interference yields the possibility of constructing a decision system of a receiving device in which the presence of impulse interference does not increase or increases almost not at all the probability of incorrect signal reception. In the idealized case, when the impulses are represented by delta-functions, it is possible to achieve complete suppression of impulse interference. In the case of actual impulses of finite length the interference can be suppressed almost completely as long as $\ll 1$ and during the time of reception of one signal element the number of interfering impulses is rather small.

Let a signal occupying frequency band F and impulse interference arrive at the input of a receiving device (Figure 8.4). We will not at first consider the effects on reception of the inevitable fluctuation interference. We will send the received signal with the interference to two multipliers to which are applied reference voltages $\cos k'\omega_0 t$ and $\sin k'\omega_0 t$ where k' is a whole number such that the frequency of $k'\omega_0$ lies outside the signal frequency band. For example, it is possible to choose $k' = k_1 - 1$ or, as was done in Figure 8.4, $k' = k_2 + 1$. The multiplier output voltages are integrated over the interval (0, T) and the resultantly derived voltages proportional to a_k, and b_k, are fed to a special circuit which computes the value of A and t_i. These data permit the interference impulse to be regenerated if it sufficiently accurately approximates a delta-function. Inasmuch as time T is spent on integration the regenerated impulse is delayed by that time in comparison with the impulse which entered the input of the receiving unit. If the incoming signal is passed through a delay line of time T and the regenerated impulse is subtracted from it, we may, in principle, obtain a signal free of impulse noise.

The given circuit is, of course, very complex for practical realization and is regarded here only as a demonstration of the possibility, in principle, of suppressing impulse noise in the case of ideal delta-impulses.

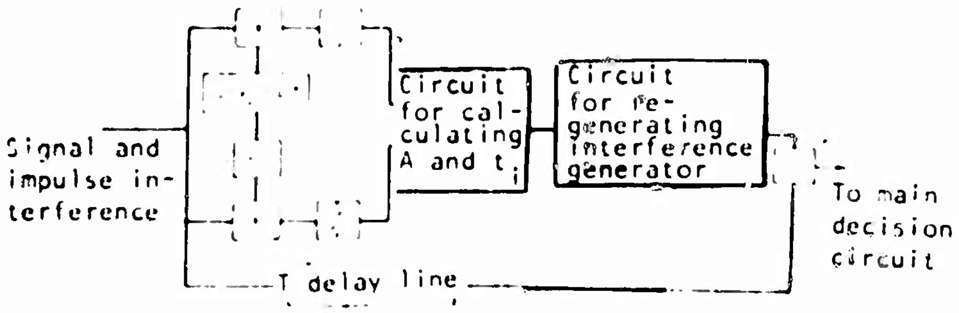

Signal and impulse interference
Circuit for calculating A and t_i
Circuit for regenerating interference generator
To main decision circuit
T delay line

Figure 8.4. Diagram Illustrating the Possibility in Principle of Compensating for Impulse Interference.

Practicable methods of complete or almost complete suppression of impulse noise will be examined below, but before proceeding to a description of them it is useful to elucidate from the example of the idealized circuit in Figure 8.4 certain general patterns which characterize all such methods. Let us begin with a consideration of the defects of this network and of the fundamental possibilities of eliminating them.

First of all, let us remark that the circuit of Figure 8.4 provides for compensation of the interference impulse only in case it is the only one throughout the length of the signal element. This drawback may be to a considerable degree eliminated by complicating the circuit. One possibility is that instead of expanding the signal plus noise into a Fourier series over an interval of duration T it be expanded over interval T/n, where n is some whole number. Here in contrast to the circuit of Figure 8.4 the reference voltage must have a frequency divisible not by ω_0, but by $n\omega_0$, and, as before, lie outside the signal frequency band; and integration must be performed over time T/n, while the delay line must be adjusted for the same period. Under this condition all the interference impulses may be compensated if there is no more than a single impulse in each T/n interval.

Another possibility for suppressing n interference impulses randomly scattered throughout the signal element length is to use n pairs of reference voltages of $\cos k'\omega_0 t$ and $\sin k'\omega_0 t$ with different values of k' and frequencies lying outside the signal frequency band. This permits us to determine 2n values of $\alpha_{k'}$, $\beta_{k'}$ which may be substituted in (8.34) to compute the 2n unknown A_m and t_m. The computation can in principle be done electronically and the compensation accomplished as in Figure 8.4.

Both of these methods permit the compensation of no more than a certain number n of the interference impulses for which the circuit is designed. It is obvious that forming a system capable of compensating any arbitrarily large number of impulses is fundamentally impossible because as n grows larger the impulse noise approximates normal white noise.

Let us return to the system in Figure 8.4 which is designed to compensate individual interference impulses and take into consideration the effect of the inevitable fluctuation noise. Its action, as is easy to see, shows itself in the fact that it is not the Fourier coefficients $\alpha_{k'}$ and $\beta_{k'}$ of the interference impulse, but the sums $\alpha_{k'} + \alpha_{k'f}$ and $\beta_{k'} + \beta_{k'f}$, where $\alpha_{k'f}$ and $\beta_{k'f}$ are the coefficients of expansion into a Fourier series of fluctuation noise over interval (0, T) at frequency $k'\omega_0$, which go to the circuit for computing the parameters A and t_i. The result is that parameters And and t_i will be inaccurately computed and the interference impulse will not be fully compensated. Furthermore, if throughout a given signal element an interference impulse does not proceed to receiver input the compensating impulse will nevertheless be formed under the effect of the corresponding component of the fluctuation noise and will be added (with sign reversed) to the signal. Since the Fourier coefficients of the white noise are mutually independent this will not result in compensation of the noise, but, on the contrary, will increase its spectral density.

It may therefore be said that the system of Figure 8.4 by compensating for impulse noise intensifies, as it were, the strength of the fluctuation noise. This increase in spectral fluctuation-noise density is ordinarily, however, not great in comparison with [illegible]$_i$.

In order to lessen this deficiency we may have recourse to a more complex system and employ a certain number r of units which compute the parameters A and t_i and which use different frequencies $k'\omega_0$. Averaging the values obtained for these parameters we may increase the accuracy with which the compensating impulse is shaped and reduce the increase in fluctuation noise intensity to a negligible value. If in so doing it must be possible to compensate n impulses, then nr pairs of reference voltages, 2nr multipliers and integrators, and r circuits for computing the parameters $A_m, t_m (m = 1, \ldots, n)$ with subsequent averaging over all r circuits will be required.

Therefore impulse noise compensation is the more effective, the broader the frequency band which is used to analyze the oscillations at receiver input. This conclusion, as we shall see from the following examples, is common to all known methods of suppressing impulse noise. The reason for this may be cited as the fact that the fundamental difference between the series of expression (8.23) and the similar series for fluctuation noise is the rigid connection between the coefficients α_k and β_k. By exploiting this connection (which manifests itself in particular as the short duration of the interfering impulse) we may by some method detect, analyze, and eliminate impulse noise. It is natural that it is possible to do this the more easily and completely, the greater the number of Fourier coefficients α_k and β_k that are subjected to analysis, i.e., the broader the frequency band that is taken into consideration in the reception process.

We would remark that the foregoing is valid only while there is no lumped noise in the expanded frequency band. Otherwise the components of the lumped noise will be added to the coefficients α_k, and β_k, used to compute parameters A and t_i, and the compensating impulse will prove to be sharply distorted. The result will be that, instead of the impulse noise being compensated, the error probability will rise under the effect of lumped noise lying outside the frequency band occupied by the signal.

From this it follows that measures to suppress impulse noise may increase the effect of lumped noise lying outside the signal frequency band. This drawback manifests itself to some degree in all methods of impulse noise suppression. It cannot usually be completely eliminated, and therefore when a receiver circuit is being designed compromise solutions must be adopted in which the impulse noise is not completely suppressed, but is so to a significant degree, while the lumped noise affects reception but little more than in a system designed with no regard to impulse noise.

Let us turn our attention to still another important feature of the network in Figure 8.4, namely, the nonlinear device for computing parameters A and t_i. This device must be nonlinear, as ensues from the nonlinear nature of (8.23) or (8.34) with respect to these parameters. The need for a nonlinear device follows also from the fact that the Fourier coefficients of the impulse noise are mutually uncorrelated and are consequently not linked to each other by any linear relationship

Under actual conditions interfering impulses are not delta-functions. Usually they can be considered the result of passage of a delta-function through a linear circuit [7]. In the general case non-gaussian interference can be described if, for any k, k-dimensional distribution functions are given. However, the task can be simplified when the impulse nature of the interference is retained. Let there be a certain number n such that the length of an interfering impulse exceeds T/n practically not at all, where T as formerly is the length of a signal element.[1] If n is sufficiently great, analysis of an element of an arriving signal $z'(t)$ can in the first approximation be replaced by an analysis of its readout values at discrete instants of time at intervals of $1/a$. The values of interference at these points can be considered independent and, therefore, for finding a likelihood function and constructing a decision principle it is sufficient to know the unidimensional distribution of interference probability. This is done in work [8], the contents of which are reflected briefly in what follows.

---

[1] This requirement is essence means that during time T/n an interfering impulse is damped so much that what is left can be ignored in comparison with the inevitable fluctuation interference present.

Let the unidimensional probability distribution density of interference be equal to $w(x)$. Limiting ourselves to values of the received signal at instants of time $k\Delta$, where $\Delta = 1/n$ and k is a whole number, it is possible to represent the likelihood function for signal $z_r(t)$ in the form

$$\Lambda_r = \prod_{k=1}^{n} w(z'_k - z_{rk}),$$

where

$$z'_k = z'\left(\frac{kT}{n}\right);\quad z_{rk} = z_r\left(\frac{kT}{n}\right).$$

For simplicity we will limit ourselves to a discussion of a binary system and then the optimal reception principle based on the criterion of maximal likelihood amounts to selection of the decision that $z_1(t)$ was transmitted if

$$\prod_{k=1}^{n} \frac{w(z'_k - z_{1k})}{w(z'_k - z_{2k})} > 1$$

or

$$\sum_{k=1}^{n} [\ln w(z'_k - z_{1k}) - \ln w(z'_k - z_{2k})] > 0. \qquad (8.56)$$

We will denote $\ln w(x) = f(x)$ and expand each term in (8.56) into a Taylor series around z'_k. This is always possible if function $w(x)$ is continuous, limited, and always different from zero, and we will assume this to be the case. Then the decision principle can be represented in the form

$$\sum_{k=1}^{n}\left[\sum_{s=1}^{r} z_{1k}^{s} \frac{(-1)^s}{s!} \frac{d^s}{dz'^s_k} f(z'_k) - \sum_{s=1}^{r} z_{2k}^{s} \frac{(-1)^s}{s!} \frac{d^s}{dz'^s_k} f(z'_k)\right] > 0$$

or

$$\sum_{k=1}^{n}\sum_{s=1}^{r} [z_{1k}^{s} - z_{2k}^{s}]\,\xi_{sk} > 0, \qquad (8.57)$$

where

$$\xi_{sk} = \frac{(-1)^s}{s!} \frac{d^s}{dz'^s_k} f(z'_k) = \xi_s\left(\frac{kT}{n}\right). \qquad (8.58)$$

Function $\xi_s(t)$ can be obtained as a result of passage of received signal $z'(t)$ through an inertialess nonlinear quadrapole with a characteristic of

$$y = \frac{(-1)^s}{s!} \frac{d^s}{dx^s} f(x). \qquad (8.59)$$

where x is the potential at the input of the quadrapole and y the potential at its input.

The sum $\sum_{r=1}^{n} z_r \xi_r$ when $n \gg 1$ can be approximated by a scalar product of functions $\xi_s(t)$ and $z_r^s(t)$ and realized in the form of a response of a filter matched with $z_r^s(t)$ to signal $\xi_1(t)$ at readout time $t = 1$.

Thus, the decision principle can be represented in the form of a finite number of branches, each of which contains a nonlinear quadrapole (8.39) and pair of filters matched with $z_1^s(t)$ and $z_2^s(t)$ (Figure 8.5).

By limiting ourselves to a finite number of branches in the circuit shown in Figure 8.5, we attain a suboptimal decision system. Specifically, if the signal power is small in comparison with the interference power in the frequency band being received (which, as a rule, is met in wide-band receiver channels), it is possible to limit ourselves to one branch and obtain the suboptimal system shown in Figure 8.6.

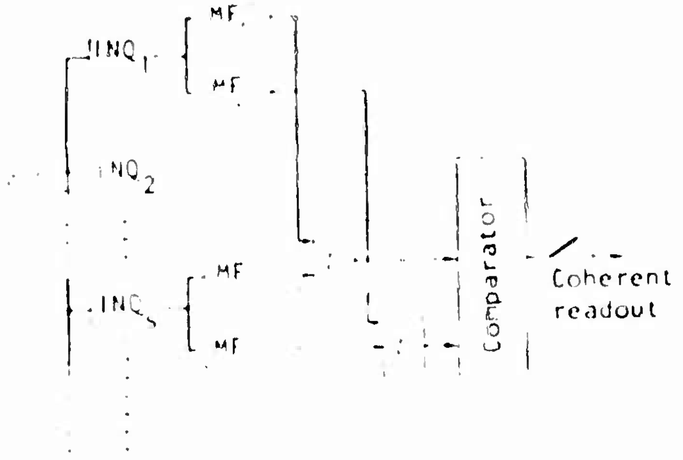


Figure 8.5. Optimal Decision System for Receiving Binary Signals in a Channel Containing Impulse Interference. INQ_s, Inertialess nonlinear quadrapole with the characteristic shown in (8.39), $MF_r^{(s)}$, Filter matched with $z_r^s(t)$.

The probability distribution density of impulse interference is in many cases well approximated by the function [7, 9, 10]

$$w(x) = A\exp(-a|x|^{\nu}), \qquad (8.40)$$

where the subscript ν can assume values from 0.5 to 2 and determines the nature of the interference and coefficients of A and a depend on its intensity. In this case the characteristic of the nonlinear quadrapole in the circuit shown in Figure 8.6 is equal to

$$y = f'(x) = -\frac{d}{dx}\ln w(x) = a\nu|x|^{\nu-1}\operatorname{sgn} x. \tag{8.41}$$

In the particular case when $\nu = 2$ distribution (8.40) becomes normal. This takes place when impulses pass through a narrow-band filter and follow one another so rapidly that the reactions they cause completely overlap. In this case, as should be expected, the nonlinear quadrapole shown in Figure 8.6 becomes linear. Furthermore, all the remaining quadrapoles in Figure 8.5, except for the first, are cut off since from (8.39) when ν 1 we have $y = 0$. Thus, the optimal decision system becomes a Kotelnikov system.

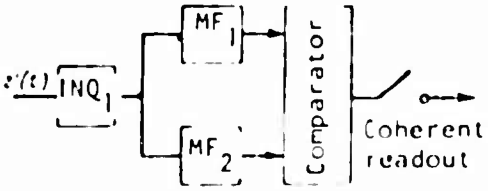


Figure 8.6. Suboptimal Decision System for Receiving Binary Signals in a Channel with Impulse Interference.

In the other extreme case of no overlapping among impulses $\nu = 1/2$ and the characteristic of the quadrapole shown in Figure 8.6 will be expressed by $y = a/2\sqrt{x}\ \operatorname{sgn} x$. When ν 1 we obtain a quadrapole with a characteristic of $y = a/2\sqrt{x}\ \operatorname{sgn} x$, i.e., an ideal limiter.

As shown in [8], the suboptimal system shown in Figure 8.6 permits significant suppression of impulse interference. This suppression is greater, the less ν is. When $\nu = 1/2$ impulse interference is completely suppressed.

8.6. Practical Methods of Protecting Against Impulse Interference

In radio communication practice various systems permitting impulse interference suppression to one degree or another are used. From what has been said above it is clear that such systems should contain a nonlinear element in a wide-band receive channel, i.e., in front of filters matched with the signal at least in passband. Systems with a limiter (Figure 8.7) which are often called WLN systems (wide-band-limiter narrow-band), suggested for the first time apparently in work [11], have found widest use. If a limiter is ideal, i.e., the limiting threshold is zero, then such a system coincides with Figure 8.6 when $\nu = 1$. In the case of an ideal limiter, if the limiting threshold is higher than the maximum of the sum of the signal and the non impulse interference, the receiver operates in a linear regime all the time with the exception of the time needed for passage of the interfering pulse (Figure 8.8). This creates more favorable conditions for protection against non-impulse (for example, lumped) interference.

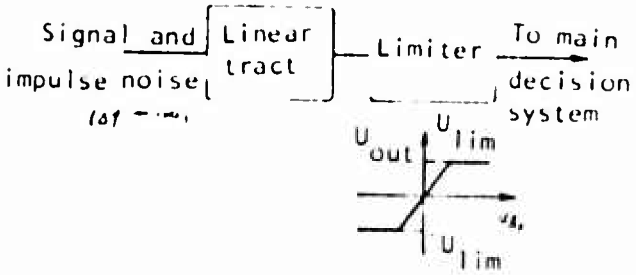


Figure 8.7. System for Protecting Against Impulse Noise by Limiting.

Let us examine the action of the system of Figure 8.7 with regard to the frequency characteristic of the linear section of the receiving unit. We will assume that the impulse duration at receiver input is substantially less than $1/\Delta f$, where Δf is the effective passband of the linear tract. This assumption is usually fulfilled in practice. Then the impulse shape at the input of the limiter is a good approximation of pulse response $g(t)$ of the linear circuit. If the signal plus the noise is fed to a limiter with a limiting level in excess of the maximum overshoot of the sum of the signal plus all non-impulse noise, the receiver will revert into a non-linear regime only throughout time τ that the interference impulse exists.

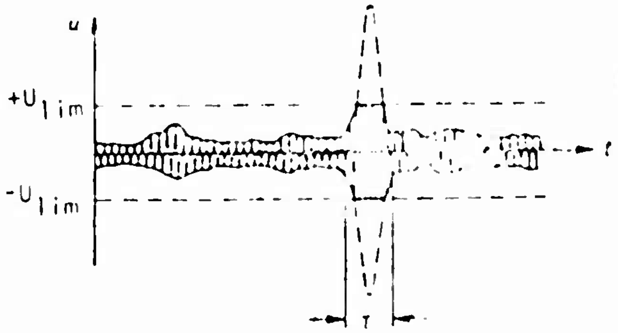


Figure 8.8. Limiting Impulse Noise.

Impulse length τ at limiter input[1] is determined by the impulse response $g(t)$ or the frequency characteristic of the linear channel. It is known that in minimum-phase circuits impulse length τ is approximately inversely proportional to Δf. It may as a first approximation be considered that

$$\tau \approx \frac{1}{\Delta f} \qquad (8.42)$$

---

[1] Impulse response $g(t)$ may, generally speaking, extend to infinity. Therefore we will understand τ to mean the duration of impulse $g(t)$ until the moment that it becomes essentially less than the sum of the signal and the fluctuation noise. We will need only an approximation of τ, therefore the indefiniteness is not significant.

To get an approximate estimate of the effect of impulse noise it may be assumed that the signal plus non-impulse noise go to the input of the decision system behind the limiter, while the interference impulses are converted into square-wave pulses of area

$$A = U_{\lim}\tau \approx \frac{U_{\lim}}{\Delta f}, \qquad (8.43)$$

which in the first approximation does not depend on the area of the original impulse. The spectral density of the interference impulse after being limited in accord with expression (8.26) is

$$\nu^2_{\lim} = 2A^2 = 2\frac{U^2_{\lim}}{\Delta f^2}. \qquad (8.44)$$

If the level of limitation is fixed, then by increasing Δf we could obtain an arbitrarily small value of $\nu^2_{\lim}$, but as Δf grows larger the level of the fluctuation and lumped noise at limiter input increases. In order for the receiver to operate linearly when there is no interference noise the limiting level $U_{\lim}$ must also be raised with the increase in passband Δf and kept above the maximum overshoot of the sum of the signal and all non-impulse noise.

Let us designate the total power of the fluctuation and lumped noise acting on the limiter input by P_n and the signal power by P_s. Then the maximum overshoot of the total signal and non-impulse noise voltage is $k\sqrt{P_n + P_s}$, where k is the peak factor of the total voltage, i.e., the ratio of the maximum (peak) voltage to the mean-square voltage.[1] Having selected a limiting level equal to this maximum overshoot we obtain

$$U^2_{\lim} = k^2(P_n + P_s) = k^2\left(\nu_0^2\Delta f + \frac{h_0^2\nu_0^2}{T}\right), \qquad (8.45)$$

where $\nu_0^2 = P_n/\Delta f$ is the total density of fluctuation and lumped noise and $h_0^2 = P_s T/\nu_0^2$ is the ratio between the signal element power and the spectral density of the fluctuation and lumped noise.

The magnitude k may be considered as the relative limiting level.

---

[1] Strictly speaking, the voltage of normal fluctuation noise has no maximum value since with a probability other than zero it may exceed any given value. Therefore, instead of the peak value the quasi-peak value should be examined, i.e., the voltage value which may be exceeded with a probability no greater than some ε. In normal fluctuation noise when ε = 0.01 the peak factor is 2.58.

The spectral density of the interference impulse is, according to expression (8.44)

$$N_u = 2P_s\left(\frac{1}{\Delta f} + \frac{k}{(\Delta f)^2}\right) \qquad (8.46)$$

and may be made arbitrarily small by selecting a sufficiently broad passband Δf.

When $\Delta f T \gg h_0^2$

$$N_{\lim} \approx \frac{h_0^2}{\Delta f T} N_0 \qquad (8.47)$$

and if furthermore $\Delta f T \gg 2k^2$ the spectral density of the interference impulse after the limiter proves to be very small in comparison to the spectral density of the other noise.

Thus, for example, when k ≈ 3 and ΔfT = 100, the spectral density of the impulse noise is only 20% of N_0. Even when $N_{\lim} \ll N_0$, however, impulse noise may perceptibly increase the error probability by acting on the decision system. The degree of this action depends on how much the interference impulse resembles the signals to which the decision system is matched. Thus, if the signals are radio signals of duration of the order of 1/Δf, then the decision system reacts to the interference impulse in almost the same way as to the signal. In this case the chief role is not played by the spectral density of the interference impulse, but by its power which is $P_{\lim} = N_{\lim}\Delta f$. From the expression (8.4[illegible]) it follows that

$$P_{\lim} \approx 2P_s\left(1 + \frac{k}{\Delta f T}\right) [illegible] \qquad (8.48)$$

i.e., the power of the interference impulse does not decrease as the passband widens, but has a value close to signal power P_s (since k^2 [illegible] noise signal and h^2 are magnitudes of approximately the same order).

In order that the decision system not react to impulse noise as to a signal, it must be matched with signals which differ as much as possible from the interference impulses. In particular, these signals must sufficiently uniformly fill the interval (0,T) assigned to the transmission of a single element, i.e., they must have an adequately small peak factor. This condition is to a significant degree satisfied by the signals of simple systems (AM, FM, PM).

Noise-like signals are the most different from impulse noise. The decision system which is matched to a noise-like signal performs coherent addition of its components, as was shown in Chapter VII. At the same time this decision systems destroys, as it were, the coherence between the components of the interference impulse. In particular, a filter matched with a noise-like signal converts the signal into an impulse, and conversely, converts the interference impulse into a noise-like voltage [12]. The frequency band occupied by a signal may in this case coincide with the effective passband Δf of the linear section of the receiver.

Under these conditions the impulse noise which has traversed the limiter acts approximately like fluctuation noise, i.e., the chief factor is not the power of the limited impulse, but the spectral density N_{lim} which, when the value of Δf is large enough, may be made arbitrarily small in accord with expression (8.17). It is this very thing which practically effectuates impulse noise suppression on condition that the number of interference impulses n throughout signal element reception is not very large and that condition $n\sigma^2_{lim} \ll \sigma^2_0$ is fulfilled.

Many different methods of increasing reception fidelity under impulse noise conditions are described in available literature. A few of them are used in practice. The majority of the proposed methods may be divided into two groups. To the first group belong the compensation methods which represent different variations of the principle which underlies the system in Figure 8.4. These methods are usually very complex or provide inadequately effective impulse noise suppression and have, therefore, been little applied.

To the second group belong the methods which limit the magnitude of interference impulse in some form or another. First of all should be mentioned the method of drastic limiting which differs from that examined in the preceding section in that after the wide-band linear channel the signal plus noise goes to a limiter with a limiting level considerably lower than the mean square value of the sum of the signal plus noise (Figure 8.9). Here, of course, large nonlinear distortions occur and, generally speaking, the information is partially lost because the limiter is an irreversible quadrupole. Nevertheless, in a number of cases the information loss is insignificant and a basic decision system included after the limiter provides reception of almost the same fidelity as in the case where the limiting level is higher than the peak level of the signal. The advantage of drastic limiting is, however, that it makes careful regulation of the limiting level or of amplification unnecessary.

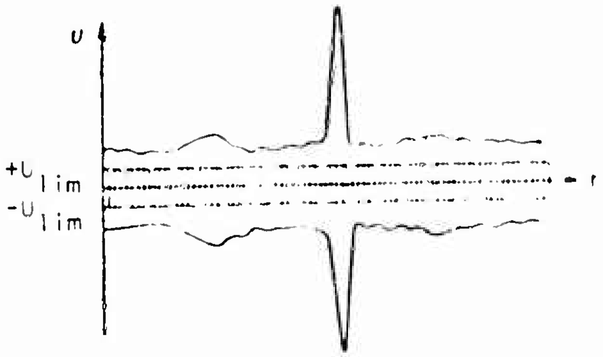


Figure 8.9. Drastic Limitation of Sum of Signal and Noise.

Instead of limiting impulse interference, it is possible to block the receiver during passage of an interference impulse with the same effect.

A detailed consideration of different methods of protecting against impulse interference is contained in work [13].

In channels with relatively infrequent impulse interference, as already noted, errors created by interference impulses can best be eliminated by using a correcting code. For this purpose it is possible to use a code with minimal redundancy, detecting that element of a signal during the reception of which the interference impulse occurred. Such detection of "unreliable" signal elements can be done by a minimum limiter which reacts to all impulses exceeding a given level. Figure 8.10 shows a diagram taken from [12] in which the main section of the receiving unit is linear and a limiter which detects impulses and transmits appropriate "erase" commands to the decoder, device is connected in parallel with it.

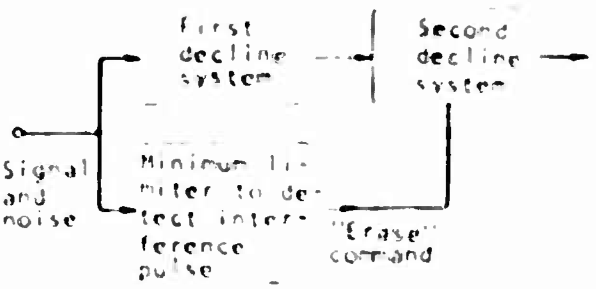


Figure 8.10. Diagram Showing Detection of Interference Impulse and Correction of Errors.

Another way to protect against infrequent impulse interference is to use time diversity. Addition of signals occurs in diversity branches in the absence of impulse interference. When an interference impulse which can be detected as shown in Figure 8.10 appears, the appropriate branch is excluded and reception is conducted in branches which are not affected at the moment of impulse interference.

8.7. Ways to Optimize a System with Joint Action of Lumped, Impulse, and Fluctuation Interference

And so, if in a channel along with fluctuation interference there is either lumped or impulse interference, they can be largely suppressed so that the probability of error is determined almost entirely by fluctuation interference. It is true that decision systems in this case are not completely optimal for fluctuation interference and the suppression of interference is not complete. However, in feasible systems it is still possible to cut the power of lumped or impulse interference by several thousand times. Thus, if the spectral density of lumped or impulse interference exceeds the spectral density of fluctuation interference by not more than 1,000 times, the fidelity of reception is determined almost entirely by fluctuation interference.

The problem is much more complex when lumped and impulse interference are present in a channel at the same time. As can be seen from preceding sections, measures taken to suppress impulse interference are largely inconsistent with conditions under which it is possible to suppress lumped noise and even contribute to enhancing their effects on a decision system. Indeed, to suppress impulse interference there must be a nonlinear device, for example, a limiter, in the wide-band part of the receiver. If strong lumped noise occurs in the passband at the input of this nonlinear device, combination frequencies, some of which may be in the frequency band of the signal will appear at the output. Thus lumped noise, after passage through a nonlinear device, is "de-multiplied" and possibilities for suppressing it are much reduced.

On the other hand, if an attempt is first made to suppress lumped noise by passing a received signal through a filter transforming $z'(t)$ into $z(t)$ (Figure 8.2), the relationships between the components are disrupted in the spectrum of the impulse interference during passage through such a filter, as a result of which the impulses are stretched out and may even overlap. Thus, impulse interference prior to limiting will approach fluctuation interference (superscript in (8.40) increases) and the effectiveness of suppression will be greatly reduced.

The existence of a limiter in the wide-band part of a receiver when there is strong lumped noise not only multiplies the noise but also suppresses the signal. As is known, when two harmonic oscillations with harmonic amplitudes are delivered to the input of a limiter, the ratios of the amplitudes at the output of the limiter change in favor of the stronger oscillation. In the extreme case of complete limiting, this ratio can be changed by a factor of 2. If the interference provides the stronger oscillations, the signal at the output of such a stage will be largely suppressed.

In the wide-band section of a receiver, regardless of whether the signal is wide-band or narrow-band, the strength of lumped noise can greatly exceed the signal strength. When the section is strictly linear this does not hinder a high level of fidelity in reception as long as the strength of the signal exceeds sufficiently the noise spectral density. If the stages of the receiver are overloaded, in the first place the ratio between signal power and spectral noise density decreases as a result of suppression and of the occurrence of combination frequencies and, second, the level of the useful signal at the input of the decision system is reduced as compared to the calculated level.

Usually the main role is played by the second factor if in designing the decision system no provision is made for a sufficiently wide dynamic range. Therefore, creating conditions under which a decision system functions correctly regardless of change in signal level within wide limits is a very important problem in receiving devices. As already noted, in the receiver of the Rake system [14], the dynamic range of the multipliers reaches 100 dB. Amplifiers, for example, between the matched filters and the comparator are often used to widen the dynamic range of the decision system.

At the present time the problem of optimizing a decision system during simultaneous effects of interference of different kinds cannot be considered solved. In practice compromise methods permitting the suppression of impulse interference to some extent and effecting simultaneous protection against lumped noise are usually used. Thus, in a WLN system with narrow band FM or RPM signals the passband of the circuit in front of the limiter is made only 3-5 times wider than the effective passband of the matched filter. This decreases the probability that strong lumped noise will reach the input of the limiter causing stretching in the length of the interfering impulses only to a magnitude on the order of 1.3-1.4, i.e., retaining the possibility of partial suppression. On the other hand, the limiting threshold is often set higher than the total level of signal and non-impulse interference so that nonlinear phenomena occur only with passage of an interfering impulse but, on the other hand, the residual spectral density of the impulse interference is higher than in the case of drastic limiting.

However, more effective ways of attaining simultaneous protection against both indicated types of interference are possible. The use of wide band signals with a small peak factor for this purpose is extremely promising. This permits using a limiter for impulse interference with a threshold higher than the peak level of the signal but still low enough for satisfactory suppression of impulse interference. The residual impulses, following passage through a filter matched with the signal, acquire the properties of fluctuation noise that has a negligibly small spectral density. At the same time the wide spectrum of the signal permits adequate suppression of narrow band lumped interference following the limiter, at least in accordance with the diagram shown in Figure 5.2. Technological difficulties are the main hindering factor in the application of such systems at the present time.

If lumped noise is distributed sufficiently infrequently along the frequency axis and the interference impulses occur sufficiently infrequently in time, it is possible to recommend the use of simultaneous frequency and time diversity. Let Q_1 frequency diversity branches be used and the signal obtained be repeated Q_2 times. The total number of diversity branches is equal to Q_1Q_2. the impulse and lumped interference the signals of all branches are added either coherently or incoherently. With the presence of lumped noise, of the Q_1 frequency branches those in which the strongest interference is found are excluded and, with the occurrence of impulse interference those from among the Q_2 time branches reached by the interference impulses. If Q_1 and Q_2 are sufficiently large and the power of the signal in one branch in the presence of fluctuation interference sufficient to provide the required level of fidelity, there will be at least one unaffected branch which can be used for reception with a probability close to unity.

Notes

1. (See Section 5.2) Optimal and suboptimal systems in the case of lumped noise have still not found wide application in practice. Usually

protection against lumped noise amounts to using a selective circuit in the receiver for the purpose of increasing the probability that strong interference will reach the input of the decision system. In this case an effort is made to obtain an amplitude frequency characteristic of the selective circuit which is as close as possible to [illegible] shaped and its passband is selected based on ideas of compromise. The narrower the passband, the less is the distortion in the signal caused by transient processes and increasing the probability of error due to fluctuation interference.

Such a circuit for a narrow-band receiver differs from the suboptimal in Figure [illegible] in that $[illegible]_k$ is not determined on the basis of an analysis of the received signal but assumed to be equal to zero for all k for which a_{rk} and b_{rk} are relatively small and unity for all remaining k.

The probability that lumped noise will cause an error in the circuit of a narrow-band receiver can be expressed as follows

$P = \int_0^{\infty} p_v$ [illegible]

where p_v is the conditional probability of error when the interference envelope is equal to V and $f(V)$ is the probability distribution density of this envelope. Obviously, p_v depends on the strength and shape of the signal and [illegible] on the passband of the receiver selective circuit.

The probabilities of p_v for various signals are computed in [1]. Without going into detail we can note that there is a threshold value [illegible] below which [illegible] and with a further increase of [illegible] rapidly approaches a constant equal for binary systems to [illegible] in the following form

$P \approx \frac{1}{2}\int$ [illegible] $F(V_0)$

where $F(V_0)$ is the probability that [illegible], in other words, the probability that lumped interference with an envelope greater than V_0 will reach the receiver passband.

For sufficiently general assumptions about the nature of lumped noise in the absence of frequency regulation it can be shown that the dependence of $F(V_0)$ on the passband is exponential:

$$F(V_0) = \exp(-\lambda\Delta f)$$

where Δf is the receiver passband, and λ is the spectral intensity of interference distribution with an envelope exceeding V_0 and depending on V_0 and on the nature of the sources of the interference in the channel.

When [illegible] 1 the probability that interference will reach the receiver passband is approximately proportional to [illegible]. Therefore, in designing a radio receiver an effort is made to reduce the passband even at the cost of increasing the probability of error caused by transient processes in the selective circuit.

2. (See Section 8.2) When there are many sources of lumped noise and also when lumped noise is manipulated and subjected to fading, the distribution of probabilities of their instantaneous values is close to normal and the distribution of [illegible] is close to exponential. As indicated in the text, in this case the optimal system amounts to that computed in Chapter III with a "whitening" filter. To construct such a system it is necessary to know the power spectrum of the manipulated and fading lumped noise. This problem is considered in work [15].

3. (See Section 8.5) Formula (8.42) is a rough approximation. Impulse length depends not only on the effective passband of the linear section, but also on the shape of its frequency characteristic, as well as on the level at which the impulse length is read out.

The product [illegible] is known [16] to assume its least value in the case where the frequency characteristic of the section is bell shaped (gaussian). It is apparent that it is precisely this frequency characteristic shape (and not at all the square-wave shape) which should be striven for in the design of receiving units for channels which contain both impulse and lumped noise. This characteristic is physically unrealizable, but it may be sufficiently well approximated by using a large number (in practice, four or five) stages of resonance amplification. The envelope of the impulse response of such an amplifier [17] is also bell shaped

$$[illegible]$$

If the impulse of expression (8.24) goes to the amplifier input, the impulse at the limiter input has the envelope

$$[illegible]$$

where U_0 [illegible] is the maximum value of the impulse.

From this it is easy to find impulse length which is read out at level U_{lim}:

$$[illegible] \qquad (8.50)$$

Therefore, formula (8.42) is accurate for a bell-shaped impulse when $U_0 \approx 3.5U_{lim}$. A correction would have to be introduced for more powerful impulses and instead of expression (8.42) we would write

$$\tau = \frac{a}{\Delta f} \qquad (8.51)$$

where $a = \sqrt{\frac{2}{\pi} \ln \frac{U_0}{U_{\text{lim}}}}$ takes on the values shown in the table:

U_0/U_{lim}	2	5	10	100	1000	10000
a	0,8	1	1.8	[illegible]	5,5	[illegible]

This correction obviously introduces no qualitative changes in the results obtained which substantiate the possibility that impulse noise may be suppressed. Nevertheless, it shows that the spectral density of the limited impulse noise [illegible]$_{\text{lim}}$ may be several times increased if the area of the initial impulse increases, say, several thousand times.

4. (See Section 8.5) When selecting a limiting level we may set ourselves a permissible probability that during the period of reception of a single element the envelope of the total voltage of the signal and non-impulse noise will exceed the limiting level. We will consider this total voltage to have normal probability distribution. The average number of envelope overshoots per unit of time depends on the average width of the spectrum. In the case of a gaussian frequency characteristic the probability [illegible] that during reception of a single signal element the envelope of the total voltage will be k times more than its mean-square value is approximately [18]

[illegible]

Setting [illegible] 0.1, we find that when $\Delta fT = 100$ it is necessary to choose a relative limiting level $k \approx 3.8$, and when $\Delta fT = 1000$, a limiting level $k \approx 4.0$. If we set [illegible] = 0.01, then when $\Delta fT = 100$ the relative limiting level $k \approx 4.6$, and when $\Delta fT = 1000$, this level is $k \approx 5.2$. These values of k must be used in formulas (8.46)-(8.47).

The dependence of k on ΔfT somewhat inhibits increasing the suppression of impulse noise when expanding the passband Δf before the limiter. Nevertheless, choice of a sufficiently large Δf can simultaneously provide an arbitrarily small probability of conversion to a nonlinear regime and an arbitrarily small spectral density of limited impulse noise [illegible]$_{\text{lim}}$.

5. (See Section 8.7) Protection against lumped and impulse interference is greatly simplified in feedback systems which will be discussed in Chapter XI. Specifically, in radio communication when it is possible to tune a receiver and transmitter to any carrier frequency within a certain band, the existence of a feedback channel permits selecting an optimal carrier in the vicinity of which the strength of the lumped noise at a given time is minimal. The selection of the carrier and suitable tuning can be automated.

Such a selection of an optimal carrier frequency can be viewed as an approximation to selection of an optimal signal when the interference has a nonuniform spectrum (see Section 5.6). Indeed, if the entire accessible range of frequencies is viewed as a communication channel with interference, the selection of the optimal signal in the first approximation amounts to concentration of its strength in that frequency sector where the intensity of the interference is minimal.

The same channel with feedback can be used for interrogating message sections the reception of which has been disrupted by impulse interference. The detection of impulse interference can be done as shown in Figure 8.10.

BIBLIOGRAPHY

1. Teplov, N. L., *"Pomekhoustoychivost' Sistem Peredachi Diskretnoy Informatsii"* [Resistance to Interference in Systems for Transmitting Discrete Information], Svyaz' Press, 1964.

2. Klovskiy, D. D., "Resistance to Interference in Binary Systems When There is Fluctuation and Lumped Interference," Elektrosvyaz', No. 2, 1965.

3. Ageyev, D. V., "Fundamentals in the Theory of Linear Selection," Nauchno-tekhnicheskiy sbornik LIIS, No. 10, 1935.

4. Komarovich, V. F. and Ye. V. Lebedinskiy, "Diversity Reception Under Conditions of Random Radio Interference," Elektrosvyaz', No. 1, 1969.

5. Petrovich, N. T. and M. K. Razmakhnin, *"Sistemy Svyazi s Shumopodobnymi Signalami"* [Communication Systems with Noise-Like Signals], Sovetskoye Radio Press, 1969.

6. Costas, Y. P., "Poisson, Shannon and the Radio-Amateur," PIRE, No. 12, 1959.

7. Gilbert, E. N. and H. O. Pollak, "Amplitude Distribution of Short Noise," BSTJ, No. 2, 1960.

8. Ponkratov, V. S. and O. Ye. Antonov, "Optimal Reception of Binary Signals Against a Background of Non-Gaussian Interference," Elektrosvyaz', No. 9, 1967.

9. Rappaport, S. S. and L. Kurg, "Optimal Decision Thresholds for Digital Signaling in Non-Gaussian Noise," IEEE Intern. Conv. Rec., Pt. 2, Vol. 13, 1965.

10. Shepelavey, B., "Non-Gaussian Atmospheric Noise in Binary-Data Phase Coherent Communication Systems," IEEE Trans., CS-11, No. 3, 1963.

11. Shchukin, A. N., "One Method for Combatting Impulse Interference," Izvestiya AN SSSR Ser. Fizicheskaya, No. 1, 1946.

12. Lerner, R. M., "Selection of Signals," In the book: Lectures on the Theory of Communication Systems, edited by Bagdadi. "Mir" Press, 1964.

13. Gol'dberg, A. P., "Characteristics of Systems for Suppressing Impulse Interference," Elektrosvyaz', No. 2, 1966.

14. Price, R. and P. E. Green, "A Communication Technique for Multipath Channels," PIRE, No. 3, 1958.

15. Kotousov, A. S., "Determining Correlation Properties of Signals in Discrete Messages at Communication Channel Output with Randomly Changing Parameters," Problems in Transmitting Information, No. 1, 1966.

16. Kharkevich, A. A., "Spectra and Analysis," Third edition, Gostekhizdat, 1962.

17. Yevtyanov, S. I., "Transient Processes in Receiving-Amplifying Circuits," Svyaz'izdat, 1948.

18. Bunimovich, V. I., "[illegible]" [Fluctuation Processes in Radio Receiving Devices], Sovetskoye Radio Press, 1956.

CHAPTER IX

MULTIPLEXING COMMUNICATION CHANNELS

9.1. Matching Channel Carrying Capacity with Source Productivity

In this chapter we will consider several problems in matching message sources with a channel. As shown in Chapter I, it is possible to transmit with a high degree of fidelity messages from a source at a fixed rate when the carrying capacity of the channel C exceeds the productivity of the source H'. The greater the C/H' ratio, the simpler coding is at a given level of fidelity. If the productivity of a source is given, by selecting a channel with a carrying capacity if $C \gg H'$, it is possible to simplify the receiving and transmitting devices, but cost of a channel will be very high inasmuch as, as a rule, it increases monotonically with the increase in carrying capacity and range of communication. If, on the other hand, an inexpensive channel with a carrying capacity only slightly exceeding the productivity of the source is selected, the devices used for encoding and decoding become more expensive.[1] Obviously, generally speaking, there must be a ratio between carrying capacity and source productivity at which the total cost of a communication system is a minimum. This optimal ratio depends on the level of technological development and, apparently, shows a tendency to lessen [1].

If the productivity of a source is much less than the carrying capacity of a channel, the requirements for economy require seeking ways to improve use of a channel. For this purpose a channel multiplexed. That is, it is used for simultaneous transmission of messages from several sources which are intended, generally speaking, for different recipients. The number of sources is called the multiplicity factor of multiplexing.

The opposite condition may exist when the productivity of a source is greater than the carrying capacity of the channel. In this case use can be made of two or more channels for transmitting a single message, i.e., resort is had to concentration of channels. If these channels are independent, their carrying capacities are added.

In the most general case n channels may be used for transmitting messages from different m sources. Thus concentration and multiplexing of channels may occur concurrently.

---

[1] Here encoding and decoding are understood in the broad sense as conversion of a message into a signal and conversion of a received signal into a message.

Capable combination of the methods of multiplexing and concentrating channels in designing communication systems permits the most effective use of channel carrying capacity, i.e., providing for a high rate of information transmission with a given permissible probability of error. Unfortunately, in many existing communication systems channel carrying capacity is greatly under-used. For example, in cable communication lines the power of a signal element often exceeds the spectral density of additive fluctuation interference by many more times than required to achieve the required level of fidelity. Under such conditions fluctuation interference leads to no errors at all. Errors are caused by impulse interference, interruptions in switching channels, and other factors which in principle, can be completely eliminated. In this regard the opinion is widely held among engineers that in wire communication fluctuation interference is of no interest and a general communication theory which devotes principal attention to gaussian interference is not needed for wire technology. Such neglect of theory is a result of exceedingly ineffective use of channel carrying capacity. If use is made of rational multiplexing methods, it is possible to increase the rate of transmission of information over such channels by many times and then the limiting factor prohibiting further increase in rate of transmission with a given level of fidelity (or vice versa) is just exactly fluctuation interference.

Unfortunately, the theory of multiplexing communication channels is still largely undeveloped and our knowledge in this field is not much greater than what was contained in the work published by D. V. Ageyev [2] in 1935 (see also [3]). The theory of concentrating channels is even less developed. In this chapter the author will make no attempt to create a well-ordered and complete theory of multiplexing but will attempt the modest task of combining certain ideas expressed in different magazine articles, or perhaps not published anywhere although generally accepted, and give the reader an idea of the problems found in communication theory. In light of the subject matter of the book, we will discuss transmission of discrete messages only.

9.2. Classification of Multiplexing Methods

At the present time there is no satisfactory classification of methods for multiplexing communication channels. The subdivision of systems of multiplexing into frequency and time multiplexing which is found in many works does not hold up under close scrutiny inasmuch as it does not encompass those methods which have found wide use, to say nothing of other possible systems which for one reason or another are not used.

In order to approach possible classification schemes, we will concentrate our attention on a signal passing in a multiplexed communication channel. In many systems this signal $z(t)$ can be represented in the form of a sum k of different k of different individual signals:

$$z(t)=\sum_{i=1}^{k}\zeta^{(i)}(t), \tag{9.1}$$

each of which carries information about the message from only one of the sources. We will call such multiplexing systems separable. The total signal $z(t)$ is often called a group signal. If each of the individual signals $\zeta^{(i)}(t)$ has m realizations, $\zeta_r^{(i)}(t)$; $r = 1,\ldots,m$; $i = 1,\ldots,k$ and the transmitted messages are independent, the group signal has m^k realizations.

Individual signals may be orthogonal in the general or intensified sense, orthogonal, opposite (when $m = 2$), or arbitrary. An especially important case occurs when realization of each individual signal is orthogonal to all realizations of the remaining individual signals. Such separable multiplexing systems will be called orthogonal. We will note that realizations of a group signal in this case are not mutually orthogonal. Only in one particular case does a group signal of a separable system form a biorthogonal system. This occurs when $k = 2$ and $m = 2$ if realizations of an individual signal $\zeta^{(i)}(t)$ are opposite $[\zeta_1^{(1)} = -\zeta_2^{(1)}; \zeta_1^{(2)}(t) = -\zeta_2^{(2)}(t)]$; and each of them is orthogonal to realizations of the other individual signal $[\zeta_1^{(1)}(t) \perp \zeta_1^{(2)}(t)]$.

The known systems of frequency and time multiplexing are examples of separable systems. In the first case individual signals usually are simple, i.e., each element is a segment of a sinusoid and the different individual signals have their own frequencies. If the difference between these frequencies are divisible by 1/T, the system of frequency multiplexing is orthogonal. Also, frequency multiplexing system can be considered approximately orthogonal if the frequency differences are much greater than 1/T. In the case of time multiplexing individual signals do not overlap in time. For this purpose the length of element T is subdivided into k parts and each individual signal is assigned its own interval. Obviously, time multiplexing systems are orthogonal. Incidentally, this orthogonality can be disrupted if during passage through a channel the individual signals are distorted so that mutual overlapping occurs.

Understandably, separable multiplexing systems are not limited to frequency and time multiplexing systems. It is sufficient to select any mk realizations and distribute them over k sources in order to construct a separable system with a code base m for each individual signal. In the particular case when these realizations are made orthogonal or biorthogonal, it is possible to construct an orthogonal separable multiplexing system.

In other systems a signal in a multiplexed channel cannot be represented in the form of (9.1) but its envelope or its instantaneous phase, instantaneous frequency, or some other parameter amounts to the sum of individual signals. In these systems which can be called quasi-separable, a transmitter signal in a multiplexed channel is obtained through modulation of a certain carrier frequency by a group signal $\Sigma\zeta^{(i)}(t)$. Thus, in the case of amplitude modulation

$$z(t) = A\left[1 + \mu_1 \sum_{i=1}^{k} \zeta^{(i)}(t)\right] \cos\omega_0 t, \qquad (9.2a)$$

in the case of phase modulation

$$z(t) = A\cos\left[\omega_0 t + \mu_\varphi \sum_{i=1}^{k} \zeta^{(i)}(t)\right], \qquad (9.2b)$$

in the case of frequency modulation

$$z(t) = A\cos\left[\omega_0 t + \mu_\omega \int \sum_{i=1}^{k} \zeta^{(i)}(t)\, dt\right] \qquad (9.2c)$$

etc.

Therefore, such multiplexed systems are often called binary modulation systems. Sometimes separable systems are relegated to them under the assumption that a transmitted signal is formed by means of single pole modulation of the carrier frequency by a group signal. However, inasmuch as single pole modulation and demodulation amount to shifting a spectrum, it is more convenient to view the shaping of a transmitted signal in separable systems as simple addition of individual signals.

There also exist systems of multiplexing in which neither the transmitted signal nor any of its parameters can be represented in the form of a sum of individual signals. They could be called inseparable but usually they are called combination. Each signal element in such a system must carry information about the messages of k sources. If each of them is encoded using a code base m, a signal element must for this purpose have m^k realizations just as in separable systems. However, there is a freer selection of these realizations. Specifically, they can form a system which is orthogonal, biorthogonal, or orthogonal in the intensified sense. It is also possible to construct a system with a given conditional frequency band of signals with as large a number of sources as desired.

In order to conclude classification of multiplexing systems, it is necessary to consider the existence of mixed systems in which the sources are subdivided into groups, within each group combination multiplexing occurs, and the obtained signals are added. Thus, combination and also separable multiplexing occur here. An example is provided by the well-known Kineplex system [4] in which 40 sources are subdivided into 20 groups of 2 sources each, the messages from each pair of sources form a combination signal, and all these signals are added just as in orthogonal systems of frequency multiplexing.[1]

As can be seen from the examples presented, multiplexing is nothing other than simultaneous encoding of messages from several sources during which process a signal common to all is formed. This encoding can occur in a discrete channel, for example, during time and combination multiplexing, or in a continuous channel (with frequency multiplexing and in most other separable and quasi-separable systems).

---

[1] Often the Kinexplex system is used to transmit messages from one and not from all 40 sources. In this case it should not be considered a multiplexed system but a communication system with a code base of $m = 2^{40}$. This permits increasing the length of an element by 40 times in comparison with a binary system for transmitting in a channel with multibeam propagation using the protective interval method (see Section 7.5).

Along with classification of multiplexing systems based on the method of shaping a signal in a channel, they can also be subdivided into synchronous and asynchronous. In synchronous systems sources emit information at the same rate or at multiple rates and each signal element has a strictly defined length T. In asynchronous systems sources can emit information at a varying rate. In separable asynchronous systems the individual signals can be synchronous but in a total group signal the starts of elements of the individual signals do not coincide. We will be mainly concerned with synchronous systems. The suggested classification of multiplexing systems is shown schematically in Figure 9.1. Some of the terminology used will become clear in what follows.

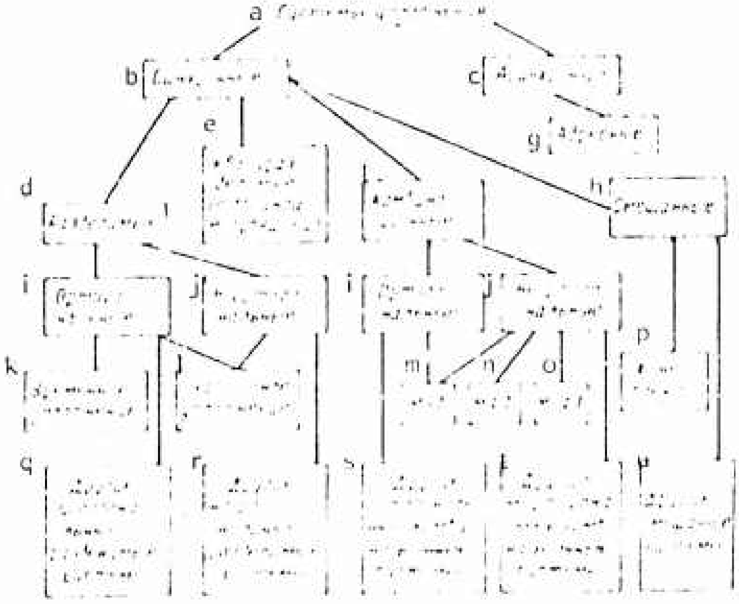


Figure 9.1. Classification of Multiplex Systems. Key: a, Multiplexed Systems; b, Synchronous; c, Asynchronous; d, Separable; e, Quasi-separable (with binary modulation); f, Combination; g, Address; h, Mixed; i, Orthogonal; j, Non-orthogonal; k, Time multiplexing; l, Frequency multiplexing; m, MNT; n, MPT; o, MRPT; p, Kineplex; q, Other orthogonal separable systems; r, Other nonorthogonal separable systems; s, Other orthogonal combination systems; t, Other nonorthogonal combination systems; u, Other mixed systems.

The great variety of systems for multiplexing permits selecting in each particular case a variation providing for the best possible use of carrying capacity of a channel while retaining a high level of fidelity in transmission.

9.3. Reception Criteria and Decision Systems

In any synchronous system for multiplexing an element of a received signal carries information about messages issuing from all k sources. Let a message from each source be encoded using a code base of m. Then, as already indicated, the number of realizations of signal is equal to m^k. If the requirement of maximizing the probability of correct reception of all transmitted messages is stipulated, the decision system must be determined by the criterion of the ideal observer. In the particular case when all realizations of a signal are equiprobable, this criterion coincides with the criterion of maximal likelihood.

Let z'(t) be a received signal. The decision system which is based on the criterion of the ideal observer must identify it with that one of the possibly transmitted signals $z_l(t)$ for which the following system of inequalities is met:

$$p(z_l|z') > p(z_r|z'),\quad r = 1,\ldots,m^k;\ r \neq l \tag{9.3}$$

After determining the most probable transmitted signal $z_l(t)$ and knowing the design of the system, it is possible to unambiguously establish which symbol was transmitted by which source.

Although decision principle (9.3) provides for a maximum probability of correct reception of all messages in a multiplexed channel, it does not always guarantee a minimum probability of error p_i in each of the messages. This can be easily understood when it is considered that in the case of incorrect identification of a received signal z'(t) with transmitted signal $z_l(t)$, not all messages will be received error-free. The criterion of the ideal observer, on the basis of which principle (9.3) was obtained, pertains to the entire received signal and minimizes the probability of false identification of signal p. regardless of how many messages are destroyed by error in the process.

If the probability of error in each of the messages must be minimized, the criterion of the ideal observer must be applied to separate messages. We will consider the i-th source in a multiplexed system. All n^k realizations of signal $z_r(t)$ can be subdivided into m subsets, each of which corresponds to one of the symbols of the i-th message. The decision system at which signal z'(t) arrives must determine the a priori probabilities of all symbols in a given message and select from them the one for which it is maximal. In other words, symbol λ_l must be recorded in the i-th message if

$$p(\lambda_l^{(i)}|z') > p(\lambda_r^{(i)}|z'),\quad r = 1,\ldots,m;\ r \neq l. \tag{9.4}$$

Here the superscript of s_i^{i} indicates the number of the message in the multiplexed system.

The most probable symbols of remaining messages are determined in the same way.

Let E_i and E_j indicate subsets of realizations of signal $x(t)$ corresponding to symbols s_i and s_j in the i-th message. Principle (9.4) can be written:

$$\sum_{s_i \in E_i} p(s_i|x) > \sum_{s_j \in E_j} p(s_j|x) \qquad (9.5)$$

for all $E_j \neq E_i$.

For many multiplexing systems principle (9.5) coincides with (9.3). However, there are systems for which principle (9.3) and (9.5) are not equivalent and lead to different decision systems and different distributions of error [5]. In these cases a question arises as to which of the two principles should be used.

An unambiguous answer cannot be given to this question. The best principle should be determined based on the requirements made of the communication system and the nature of the transmitted messages. For example, let redundancy be eliminated in large measure during encoding of the messages. Then any error occurring during reception renders almost all messages useless and in some cases can even lead to irreparable consequences. In this situation it would be incorrect to strive to decrease the average number of errors in a message and the probability of error-free reception of an entire message should be increased. For example, a decision system which provides in 50% of all cases for error free reception of messages and in the remaining 50% of all cases one incorrectly received symbol will be worse than another decision system which provides for 90% error-free reception and in the remaining 10% of all cases 100 incorrect symbols, although in the second case the average number of errors is 20 times greater than in the first. Obviously, in such situations it is advisable to use principle (9.3), i.e., to minimize p_c.

In other cases, for example, in transmitting text containing significant redundancy, it is more advisable to use principle (9.5), minimizing p_1, inasmuch as a small number of errors in a message can be corrected based on context.

For a comparison of systems with a different multiplexing factor it is possible to use either probability of error p_1 in separate messages or the equivalent probability of error p_e. By analogy with (2.6) we see that the latter is related to p_c by the monotonic relationship $p_e = 1 - (1 - p_c)^{1/k \log m}$. Therefore, the condition of minimum p_e coincides with the condition of minimum p_c.

If we drop the method of obtaining a signal by multiplexing messages, it is possible to consider a signal $x(t)$ transmitted over a multiplexed channel in any synchronous system as obtained through encoding all transmitted messages using a code with a base of m^k. Then p_Σ represents simply the probability of error in a communication system with a code base of m^k and the task of computing it is nothing other than that considered in preceding chapters. It is true that it is not solved in all cases. Below we will give a solution of it for a few multiplexing systems. We will also give expressions for p_i.

Regardless of the decision principle, the probability of error in a total signal p_Σ and the probability of error in the i-th message are related by the following inequalities

$$\max_i p_i \le p_\Sigma \le \sum_i p_i. \qquad (9.6)$$

Indeed, inasmuch as p_Σ represents the probability that at least one of the messages is received correctly, it may not be less than the probability of error p_i in any of the messages and at the same time may not be greater than the sum of the probabilities of error in all messages. The first inequality becomes an equality if the errors in the messages occur simultaneously; the second inequality becomes an equality if with any incorrect identification of signal the error occurs in only one of the messages.

We will use p_i' and p_i'' to denote the probabilities of error in the i-th message when using decision principles (9.5) and (9.5) respectively. We will use p_Σ' and p_Σ'' for the probabilities of incorrect identification of a multiplexed signal. From the essence of the criteria used in deducing the decision principles, we have

$$\left.\begin{array}{l} p''_\Sigma \le p'_\Sigma \\ p'_i \le p''_i \end{array}\right\} \qquad (9.7)$$

Combined use of inequalities (9.6) and (9.7) permits evaluation of the change in probabilities of error in transition from one decision principle to another. Thus

$$p'_{ia} \le \max p'_i \le p'_\Sigma \le p''_\Sigma \le \sum p''_i,$$

where $p_{i\,av} = 1/k \sum_i p_i$ is the average probability of error in all messages, whence

$$p''_{ia} \le p'_{ia} \le kp''_{ia}. \qquad (9.8)$$

On the other hand,

$$p''_\Sigma \le kp''_{i\,a} \le kp'_{i\,a} \le kp'_\Sigma,$$

whence

$$p'_{\Sigma} \approx p''_{\Sigma} \approx kp'_{\Sigma} \tag{9.9}$$

For some multiplexing systems these equalities will be made more exact in what follows.

9.4. Separable and Quasi-Separable Multiplexed Systems

Orthogonal Separable Systems

In orthogonal separable systems realizations of individual signals $s^{(i)}(t)$ in (9.1) for different messages are orthogonal. Different realizations of one individual signal ($s_r^{(i)}$ for various r) may in this case not be orthogonal. We will assume that orthogonality is retained even after passage of a signal through the channel. This means, for example, that for a channel with randomly changing phase orthogonality must be met in an intensified sense. The condition of orthogonality retention, at least in the first approximation, can be met for all channels used in practice through a suitable selection of signals.

If orthogonality is provided in such a way that the signals do not overlap in time, a system with time multiplexing results. For each sourće part of the length of signal element T equal to T/k is separated. In channels with a limited passband or multibeam propagation it becomes necessary for the purpose of retaining orthogonality to use the entire interval but only that part of it equal to T/k - τ_m, where τ_m is the maximal stretching of the signal as it passes through the channel.

In the case of frequency multiplexing, if the individual signals are simple segments of sinusoids with frequencies multiples of 1/T, orthogonality is disrupted with rather fast fading. Incidentally, under usual conditions of a shortwave radio channel when T is on the order of tens of milliseconds or less, these disruptions in orthogonality can be ignored. With more rapid fading or with a greater length of signal element, in the case of frequency multiplexing approximate orthogonality is achieved by using narrow-band individual signals, spreading them in frequency so that the spectra for all practical purposes do not overlap even when widening of the spectrum due to fading is taken into account.

Of course other orthogonal separable systems of multiplexing are possible although they have not yet found practical application.

We will consider a decision circuit for an orthogonal separable system based on the decision principle of (9.3). To simplify the problem we will limit ourselves to a gaussian channel with constant parameters. Let us also assume that all messages are statistically independent and that the symbols of the messages are equiprobable. Inasmuch as (9.3) coincides with the decision principle for a signal which is not multiplexed when the code base in m^k, the decision that group signal $z_i(t)$ was transmitted must be reached in accordance with (3.24a) if

$$\int_0^T [z'(t) - \mu z_1(t)]^2\,dt < \int_0^T [z'(t) - \mu z_r(t)]^2\,dt \qquad (9.10)$$

for $r = 1,\ldots,m^k$; $r \neq 1$.

According to (9.1), for a separable system

$$z_r(t) = \sum_{i=1}^{k} \zeta_{ri}^{(i)}(t),$$

where the subscript ri indicates that individual signal which corresponds to the i-th message in group signal $z_r(t)$. Substituting this expression in (9.10) we obtain

$$\int_0^T \left[z'(t) - \mu \sum_{i=1}^{k} \zeta_{1i}^{(i)}\right]^2 dt < \int_0^T \left[z'(t) - \mu \sum_{i=1}^{k} \zeta_{ri}^{(i)}\right]^2 dt,$$

but

$$\int_0^T \left[z'(t) - \mu \sum_{i=1}^{k} \zeta_{ri}^{(i)}\right]^2 dt = \int_0^T \left[z'^2(t) + \sum_{i=1}^{k} \mu^2 \zeta_{ri}^{(i)2}(t) + 2\mu^2 \sum_{i=1}^{k} \sum_{\substack{j=1 \\ j \neq i}}^{k} \zeta_{ri}^{(i)}(t)\zeta_{rj}^{(j)}(t) - 2\mu \sum_{i=1}^{k} z'(t)\zeta_{ri}^{(i)}(t)\right] dt. \qquad (9.11)$$

After substituting (9.11) in (9.10), considering the condition of orthogonality and assuming that all signals $z_r(t)$ have the same power, we obtain an equivalent decision principle in the form

$$\sum_{i=1}^{k} \int_0^T z'(t)\zeta_{1i}^{(i)}(t)\,dt > \sum_{i=1}^{k} \int_0^T z'(t)\zeta_{ri}^{(i)}(t)\,dt, \qquad (9.12)$$

where the inequalities must be met for all combinations of individual signals $\zeta_{ri}^{(i)}(t)$ which differ from that combination which forms group signal $z_1(t)$. Consequently, (9.12) must also be met for group signal $z_r(t)$ which differs from $z_1(t)$ only in the symbol in a certain j-th message. Consequently, system of inequalities (9.12) is equivalent to system (9.13).

$$\int_0^T z'(t)\zeta_1^{(j)}(t)\,dt > \int_0^T z'(t)\zeta_r^{(j)}(t)\,dt, \qquad (9.13)$$

$$r = 1,\ldots,m;\ r \neq l;\ j = 1,\ldots,k$$

This principle must be realized in the decision circuit containing mk filters matched with all realizations of individual signals $\zeta(t)$ (Figure 9.2) to which the received signal is delivered. At instant of readout the voltages in each group of m filters matched with realizations of individual signals $\zeta^{(i)}(t)$ are compared and decisions reached for each message separately.

It can easily be seen that principle (9.5) can be relegated to such a decision circuit.

We can consider the reception of signals in accordance with (9.3) and (9.5) for a channel with a randomly changing phase altogether similarly and also for any channel with variable parameters on condition that individual signals of various messages remain orthogonal at the output of the channel. In all cases the decision circuit is subdivided into k separate circuits for each message, each of which coincides with the decision circuit for individual signal $s^{(i)}(t)$ used without multiplexing.[1] This could have been assumed earlier inasmuch as the signal does not affect the result of optimal processing if they are orthogonal.

Thus, for orthogonal separable systems principles (9.3) and (9.5) are equivalent and, consequently, for them inequalities (9.7) become equalities.

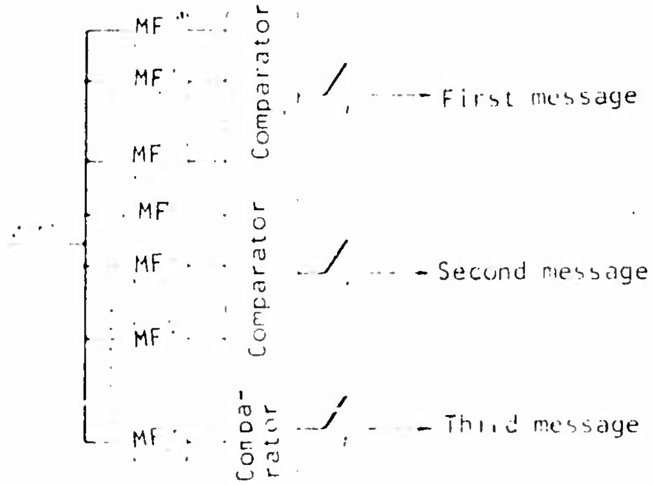


Figure 9.2. Decision Circuit for a Separable System.

With the assumptions made the probability of error p_i during reception of the i-th message does not depend on errors in remaining messages inasmuch as orthogonal signals do not affect the voltage in matched filters at instant of readout. Therefore, p_i can be calculated using the same formulas which were obtained for unmultiplexed channels with the assumption that only individual signal $s^{(i)}(t)$ is transmitted and understanding h^2 to be the ratio between the power of the individual signal and the spectral density of the additive interference. If all individual signals are isomorphic and the interference amounts to normal white noise, for all messages the probabilities of error are the same. With a given probability of error p_i and fixed spectral density of interference, the power of the group signal must be k times greater than the power in an unmultiplexed channel.

In a channel without fading errors in different messages, obviously, are not calculated. Therefore, the probability of error p_Σ in the total signal is

---

[1] This result holds true only when all messages are statistically independent.

$$p_e = 1 - (1 - p_i) . \qquad (9.14)$$

From (9.14) it is apparent that when m = 2 the equivalent probability of error p_e coincides with p_i.

In a fading channel and also in the case of lumped or impulse interference errors in different messages, generally speaking, are correlated and the dependence between p_e and p_i is more complex. The degree of this correlation depends on the properties of the channel and the specific multiplexed system. Thus, in the case of selective fading and frequency multiplexing, errors in the i-th and j-th messages are weakly correlated if the spectra of signals $^{(i)}$ and $^{(j)}$ are sufficiently spread. In the case of purely impulse interference, errors are strongly correlated for frequency multiplexed systems and are practically not correlated for time multiplexed systems, and in the case of lumped interference, vice versa.

We will determine the least possible band occupied by a group signal in an orthogonal separable multiplexed system. This parameter is important because it permits us to judge the effective width of spectrum and this is especially important in multiplexing a channel when limitations are imposed on the transmission frequency band.

A set of elements of signal of length T occupying a possible band of frequencies F is isomorphic to vectorial B-dimensional space where B = 2FT is the signal base. Inasmuch as the orthonomal basis of such a space contains B vectors, the maximum number of mutually orthogonal signals is equal to 2FT. We will note that there is an infinite number of orthogonal systems, each of which has its own orientation of basis vectors.

It can easily be shown that the maximal number of signals mutually orthogonal in the intensified sense is half this magnitude. We will consider a complete system S of signals orthogonal in the intensified sense which have been normalized in power. Let $z_r(t)$ be one of these signals. Signal $\bar{z}_r(t)$ conjugate with it does not enter this system since it does not satisfy the condition of orthogonality in the intensified sense with $z_r(t)$. However, it is orthogonal to $\bar{z}_r(t)$ in the ordinary sense and also to all remaining signals of the system by definition. Furthermore, two signals conjugate with any two other signals of a system are orthogonal with one another and consequently do not coincide. Therefore, by adding conjugate signals to system S, we obtain a system of signals which are pairwise orthogonal in the ordinary sense. This system is complete, for otherwise it would be necessary to add to it a signal $z_s(t)$ orthogonal to all signals of the initial system S and signals conjugate with them, i.e., orthogonal to all signals of system S in the intensified sense, and this contradicts the supposition that S is a complete system of signals orthogonal in the intensified sense. Consequently, system S contains exactly half the signals of a complete system orthogonal in the ordinary sense, i.e., FT signals.

From (2.11) it is apparent that when [illegible] coincident probability [illegible] p_c coincides with p_1.

In a fading channel and also in the case of lumped or impulse interference errors in different messages, generally speaking, are correlated and the dependence between p_c and p_1 is more complex. The degree of this correlation depends on the properties of the channel and the specific multiplexed system. Thus, in the case of selective fading and frequency multiplexing, errors in the i-th and j-th messages are weakly correlated if the spectra of signals $^{(i)}$ and $^{(j)}$ are sufficiently spread. In the case of purely impulse interference, errors are strongly correlated for frequency multiplexed systems and are practically not correlated for time multiplexed systems, and in the case of lumped interference, vice versa.

We will determine the least possible band occupied by a group signal in an orthogonal separable multiplexed system. This parameter is important because it permits us to judge the effective width of spectrum and this is especially important in multiplexing a channel when limitations are imposed on the transmission frequency band.

A set of elements of signal of length T occupying a possible band of frequencies F is isomorphic to vectorial B-dimensional space where B = 2FT is the signal base. Inasmuch as the orthonomal basis of such a space contains B vectors, the maximum number of mutually orthogonal signals is equal to 2FT. We will note that there is an infinite number of orthogonal systems, each of which has its own orientation of basis vectors.

It can easily be shown that the maximal number of signals mutually orthogonal in the intensified sense is half this magnitude. We will consider a complete system S of signals orthogonal in the intensified sense which have been normalized in power. Let $z_r(t)$ be one of these signals. Signal $\bar{z}_r(t)$ conjugate with it does not enter this system since it does not satisfy the condition of orthogonality in the intensified sense with $z_r(t)$. However, it is orthogonal to $z_r(t)$ in the ordinary sense and also to all remaining signals of the system by definition. Furthermore, two signals conjugate with any two other signals of a system are orthogonal with one another and consequently do not coincide. Therefore, by adding conjugate signals to system S, we obtain a system of signals which are pairwise orthogonal in the ordinary sense. This system is complete, for otherwise it would be necessary to add to it a signal $z_s(t)$ orthogonal to all signals of the initial system S and signals conjugate with them, i.e., orthogonal to all signals of system S in the intensified sense, and this contradicts the supposition that S is a complete system of signals orthogonal in the intensified sense. Consequently, system S contains exactly half the signals of a complete system orthogonal in the ordinary sense, i.e., FT signals.

Now it is possible to determine the minimal value of the conditional frequency band of an orthogonal separable multiplexed system if the number k of multiplexing messages and the code base m for each of them are given. When m = 2 a system is possible in which two realizations of an individual signal $\zeta^{(i)}(t)$ are opposed, and orthogonality with the other individual signals is met in the ordinary sense. Then $k \leq 2FT$, whence

$$F_{\min} = \frac{k}{2T}. \tag{9.15}$$

Such a system permits only coherent reception. With incoherent reception orthogonality must be met in the intensified sense. Two realizations of an individual signal can in this case be opposed if relative phase modulation is used. Then

$$F_{\min} = \frac{k}{T}. \tag{9.16}$$

This formula remains valid also when $m > 2$ if m realizations of individual signal $\zeta^{(i)}(t)$ are formed in the form of a linear combination of one realization of $\zeta_1(t)$ and realization $\bar{\zeta}_1(t)$ conjugate with it. Most frequently, however, systems are used in which all realizations of individual signals are mutually orthogonal in the intensified sense, for example, FK systems with frequency or time multiplexing. Since the total number of realizations is mk, for such systems

$$F_{\min} = \frac{mk}{T}. \tag{9.17}$$

Thus, if the length of element T is given, in all orthogonal separable systems the conditional frequency band F is proportional to the multiplexing factor.

Nonorthogonal Systems

In nonorthogonal separable multiplexed systems the probability of error p_i for the i-th message depends on what symbols the remaining messages contain. Thus, in using the decision circuit of Figure 9.2 the voltages at the output of the matched filters at instant of readout depend not only on the individual signal of the corresponding message $\zeta^{(i)}(t)$ but on all remaining individual signals.

Let signal $z'(t) = \mu z(t) + n(t)$, where $z(t)$ is a group signal (9.1), arrive at the input of a filter matched with a certain realization of individual signal $\zeta_r^{(j)}(t)$. The voltage at filter output at instant of readout is proportional to

$$\begin{aligned} u = \int_0^T z'(t)\zeta_r^{(j)}(t)\,dt = \mu\int_0^T \zeta^{(j)}(t)\zeta_r^{(j)}(t)\,dt + \\ + \sum_{\substack{i=1\\ i\neq j}}^{k} \mu\int_0^T \zeta^{(i)}(t)\zeta_r^{(j)}(t)\,dt + \int_0^T n(t)\zeta_r^{(j)}(t)\,dt \end{aligned} \tag{9.18}$$

where $z^{(j)}(t)$ is a realization of the j-th individual signal contained in $z(t)$. The first term in (9.18) represents the useful voltage containing information about the transmitted message and the last term is the action of the additive interference. The sum forming the second term is a random value (as a consequence of the randomness of realization $z^{(i)}(t)$) not carrying information about a transmitted symbol in the j-th message, i.e., it amounts to additional system interference. It is usually called transient interference.

Obviously, the existence of transient interference increases the probability of error more, the stronger is the deviation from orthogonality and the higher is the multiplexing factor. For each specific system the probability of error p_i can be calculated in final form or reduced to quadratures. The distribution of probabilities of transient interference can be considered normal for an approximate evaluation of the probability of error when the multiplexing factor is large. Without resorting to calculations of this we will note that the probabilities of error p_i for different messages, generally speaking, are different.

The only advantage of nonorthogonal separable systems over orthogonal systems is the possibility of obtaining an arbitrary multiplexing factor with a given signal base and this permits multiplexing channels having a given frequency band F with an arbitrary number of messages. However, the smaller the FT/k ratio is, the more orthogonality is disrupted in individual signals, i.e., fidelity drops as a consequence of transient interference. As will be shown below, the best use of carrying capacity of a narrow-band channel is provided by nonorthogonal combination systems. Therefore, use is never intentionally made of nonorthogonal separable systems. They occur as a result of disruption in orthogonality in a separable multiplexing system which is in intent orthogonal. As already noted, such disruptions can occur in channels with fading, with multibeam propagation, or with a long impulse reaction. In designing separable multiplexed systems it is important to select individual signals so that under the conditions existing in a given channel or a channel with a long impulse reaction use should not be made of time multiplexing. Use should be made of frequency multiplexing in a channel with fast fading.

Elementary calculations show that with a permissible probability of error $p_i \leq 10^{-6}$ it is possible in the first approximation to ignore disruptions in orthogonality if the dispersion of the second term (sum) in (9.18) does not exceed 1% of the square of the energy of the individual signal.

Quasi-Separable Systems

In principle quasi-separable multiplexed systems with double modulation have no advantage over separable systems. Nevertheless, they are widely used, especially in radio relay and tropospheric channels because in double modulation the requirements of stable frequency are eased. Quasi-separable systems also have several merits of a technological and organizational nature. If there is an apparatus which permits transmitting telephone signals over a given wide and radio relay channel, for example, by using frequency modulation, and also an apparatus which permits shaping a group signal for frequency multiplexing of a telephone channel. Instead of constructing a special apparatus for multiplexing the wideband channel it is simpler to resort to frequency modulation using the existing group signal.

An optimal decision scheme must be based on application of principle [illegible] or [illegible] to signal [illegible] and can be effected using filters matched with different realizations of this signal. However, such a system is never used since this would mean retreat from those simplifications for the sake of which double modulation is used. Instead of this, an ordinary demodulator is used to extract from a received signal the separated group signal which is then sent to a decision circuit for a suitable multiplexed system as shown in Figure 9.3.

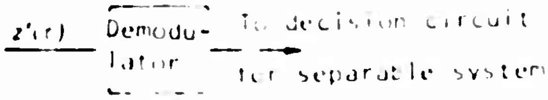


Figure 9.3. Reception of Signals in a Quasi-Separable System.

Without presenting calculations for the probability of error, we will note one widespread misconception leading to an increase in resistance to interference in quasi-separable systems. As is known [6, 7], in the reception of continuous signals with wideband (for example, phase or frequency) modulation, the signal to noise ratio at demodulator output can be much greater than at receiver input. It might seem that calculation of the probability of error in a quasi-separable system could be done as in a separable system, substituting in the formulas the ratio between signal power and spectral noise density obtained in the group signal following the demodulator. A simple example shows the error of this approach.

For simplicity let k = 1, i.e., let there be transmitted only one message. This can be considered as a particular case of a multiplexed system. We will form a "group" signal through frequency modulation and we will use this signal to modulate the oscillations of the carrier frequency in phase.* We obtain two realizations of the signal

$s_1(t) = A\cos$ [illegible]

$s_2(t) = A\cos$ [illegible]

$(0 \leq t \leq T)$ [illegible]

---

*We are considering phase and not frequency modulation in order to avoid the necessity of considering the nonuniformity of noise spectrum at the output of the frequency demodulator. This nonuniformity leads to a situation wherein in systems with frequency multiplexing and frequency modulation the probabilities of error p_1 differ for different messages.

where A is the amplitude; ω_0 is the carrier frequency, [illegible] is the phase modulation coefficient and [illegible] and [illegible] are frequencies corresponding to symbols [illegible] and [illegible].

These signals, generally speaking, are not orthogonal since their scalar product is

$$\int_0^T s_1(t)s_2(t)\,dt = [illegible]\left[\int_0^T \cos[illegible]\,\Omega_1 t [illegible]\,\Omega_2 t\,[illegible] + \int_0^T \cos[illegible]\,\Omega_1 t\,[illegible]\,\Omega_2 t\,[illegible]\right]$$

and although the first integral in the case when [illegible] is equal practically to zero, this cannot be said about the second integral.

If optimal coherent reception of signals (9.19) exists in a channel with [illegible], the probability of error cannot be less than for opposed signals, i.e.,

$$p = \frac{1}{2}[1 - \Phi(\sqrt{2}\,h)], \qquad [illegible]$$

where h^2 [illegible] is the ratio between the power of the signal (9.19) and spectral density of interference at receiver input.

We will now use another demodulation procedure, namely, we will first perform phase demodulation and separate the [illegible] group signals

$$s_1 = [illegible]\,\Omega_1 t, \qquad [illegible]$$

$$s_2 = [illegible]\,\Omega_2 t.$$

[illegible] will then [illegible] probability of error will be equal to

$$p = \frac{1}{2}\exp\left(-\frac{h_1^2}{2}\right), \qquad [illegible]$$

where h_1^2 is the ratio between the power of the group signal (9.2[illegible]) and the spectral noise density at the output of the phase demodulator. According to phase demodulation theory [illegible], the generalized gain is

$$[illegible] \qquad [illegible]$$

It can easily be seen that the probability of error computed in this way in a nonoptimal system can be many times less than in an optimal system. Indeed, let [illegible] and [illegible]. Then from (9.2[illegible]) [illegible], the probability of error [illegible] in the optimal system according to (9.20) is $3.5\cdot10^{-4}$ and in a nonoptimal system, according to (9.22), it is $5\cdot10^{-10}$.

This seeming paradox arose because the threshold properties of phase modulation were not taken into consideration. Indeed, ratio (9.23) is valid only when the noise envelope at receiver input is much less than the signal envelope. But errors do not occur under these conditions. If in a channel, it only for a very short time, interference exceeds the signal or becomes comparable in level with the signal after demodulation the ratio between signal and interference not only does not increase but even decreases. In other words, the signal at these instants is suppressed by interference and preconditions are established for incorrect symbol reception.

Thus, computation of probability of error in a nonoptimal system with phase modulation, which led to the result $p = 5\cdot10^{-10}$, was performed incorrectly. For exact analysis it is necessary to use instead of (9.23), the more complex dependence between the strength of the signal at the input and the strength of the signal at the output of the demodulators in the deduction of which no suppositions were made to the effect that the interference at receiver input is much weaker than the signal, and to consider that in (9.22) h_{gr} is variable. In this case it is apparent that the probability of error must be greater than in the case of an optimal decision system.

9.[illegible] Combination Multiplexed Systems

Orthogonal Systems

For transmission of k messages, each of which is encoded the code of base m, a signal element in a multiplexed channel must have $m^* = m^k$ realizations. If all these realizations are mutually orthogonal, a combination orthogonal multiplexed system results.

It is possible to determine a minimal conditional frequency band of such a signal just as in the case of orthogonal separated systems by equating m^* to a signal base of 2FT or FT for orthogonal signals in the general or the intensified sense respectively. As a result we obtain

$$F_{min} = \frac{m^*}{2T} \qquad (9.24)$$

for orthogonality in the ordinary sense and

$$F_{min} = \frac{m^*}{T} \qquad (9.25)$$

for orthogonality in the intensified sense.

Combination systems which are orthogonal in the intensified sense present the greatest interest for actual practice. As an example is provided by the widely used double frequency telegraphy (DFT) system in which $m = 2$, $k = 2$ and four signal realizations represent segments of sinusoids with different frequencies.

Orthogonality in the intensified sense here is provided if these frequencies are divisible by 1/T. For approximate orthogonality in the intensified sense it is sufficient that the frequency differences greatly exceed 1/T. Orthogonal systems of multiple frequency telegraphy (MFT) can be constructed similarly. In these systems m^k signals are used in the form of segments of sinusoids at different frequencies. If the difference between adjacent frequencies is equal to 1/T, a minimal conditional frequency band is realized (9.25).

Comparing (9.25) with (9.17), we see that only when $m = k = 2$ do separable and combination orthogonal systems occupy the same frequency band (the trivial case when $k = 1$ is not considered). In all remaining cases orthogonal combination systems occupy a wider frequency band than separable systems and furthermore this difference increases greatly with an increase in the multiplexing factor k.

A decision circuit for a combination system based on principle (9.3), i.e., minimizing the total probability of error p, does not differ from the decision circuit for an unmultiplexed system with a code base of m^k. It is only necessary, after identifying a signal realization which has been transmitted, to form the symbols corresponding to it for all messages.

A decision circuit based on rule (9.5) minimizing p_i is much more complex. In accordance with (9.5), it is necessary to summate the values of the a posteriori probabilities or magnitudes proportional to them. In the case of equiprobable symbols it is possible to summate magnitudes which are proportional to likelihood functions. But in ordinary decision circuits (for example, in a quadrature circuit or at the output of a matched filter) voltages are obtained which are not proportional to a likelihood function but only depend monotonically on it. Inasmuch as in a circuit based on principle (9.3) it suffices to compare likelihood functions, this can be replaced by a comparison of magnitudes depending monotonically on them. When using principle (9.5) it is necessary to convert an obtained voltage (for example, the value of the envelope at matched filter output) into a magnitude proportional to the likelihood function. In a channel without fading in the case of incoherent reception, according to (4.27) for this purpose it is necessary to obtain the function $I_0(V$ [illegible]$)$.

Figure 9.4 shows functional decision circuits for a combination system which are constructed in accordance with the two rules mentioned. For simplicity the case when $m = 2$, $k = 2$ (for example, DFM) is taken. The reader can easily construct similar circuits for any m and k.

The probabilities of error in a combination system orthogonal in the intensified sense in the case of incoherent reception can easily be computed if use is made of principle (9.3) which minimizes the total probability of error p. For computation of p it is only necessary to use the previously obtained formulas pertaining to ordinary systems orthogonal in the intensified sense, replacing the code base [illegible] in them with [illegible] For example, [illegible] slow Rayleigh fading, based on [illegible]

[illegible]

We will note that to compute the probability of error in a message, [illegible] the same decision principle, the system under consideration is symmetrical. If the [illegible] signal is received incorrectly, it can with the same probability be identified with any of the remaining [illegible]. Among these realizations there are [illegible] to which corresponds the [illegible] in the [illegible] message which was actually transmitted. If a received signal is identified with one of these realizations, there will [illegible]. Hence the [illegible] probability of error [illegible] when the [illegible] is incorrectly identified is equal to

[illegible]

In the particular case for [illegible]

[illegible]

It follows from the symmetry of the [illegible] that the [illegible] probabilities of error do not depend on the [illegible] and [illegible] the [illegible] with [illegible] of different messages [illegible] with one another.

It is much more difficult to [illegible] of error [illegible]. We will first consider [illegible] which [illegible]

[illegible]

where the symbol [illegible] denotes the [illegible]

[illegible] the transition [illegible] the probability [illegible] message [illegible]

[illegible]

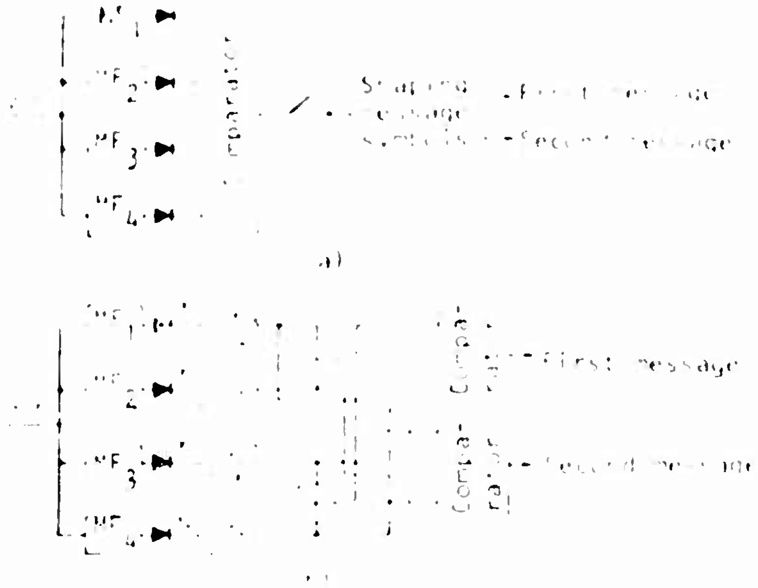
[illegible]

$$[illegible]$$

[illegible]

$$[illegible]$$

[illegible]



MF1
MF2
MF3
MF4
Comparator
Starting message symbols
First message
Second message
(a)
MF1
MF2
MF3
MF4
Comparator
Comparator
First message
Second message
(b)

Figure 9.4. [illegible] circuits for [illegible] of Signals in a Combination System [illegible] the Total Probability of Error (a) and Probability of Error in Each Message (b).

[illegible]

Figure [illegible] shows the dependence of probability of error in the [illegible] channel with Rayleigh fading for orthogonal combination systems. [illegible] the broken lines show the dependences for an orthogonal separable [illegible] case of relative phase modulation of individual signals. In the [illegible] orthogonal combination systems have a clear advantage.

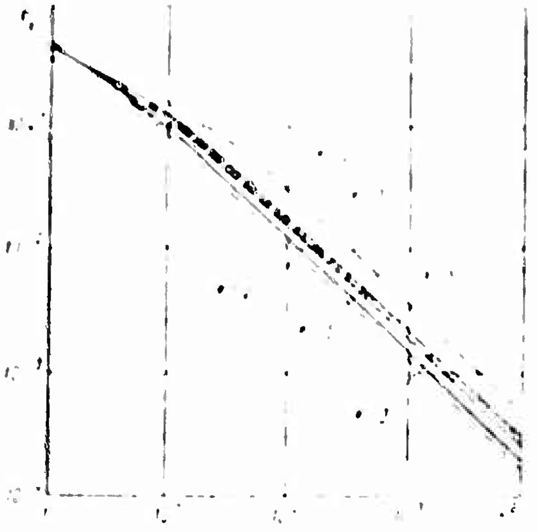


Figure 9.5. Probabilities of Error p_i in a Channel with Rayleigh Fading. ———, Combination orthogonal systems; ------, Separable orthogonal RPT systems.

MPT Systems

In nonorthogonal combinations systems $r = m^k$ realizations of the total signal can be selected entirely arbitrarily. Therefore, it is difficult to find laws which will be valid for any system. We will limit ourselves to a consideration of three variations of nonorthogonal combination systems--MPT, MKPT, and MFT.

In multiple systems with phase modulation (MFT) realizations of a signal amount to segments of a sinusoid of a certain frequency with an initial phase assuming m^k different values

[illegible]

[illegible]

We will expand [illegible] received signal [illegible] It can easily be seen that all [illegible]

[illegible]

The area of correct reception [illegible] within the limits

[illegible]

Thus, in the space of the received [illegible] components with a frequency [illegible] is subdivided into 2^k sectors [illegible] for the i-th message there are 2^{k-1} sectors corresponding to the symbol "0" and 2^{k-1} sectors corresponding to the symbol "1".

If we now refer to principle (9.[illegible]) [illegible] it is easy to see that in the i-th message symbol "0" must be registered at the [illegible] when the signal received lies within one of the 2^{k-1} sectors corresponding to this symbol. In determining symbols of all k messages in this way, we finally find that the aggregate of all received symbols is unambiguously determined by that sector in which the value of [illegible] lies, i.e., we arrive at principle (9.[illegible]) [illegible] "PI system", as in the [illegible] systems and in [illegible]

probabilities of error for all messages, or to achieve the most uniform probability of error for all messages, etc. The well known Gray keying code [8] provides for the least average probability of error. For this code sectors corresponding to one symbol in the first and second messages are arranged very compactly, two groups each occupying 90° are formed for the third message, four groups of 45° each are formed for the fourth message, etc., as shown in Figure 9.8. A comparison of angles with aggregates of symbols for n = 4 is given in table 9.1.

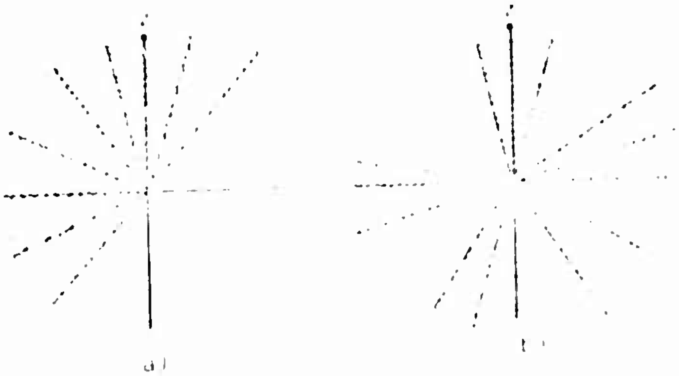
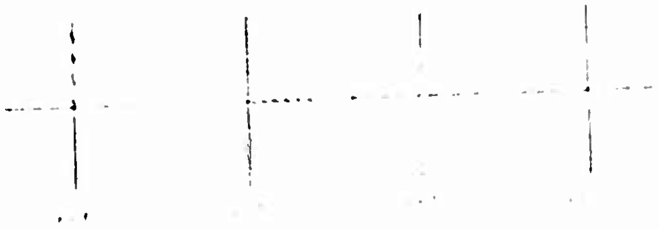


Figure 9.7. Effect of [illegible] Code [illegible] Error [illegible] in the [illegible] Message [illegible]

[illegible]



[illegible]

[illegible]

[illegible]

[illegible]

$$P_2 = 1 - \frac{1}{2}\Phi\left(\sqrt{2}\,h\sin\frac{\pi}{2^n}\right) - \frac{1}{2} - V\left(\sqrt{2}\,h\sin\frac{\pi}{2^n},\ \sqrt{2}\,h\cos\frac{\pi}{2^n}\right), \tag{9.34}$$

where V(x,y) is a Nicholson function [10].

Specifically, when k = 1,

$$p_1 = \frac{1}{2}[1 - \Phi(\sqrt{2}\,h)] \tag{9.35}$$

which coincides with (5.4), and when k = 2 (DPI)

$$p_2 = \frac{3}{4} - \frac{1}{2}\Phi(h) - \frac{1}{4}\Phi^2(h), \tag{9.36}$$

which can also be obtained from (5.70a).

TABLE 9.1

[illegible]	Number of message			
	1	2	3	4
0	0	0	0	0
[illegible]	0	0	0	1
[illegible]	[illegible]	0	1	1
[illegible]	0	0	1	0
[illegible]	0	1	1	0
[illegible]	0	1	1	1
[illegible]	0	1	0	1
[illegible]	0	1	0	0
[illegible]	1	1	0	0
[illegible]	1	1	0	1
[illegible]	1	1	1	1
[illegible]	1	1	1	0
[illegible]	1	0	1	0
[illegible]	1	0	1	1
[illegible]	1	0	0	1
[illegible]	1	0	0	0

With large multiplexing factors and small probabilities of error it is possible to use the rather precise evaluation

$$p = 1 - \Phi\left(\sqrt{2}\,h\sin\frac{\pi}{2^n}\right) \tag{9.37}$$

In the case of slow Rayleigh fading for DPI, by averaging (9.36) with respect to h as was done in Chapter V, we obtain

$$p_\Sigma \approx [illegible] \qquad (9.38)$$

When $k = 2$ and $h_0^2 \gg 1$ from (9.31), as was shown by N. F. Khvorostenko, we can obtain the evaluation

$$p_\Sigma \approx [illegible] \qquad (9.39)$$

For computation of probability of error p_i in a message of a DPT system we will note that it can be viewed as an orthogonal (in the general sense) separable system. Indeed, signal (9.31) when $m = k = 2$ can be represented in the form

$$\begin{gathered} s_r(t) = A\cos\left[\omega t + \frac{\pi}{2}(r-1)\right] = \frac{1}{\sqrt{2}}A\cos\left[\omega t + \frac{\pi}{4} + (r_1 - 1)\pi\right] + \frac{1}{\sqrt{2}}A\cos\left[\omega t - \frac{\pi}{4} + (r_2 - 1)\pi\right] = \\ s^{(1)}_{r_1}(t) + s^{(2)}_{r_2}(t), \\ r = 1, 2, 3, 4;\ r_1 = 1, 2;\ r_2 = 1, 2. \end{gathered} \qquad (9.40)$$

Thus, signal $s(t)$ breaks down into the sum of two mutually orthogonal individual signals $s^{(1)}(t)$ and $s^{(2)}(t)$, each of which has two opposed realizations and carries information about its message. It follows from this that errors in both messages in the absence of fading are independent, and furthermore, based on considerations of symmetry, $p_1 = p_2 = p_i$. Therefore, the probability of correct reception of both messages is

$$1 - p_\Sigma = (1 - p_1)(1 - p_2) = (1 - p_i)^2, \qquad (9.41)$$

whence

$$p_i = 1 - \sqrt{1 - p_\Sigma}. \qquad (9.42)$$

Considering (9.36) we obtain

$$p_i = 1 - \frac{1}{2}[1 + \Phi(h_0)] = \frac{1}{2}[1 - \Phi(h_0)], \qquad (9.43)$$

which coincides with the probability of error for a binary FK system in the case of coherent reception. With Rayleigh fading, by averaging (9.43) we obtain

$$p_i = \frac{1}{2}\left[1 - \sqrt{\frac{h_0^2}{h_0^2 + 2}}\right]. \qquad (9.44)$$

When $k = 3$ and the Gray code is used, the probabilities of error p_i in the absence of fading can be calculated from Figure 9.9. The heavy arrows show vectors depicting realizations of signal $s(t)$ and the broken lines indicate boundaries between different solutions. Let the signal 000 be transmitted. An error occurs in the first message if an interference component directed along arrow a exceeds the magnitude A sin π/8, inasmuch as the sum of the signal and the interference is on the other side of the boundary labeled I and separating the areas corresponding to "0" and "1" in the first message. The interference component which is orthogonal to arrow a has no effect on errors in the first message. The probability of such an error, as can easily be calculated, is

$$p'_1 = \frac{1}{2}\left[1 - \Phi\left(\sqrt{2}\,h\sin\frac{\pi}{8}\right)\right].$$

If signal 010 is transmitted, the probability of "0" becoming "1" in the first message is the same.

In case signal 001 or 011 is transmitted, the interference component running along b (or c) must exceed A cos π/8 and this happens with a probability of

$$p''_1 = \frac{1}{2}\left[1 - \Phi\left(\sqrt{2}\,h\cos\frac{\pi}{8}\right)\right].$$

Assuming that all signals are transmitted uniformly, we find that the total probability of error in the first message is equal to

$$p_1 = \frac{1}{2}(p'_1 + p''_1) = \frac{1}{4}\left[2 - \Phi\left(\sqrt{2}\,h\sin\frac{\pi}{8}\right) - \Phi\left(\sqrt{2}\,h\cos\frac{\pi}{8}\right)\right] \qquad (9.45)$$

It can easily be seen that the probability of error p_2 in the second message will be the same.

To determine the probability of error p_3 in the third message we will consider as an example signal 001. The "1" in the third message will become a "0" if the interference component running along arrow e exceeds A sin π/8, or[1] the component orthogonal to it running along arrow f exceeds A cos π/8. A similar condition occurs with transmission of any other signal. Inasmuch as the interference components are independent, we can easily calculate [11]

$$\begin{aligned} p_3 &= 1 - \frac{1}{4}\left[1 + \Phi\left(\sqrt{2}\,h\sin\frac{\pi}{8}\right)\right]\left[1 + \Phi\left(\sqrt{2}\,h\cos\frac{\pi}{8}\right)\right] - \\ &\quad - \frac{1}{4}\left[1 - \Phi\left(\sqrt{2}\,h\sin\frac{\pi}{8}\right)\right]\left[1 - \Phi\left(\sqrt{2}\,h\cos\frac{\pi}{8}\right)\right] = \\ &= \frac{1}{2}\left[1 - \Phi\left(\sqrt{2}\,h\sin\frac{\pi}{8}\right)\Phi\left(\sqrt{2}\,h\cos\frac{\pi}{8}\right)\right]. \end{aligned} \qquad (9.46)$$

---

[1] Here we have in mind an excluding "or" since if both interference components exceed the indicated magnitude, as apparent from Figure 9.9, the symbol of the third message will be received correctly.

[illegible] second [illegible]

Figure 9.[illegible] [illegible]

[illegible]

When [illegible] calculation of the probability of [illegible] complex. However, with the [illegible] small, we can [illegible] use of the fact [illegible] fixed realization of a signal the error [illegible] of the message. Therefore, [illegible] error $P_{1\,av}$ is approximately [illegible] and (9.50), in a channel with [illegible]

$$P_{1\,av} \approx \frac{1}{2}\left\{1 - \Phi\left(\sqrt{\text{[illegible]}}\right)\right\}$$

and in a slow Rayleigh fading

$$P_{1\,av} = \frac{1}{2h_0^2 \text{[illegible]}}\left[\text{[illegible]}\right]$$

An analysis of the expressions [illegible] factor the average probability of error [illegible] from orthogonal combination systems, [illegible] in Figure 9.10 where the [illegible] lines [illegible] dependence of the average probability of error on [illegible]. Therefore, such systems can best be used for channels in which the [illegible] capacity is conditioned by a high signal-to-noise ratio and the [illegible] small.

MRPT Systems

In multiple relative phase telegraphy (MRPT) systems information is contained in the difference in phase between adjacent signal elements. In other words, they are distinct from MPT systems in that reading the phase [illegible] element is done not from a constant "reference phase" but from the phase of the preceding signal element. The difference in phase between adjacent elements is m = 2^k different values. When m = 2 these values are equal to

$$\Delta_n = \frac{2\pi}{m} n + \Delta_0 \qquad (9.[illegible])$$

where n = 1, ..., 2^k and Δ_0 is an arbitrary constant phase difference which in existing systems is equal to zero but sometimes differs from zero and then is used to simplify shaping the signal and also for synchronization [1.]. The value of Δ_0 does not affect resistance to interference.

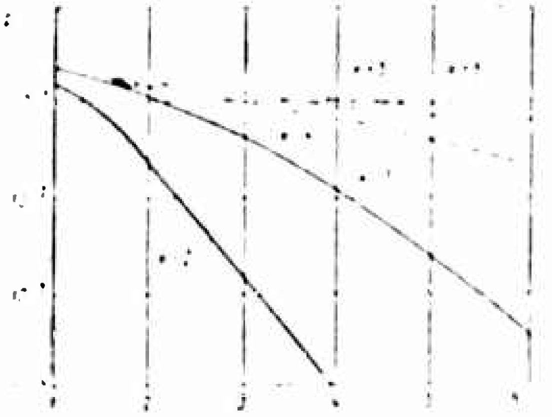


Figure 9.10. Average Probabilities of Incorrect Reception of a Message Symbol in an RPT System in a Channel without Fading.

As compared with PT systems RPT systems have the advantage that with the use of coherent reception they are not as sensitive to spontaneous shifts of the phase of the reference voltage. The latter cause "reverse operation" of PT systems and also a shift in the information being transmitted from one message to another, while in the case of RPT systems they cause at worst the same error in each of the messages. Furthermore, in RPT systems incoherent reception is possible, no reference voltage with [illegible] being necessary.

In the case of coherent reception the initial phase of a transmitted oscillation of a signal is identified just as in a PT system and then by comparison with the phase determined in the preceding element the transmitted aggregate of symbols of all messages is found. It follows that just as in PT systems the decision principles based on minimization of [illegible] are equivalent. This is true also for incoherent reception [13].

The probabilities of error in the case of coherent RPT reception can be evaluated by using the results obtained above for PT. It must be borne in mind that an isolated error in determination of phase of an arriving signal causes two adjacent errors in determining the difference of adjacent elements. In the case of n adjacent errors in determination of phase of signal the number of errors in determining the phase differences can assume values from 2 if all errors in determination of phase coincide in magnitude and sign to n + 1 if the difference in phase determination vary. Therefore, the probability of total error in the case of coherent [illegible] reception p [illegible] can be evaluated by the inequality

$$[illegible] \qquad (9.50)$$

with small probabilities of error when [illegible] the errors in phase determination are isolated, this inequality is [illegible] evaluation. Therefore, [illegible] in accordance with (9.[illegible]),

$$[illegible]$$

which coincides with [illegible] in accordance with (9.[illegible]),

$$[illegible]$$

These approximate equalities [illegible]

[illegible]

$$[illegible]$$

We will now turn to a consideration of [illegible] signal reception. As shown in section [illegible]

[illegible] signals with relative phase modulation over the interval [illegible] then one of two signal realizations in which the initial phase [illegible] responds to each signal [illegible].

Realization of a "[illegible]" of an [illegible] signal assumes the form

[illegible] (9)

where [illegible] the difference [illegible] transmitted [illegible] symbols and [illegible]

[illegible]

[illegible] phase shift [illegible]

[illegible] general [illegible] for the [illegible] coherent reception [illegible] circuits [illegible] signal element [illegible]

[illegible] (9.[illegible])

[illegible]

We will assume that [illegible] where [illegible]

[illegible]

where [illegible] and [illegible] are [illegible] over the interval [illegible] respectively [illegible]

Similarly,

$$a\int_0^T x'(t)\cos(\omega t + \Delta_r + \varphi)\,dt = \frac{aT}{2}[V'_n\cos(\Delta_r + \varphi)$$

$$- W'_n\sin(\Delta_r + \varphi)],$$

where V'_n and W'_n are the same kind of Fourier coefficients for $x'(t)$ over the interval (0,T)

$$\int_T [illegible] \qquad a\int_T [illegible]$$

$$\frac{aT}{2}[V_n [illegible] W_n [illegible]]$$

$$\int_0^T [illegible] \qquad \frac{aT}{2}[V'_n\sin(\Delta_r + \varphi) + W'_n\cos(\Delta_r + \varphi)]$$

Thus, [illegible] (9.54) can be written in the following form:

$$[illegible]$$

[illegible] collecting like terms, yields

$$(V_nV'_n + [illegible])\cos\Delta_r - (V_nW'_n - W_nV'_n)\sin\Delta_r >$$
$$> (V_nV'_n + W_nW'_n)\cos\Delta_r - (V_nW'_n - W_nV'_n)\sin\Delta_r \qquad (9.55)$$

[illegible]

$$[illegible] \qquad (9.56)$$

[illegible]

$$\cos[illegible]\Delta_r [illegible]\sin[illegible]\Delta_r,$$

$$\cos(\hat{\Phi} - \Delta_r) [illegible] \cos(\hat{\Phi} - \Delta_r).$$

[illegible] that [illegible]

$$|\hat{\Phi} - \Delta| [illegible] |\hat{\Phi} - \Delta|, \qquad (9.57)$$

which [illegible] although the variables [illegible] here have a different meaning. We will note that the mathematical expectation of [illegible] is equal to the transmitted phase difference [illegible]

As already noted in Section 4.6 for reception of RPT signals it is possible to use decision circuits usual for optimal incoherent reception, for example, a quadrature circuit with matched filters and envelope detectors, and others. The same pertains to MRPT. However, here other decision circuits are possible in which separate decisions are reached for k messages. For example, a decision circuit is suggested in [14] which is based on the algorithm shown in (9.57) for any multiplexing factor on condition that the Gray code is used.

As can easily be seen, when the Gray code is used it follows from (9.57)* that the symbol "0" in the first message is recorded if $0 \leq \varphi < \pi$, in the second message if $-\pi/2 \leq \varphi < \pi/2$, in the third message if $-\pi/2 \leq 2\varphi < \pi/2$, in the fourth message if $-\pi/2 \leq 4\varphi < \pi/2$, and generally in the i-th message $(i \geq 2)$ a "0" is registered if $-\pi/2 \leq 2^{i-2}\varphi < \pi/2$. In other words, in the first message "0" is registered if $\sin \varphi \geq 0$ and in remaining messages if $\cos(2^{i-2}\varphi) \geq 0$.

This permits constructing the decision circuit shown in Figure 9.11. The signal received passes through the filter MF matched with a segment of a sinusoid with a frequency of ω_0 and a length of T. At readout instants divisible by T the sine and cosine components of the output voltage of this filter are divisible by A_n and B_n. This voltage is shifted in phase by $\pi/2$ and is multiplied by the same voltage delayed by T. After integration a voltage is obtained which is proportional, as can easily be seen, $A'_n B''_n - A''_n B'_n$, i.e., coinciding in sign with $\sin \varphi$. The same operation without a change in phase occurs in the second multiplier, i.e., a voltage is obtained which coincides in sign with $\cos \varphi$. At each subsequent multiplier a direct and delayed voltage arrive after multiplying the frequency by 2. In the same way the signs of $\cos(2^{i-2}\varphi)$ are determined and these are used to reach decisions for all messages.

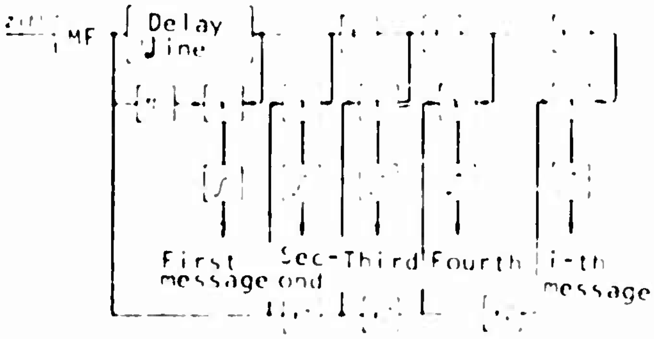


Figure 9.11. Autocorrelation Decision Circuit for Messages in an MRPT System.

---

*Here it is assumed that $\omega_0 T = 2\pi k$.

**In [14] no mention is made of a matched filter. Therefore, a circuit is suggested there which, strictly speaking, is not optimal.

We will draw attention to one decision circuit for the DRPT (k = 2) which was first used in an MC 1 system [9]. Let $\varphi_1 = 0$ correspond to symbols 00, $\varphi_2 = \pi/2$ to symbols 01, $\varphi_3 = \pi$ to symbols 11, and $\varphi_4 = 3\pi/2$ to symbols 10. Substituting these values of φ_r in (9.55), we find that symbols 00 must be registered if

$$A'_nA''_n + B'_nB''_n > |A'_nB''_n - A''_nB'_n|$$

symbols 01 if

$$-(A'_nB''_n - B'_nA''_n) > |A'_nA''_n + B'_nB''_n|;$$

symbols 11 if

$$-(A'_nA''_n + B'_nB''_n) > |A'_nB''_n - B'_nA''_n|;$$

and symbols 10 if

$$A'_nB''_n - B'_nA''_n > |A'_nA''_n + B'_nB''_n|$$

It can easily be seen from these inequalities than in the first message symbol "0" must be registered if

$$A'_nA''_n + B'_nB''_n + B'_nA''_n - A'_nB''_n > 0, \quad (9.58)$$

and in the second message if

$$A'_nA''_n + B'_nB''_n - B'_nA''_n + A'_nB''_n > 0 \quad (9.59)$$

The quadrature decision circuit* shown in Figure 9.12 [9] which needs no further explanation is constructed on the basis of this algorithm. We will only note that one of its merits is that the memory device must "remember" only the magnitude of a constant voltage in distinction from the delay lines in the circuit of Figure 9.11 and other authocorrelation circuits where the phase of a variable voltage is remembered.

We will begin the determination of error probabilities in the case of incoherent reception with a DRPT system. An evaluation of the total probability of error p for this system was obtained in (4.111).

Let symbols 00 be transmitted by difference in phase $\varphi_1 = \pi/4$, symbols 01 by $\varphi_2 = 3\pi/4$, symbols 11 by $\varphi_3 = 5\pi/4$, and symbols 10 by $\varphi_4 = 7\pi/4$. Then principle (9.57) can be represented in the following form: symbol "0" in the first message is registered if cos $\varphi > 0$ and in the second message if sin $\varphi > 0$. Based on (9.56) the principle for registering symbol "0" in the first message can be written as follows:

---

[*] In [9] this circuit is called a correlation circuit.

$$A'_nA'_n + B'_nB''_n > 0 \qquad (9.60)$$

To compute the probability of error in the first message the probability of disrupting the inequality of (9.60) should be found on condition that phase difference Δ_1 or Δ_2 was transmitted. We will use $F(\Delta_i)$ to indicate the probability that inequality (9.60) will not be met if difference in phase Δ_i was transmitted. In work [11] it is shown by investigation of the distribution of probabilities of quadratic form (9.60) that in the absence of fading

$$F(\Delta_i) = Q\left(\sqrt{2}\,h\sin\frac{\Delta_i}{2},\ \sqrt{2}\,h\cos\frac{\Delta_i}{2}\right) - \frac{1}{2}e^{-h^2}I_0(h^2\sin\Delta_i) \qquad (9.61)$$

where $Q(x,y)$ is a Q-function (4.53). A detailed derivation of formula (9.61) is set forth in monograph [9].

The symbol "0" in the first message can be transmitted by differences in phase $\Delta_1 = \pi/4$ or $\Delta_2 = 3\pi/4$. Inasmuch as

$$F(\pi/4) = F(3\pi/4)$$

the probability of error in the first [illegible] message with transmission of symbol "0" is equal to

$$p_1 = F\left(\frac{\pi}{4}\right) = Q\left(\sqrt{2}\,h\sin\frac{\pi}{8},\ \sqrt{2}\,h\cos\frac{\pi}{8}\right) - \frac{1}{2}e^{-h^2}I_0\left(h^2\sin\frac{\pi}{4}\right) \qquad (9.62)$$

From considerations of symmetry it is clear that the probability of error in the transmission of "1" will be the same and also that $p_2 = p_1$.

For the same system in the case of slow Rayleigh fading

$$p_1 = p_2 = \frac{1}{2}\Bigl[1 \text{ [illegible]} \Bigr] \qquad (9.63)$$

This formula can be obtained by averaging (9.62) with respect to the Rayleigh random variable h or by investigating the quadratic form (9.60).

It is rather easy to calculate probabilities of error in the first and second messages [illegible] when the [illegible] Δ_1 [illegible] Δ_2 [illegible] In this process in the first message differences in phase from Δ_1 [illegible] 8 to Δ_1 [illegible] correspond to the symbol "0". According to (9.57) such a decision must be reached if [illegible] which agrees with principle (9.60). But the probability of disruption of the inequality (9.60) now depends on which of the four phase differences was transmitted, [illegible] on the symbols of other messages. The average probability of error in the first message [illegible].

[illegible] (9.64)

It is easy to see that the probability of error in the second message will be the same. As far as the third message is concerned, the probabilities of error p_3 for it, as in the case of coherent reception, is greater than p_1 and p_2. The general methods of calculating p_3 and also the probabilities of error for the different messages of an MPFT system when [illegible] are set forth in [11]. [illegible], however, the [illegible] to [illegible] results by analogy.

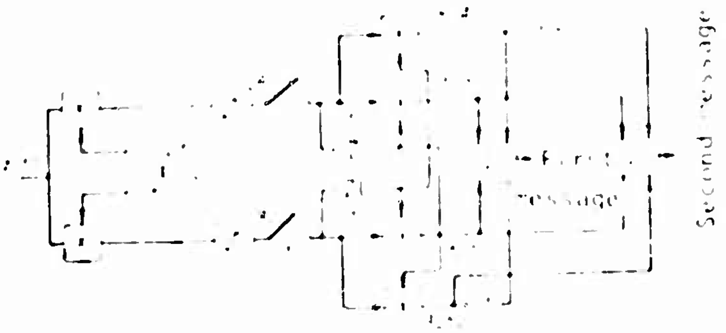


Figure 9.12. Quadrature (Correlation) Circuit for Receiving DPRT Signals: Z, Memory device for time T, [illegible], Summators and I, Polarity Inverter

[illegible] relationship of the [illegible] with [illegible] increase in the multiplexing [illegible] greatly and also the [illegible] coherent [illegible] incoherent reception increases. [illegible] with [illegible] increase [illegible] with respect to power.

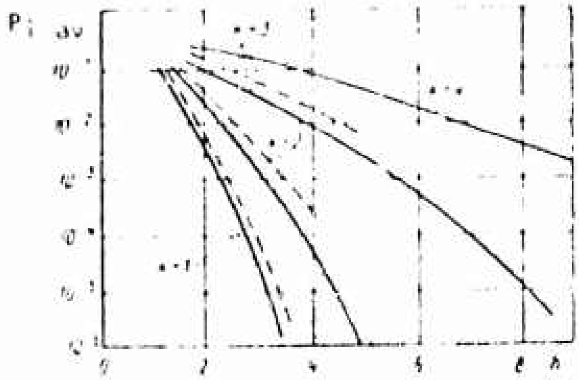


Figure 9.13. Average Probabilities of Error in Messages of an MRPT System in the Case of Coherent (Solid Lines) and Incoherent (Broken Lines) Reception.

With an increase in the rate of fading, the probability of error in MRPT systems increases and, just as in a single RPT system, does not approach zero with an increase in h. The dependence of probabilities of error on rate of fading can be found by the method described in Chapter V. The results are set forth in works [9, 11, 15, 16]. In work [15] the resistance to interference in DRPT systems in the case of diversity reception is investigated.

Nonorthogonal MFT Systems

MFT systems may be nonorthogonal if the differences in frequencies of signal realizations are not divisible by 1/T. Practically speaking, the use of such systems is meaningful when the difference between adjacent frequencies are less than 1/T. This permits obtaining a system for multiplexing a channel with a smaller passband than needed for a given factor in the case of an orthogonal MFT system. Of course, due to this the probability of error increases and also symmetry is disrupted, i.e., the fidelity ceases to be independent of the selected keying code. Thus, these systems as if occupy an intermediate position between MRPT and orthogonal MFT.

The theory of nonorthogonal MFT systems has been developed very little and calculation of error probabilities for them is a very complex matter. We will limit ourselves to a qualitative review and comparison of resistance to interference in nonorthogonal MFT systems and MRPT. For this purpose we will define the parameter of nonorthogonality between two realizations of a signal:

$$\rho_{ik}=\frac{1}{P_c T_c}\left[\left(\int s_i(t)\, s_k(t)\, dt\right)^2+\left(\int s_i(t)\, \tilde{s}_k(t)\, dt\right)^2\right]^{1/2}, \tag{9.65}$$

where integration is done over an interval of signal element length, i.e., from 0 to T for MFT or from -T to T for MRPT and $T_c = T$ for MFT and $T_c = 2T$ for MRPT.

As was shown in Chapter IV, this parameter determines the resistance to interference of k-ary nonorthogonal systems in the case of incoherent reception. It can be assumed that of the two combination systems the system in which the maximal magnitude of this parameter $\max p_{rl}$ is less is the more resistant to interference, all other things remaining equal. Two systems with the same value of $\max_{r,l} p_{rl}$ are approximately isomorphic.

For MRPT systems (9.55)

$$p_{rl}=\frac{1}{T}\left\{\left[\int_{-T}^{0}\cos^2(\omega t+\theta)\,dt+\int_0^T\cos(\omega t+\Delta_r+\theta)\times\cos(\omega t+\Delta_l+\theta)\,dt\right]^2+\left[\int_{-T}^{0}\cos(\omega t+\theta)\sin(\omega t+\theta)\,dt+\int_0^T\cos(\omega t+\Delta_r+\theta)\sin(\omega t+\Delta_l+\theta)\,dt\right]^2\right\}^{1/2}=\frac{1}{2}\{[1+\cos(\Delta_r-\Delta_l)]^2+\sin^2(\Delta_r-\Delta_l)\}^{1/2}=\left\{\frac{1}{2}[1+\cos(\Delta_r-\Delta_l)]\right\}^{1/2}=\left|\cos\frac{\Delta_r-\Delta_l}{2}\right| \tag{9.66}$$

For an MFT system (assuming $f_r T \gg 1$)

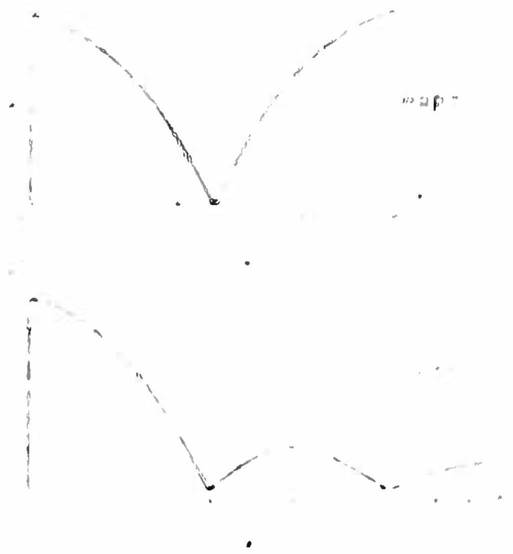
$$p_{rl}=\frac{2}{T}\left\{\left[\int_0^T\cos(\omega_r t+\theta_r)\cos(\omega_l t+\theta_l)\,dt\right]^2+\left[\int_0^T\cos(\omega_r t+\theta_r)\sin(\omega_l t+\theta_l)\,dt\right]^2\right\}^{1/2}\approx\frac{1}{(\omega_r-\omega_l)T}\times\{2[1-\cos(\omega_r-\omega_l)T]\}^{1/2}=\left|\frac{\sin\frac{\omega_r-\omega_l}{2}T}{\frac{\omega_r-\omega_l}{2}T}\right| \tag{9.67}$$

Figure 9.14a shows the dependence of p_{rl} on $\Delta_r - \Delta_l$ for MRPT and Figure 9.14b shows the dependence of p_{rl} on $(f_r - f_l)T$ for MFT.

In MRPT systems the values of transmitted differences in phases are limited by the magnitude 2π. In the case of k-fold multiplexing the least value of $|\Delta_r - \Delta_l|$ cannot be greater than $2^{-k+1}\pi$. Therefore, $\max p_{rl}$ increases rapidly with an increase in the multiplexing factor. The values of $\max p_{rl}$, as can be seen from Figure 9.14a, are approximately equal: when k = 2, 0.71; when k = 3, 0.92; when k = 4, 0.98. When $k \gg 1$, assuming in (9.66) that $\Delta_r - \Delta_l = 2^{-k+1}\pi$, we can easily obtain

$$\max_{r,l} p_{rl} \approx 1-\frac{\pi^2}{2\cdot 4^k} \tag{9.68}$$

The rapid increase in this parameter causes a sharp drop in the resistance to interference with an increase in the multiplexing factor.



[illegible]

[illegible]

[illegible]

[illegible] for the same signal [illegible] in an "element" in an [illegible] system [illegible] double that in a MFI [illegible] concluded that the [illegible] systems will be approximately isomorphic [illegible] the multiplicity factor [illegible] the same, the signal power in MFI is twice as great as in [illegible], and the frequency band in the MFI system is F [illegible] $\sqrt{3}$ F.

We should note that when [illegible] = 1 an MFI system is isomorphic to an orthogonal [illegible] system with a power twice as great [illegible] in which F [illegible]. When [illegible] = 2 a MFI system is approximately isomorphic to an orthogonal MFI system with power twice as great when F ≈ 1.8 F, since in both systems in this case $\max_{pr}$ [illegible] 0.7 (see Figure 9.14). With a further increase in the multiplexing factor, the frequency band occupied approximately by an isomorphic MFI system, rapidly approaches F = $\sqrt{3}$ F.

By increasing the frequency band F it is possible to decrease the parameter of nonorthogonality and increase the resistance to interference of an MFI system. Approximate calculations show that an increase in the frequency band from $\sqrt{3}$ F to $2\sqrt{3}$ F with a large multiplicity factor decreases the probability of error in [illegible] less degree than doubling the signal power. Therefore, it is possible in [illegible] which [illegible] system, occupying a [illegible] with respect to inter[illegible] the same [illegible] system. Further [illegible] to interference. Thus, with [illegible] which surpass [illegible] that, [illegible]

[illegible]

[illegible]

[illegible]

[illegible] It can conveniently be used [illegible] systems. In this [illegible] each aggregate k of symbols of all the transmitted messages is formed into [illegible] of a secondary code with a base of m^k to which a certain realization of the common signal corresponds. The transmitted symbols of the secondary code are sequentially combined into code combinations in which are also included the necessary redundancy check symbols. Correction or detection of errors is also done in the secondary code after which the corrected symbols are reformed into an aggregate of symbols of the primary code distributed over their messages.

Although the theory of interference-resistant codes with a base of m^k, where m is a simple number (specifically m = 2), has been rather well worked out, the method of coding prescribed has not found wide use in practice. Apparently, the complexity of technological realization is not justified by the small advantages which can be expected from its use.

A third and more practical method of coding is used for separable and quasi-separable systems and has been given the name of parallel coding. It amounts to combining the symbols of all transmitted messages into one code combination and the redundancy symbols added to it are transmitted in the form of additional code messages. For example, if a multiplexed system is designed for the transmission of seven messages and actually only four are transmitted, instead of the remaining three messages, it is possible to transmit check symbols formed in accordance with principle (2.50) and to form a systematic code (7.4). This permits correcting an error in any one of the messages or detecting simultaneous errors in two messages, no matter how frequently they occur.

Parallel coding can be done just as simply as ordinary sequential coding and has many advantages. With parallel coding it is simpler to effect entirety reception, the methods of which will be considered in the following chapter. In a channel with selective fading, in the case of frequency multiplexing, errors in different messages are usually less correlated than errors in the sequential symbols of one message. Therefore, interference-resistant codes which are calculated for a channel without memory are more effective in parallel coding than in sequential coding.

Of course, various mixed methods of coding are possible, for example, coding by using an integrative code in which the first stage of coding is parallel and the second sequential.

9.7. Discussion of Results

In summing up the results of this brief review of methods of multiplexing communication channels, we should first of all note that they are greatly varied as to methods of realization and as to the basic information parameters such as resistance to interference and frequency band occupied. Furthermore, the methods of interference resistant coding in multiplexed channels are also greatly varied.

The one who is developing a system faces the difficult task of selecting the most suitable solution, the task being made even more difficult by the fact that along with information parameters he must give thought to many other technological, economic, and organizational factors. It would be foolhardy to try to give recommendations for the selection of a multiplexed system applicable in all cases. However, several general ideas can be expressed here.

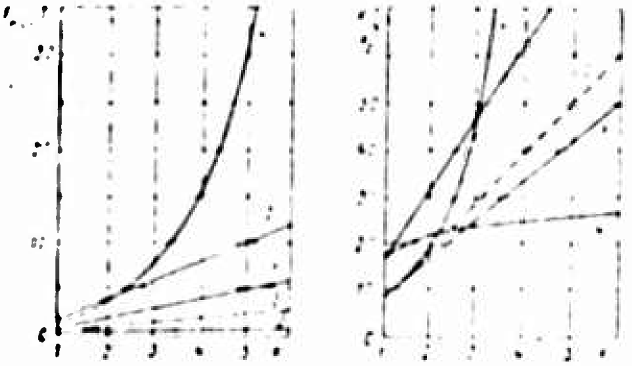
Figure 9.15a shows the dependence of frequency band occupied by a signal and multiplexing factor for the principal systems considered. It was assumed for separable systems that for individual signals relative phase telegraphy is used. Figure 9.15b shows for the same systems the relationship between signal power required to obtain a given level of fidelity (characterized by the average probability of error in a message $p_{1\ av} = 10^{-4}$) and the multiplexing factor. It is very clear from these figures that systems retaining a high level of resistance to interference, when multiplexing factor increases, require great widening of the frequency band occupied by these signals and, on the other hand, in systems retaining a given frequency band the required power rapidly increases with an increase in the factor.

As was noted in our chapter, the problem of multiplexing a channel arises in those cases when the carrying capacity of the channel greatly exceeds the productivity of each of the sources of messages which are to be transmitted. In the simplest case let each source emit messages which are encoded by a sequence of equiprobable and independent binary symbols at the rate of v per second, let the channel have constant parameters and the passband be delimited by the value F (or it may pass signals with a frequency band not exceeding F), let there be in it normal white noise with a spectral density of ν^2, and let the power of the signal at channel output not exceed P. This channel can multiplex k messages if the following inequality is met:

$$kv < F\log_2\left(1+\frac{P}{\nu^2 F}\right). \tag{9.70}$$

where the left side represents the total productivity of the sources and the right part the carrying capacity of the channel. This inequality can be re-written as follows:

$$\frac{2F}{v}\log_2\sqrt{1+\frac{P}{\nu F}} \geq L \qquad (9.71)$$



a)

b)

Figure 9.15. a) Dependence Between Minimal Possible Frequency Band of Signal on Multiplexing Factor; b) Dependence in the Absence of Fading on Multiplexing Factor in the Case When $p_i = 10^{-4}$ 1, Separable orthogonal system, individual RPT signals; 2, Separable orthogonal system, individual signals; 3, MRPT; 4, Orthogonal MFT systems; 5, Mixed system--frequency multiplexing with combination DRPT signals ("Kineplex" type).

The selection of multiplexing method depends mainly on the ratio between the two factors on the left (9.71). For example, let $\frac{2F}{v} \gg \log_2\sqrt{1+\frac{P}{\nu F}}$ i.e., the high carrying capacity of the channel is conditioned mainly by the wide passband, and the available signal power is not great. This occurs, for example, in transmitting telegraph messages over tropospheric radio channels, over channels with passive relay using earth satellites, or in transmitting telemetric messages over low-power radio channels when the rated speed of the transmitters is much less than the passband, etc. Obviously, in this case we should select a multiplexed system which provides for a high level of fidelity with a relatively small ratio of signal power to noise spectral density, if only due to the wide band of frequencies used. In principle an orthogonal MFT system or any other orthogonal combination system is such a system. However, it should be considered that with a large multiplexing factor, designing combination systems is technologically different. Therefore, use is sometimes made under these conditions of separable or mixed multiplexing systems in which the total signal is the sum of several signals, each of which carries several messages and is obtained by the combination method.

In the other extreme case $\frac{2F}{v} \ll \log_2\sqrt{1+\frac{P}{\nu F}}$, i.e., the high carrying capacity is conditioned by a large signal-to-noise ratio and the passband is relatively small. This case is characteristic of [illegible] channels, especially in the case of the [illegible] some radio relay channels, [illegible] In this case [illegible] system [illegible] plexing, [illegible] systems, for example, [illegible]

In [illegible] case when [illegible] of the [illegible]

[illegible] the frequency [illegible] signal power [illegible] most applicable.

The most difficult problem arises when [illegible] interference [illegible] fast fading, multipath propagation, etc. [illegible] possible multiplexing system only in light of all channel characteristics and the dependences caused by them between errors in the reception of separate symbols. For example, in a channel with multipath propagation, time multiplexing should not be used inasmuch as in this case shortening the length of a signal element increases the inter-symbol interference and greatly reduces the fidelity of reception. If it is desirable in such a channel to use a separable multiplexed system, a preference for frequency and not time multiplexing should be expected in order not to reduce the length of an element. If in a channel single impulse interference predominates, it is desirable to use time multiplexing and not frequency or combination since with time multiplexing each short interfering impulse can distort one symbol in one message while with frequency or combination multiplexing it can cause errors in all transmitted messages.

It should be stressed, generally, that with any type of multiplexing, separable or combination, it is possible to use arbitrary signal shapes. Therefore, in such channels where the shape of the signal affects the resistance to interference (see Chapter XII) it should be matched with channel characteristics.

The magnitude 2F in (9.7) is often called the Nyquist carrying capacity of the channel. Its essence is as follows. According to the Kotelnikov theorem, a signal passing through a channel with a passband strictly limited to F is completely characterized by its values readout at intervals of time equal to $(2F)^{-1}$. If only binary signals are sent in such a channel, it is apparent that 2F characterizes the maximum possible rate of transmission. It can easily be seen that this rate does not change if use is made of frequency multiplexing after dividing band F into k equal parts since in each part the Nyquist carrying capacity will be equal to 2F/k. It is possible to exceed the Nyquist rate by refraining from binary coding, specifically by using a combination multiplexing system.

Until recently the opinion was widely held that multiplexing should be applied only to channels possessing an excess Nyquist carrying capacity. Therefore, the excess in the signal-to-noise ratio usually remained unused. Only [illegible] than the Nyquist rate.

As we [illegible] multiplexing, and [illegible]

Notes

[illegible] multiplexing [illegible] Russian language.

We will note that [illegible] multiplexing which [illegible] sharing of signal [illegible] quency multiplexing are often [illegible] systems [illegible] which pertains to the "[illegible]" [illegible]

In the [illegible] can be used for [illegible] the method of [illegible] pseudo-[illegible] systems [illegible] of information [illegible] tant for channels using multi-beam propagation [illegible] method can be used when the [illegible] The power of a signal [illegible] due to the pro[illegible] signal with a large [illegible] obtaining 2 [illegible] stance to interference, [illegible]

2. [illegible] are asynchronous systems which [illegible] systems which have [illegible] known advantages. The first [illegible] different from the [illegible] systems described with the exception of the [illegible] in the different messages are not related. [illegible] orthogonal and it is inevitable that in the [illegible]

interference which can be combatted only by using large protective intervals in frequency, i.e., by achieving approximate orthogonality at the price of poor use of the frequency band.

In the second group are included principally asynchronous address systems [17]. For the most part they are used for the transmission of continuous messages but in principle they can be used successfully for multiplexing a channel with discrete messages. Asynchronous address systems amount to divisible systems in which a message transmitted to a certain addressee (recipient) is characterized by its own signal realization. Receivers of each addressee are matched in a certain sense with the addressee's realization of the signal and, to the extent possible, do not react to other realizations.

If such a system were synchronous, it would be possible to make all realizations orthogonal and to completely eliminate transient interference. The point in using an asynchronous system is that each source can input its message into the channel regardless of other sources. In this process the sources, just as the recipients, can be physically located in different places and can use in radio communication the same frequency band or the same radio relay (for example, one located on an artificial earth satellite).

In principle it would be possible to construct such an asynchronous system on the basis of frequency multiplexing by assigning a frequency band to each addressee, to which the addressee's receiver would be tuned. Then each source would have to tune its transmitter to the band assigned to its addressee. In essence radio communication in the shortwave range, which is not usually regarded as a multiplexing system but more as an address system, is organized in this way. The formation of signal realizations using a so-called frequency time matrix is a peculiarity of systems which have been named discrete-address systems. Each signal realization amounts to a sequence of several radio impulses with a different frequency filler. The addresses are distinguished as intervals of time between impulses as well as by the frequencies of the filters. This permits a very simple device for selecting an addressee. Reception is accomplished using a nonlinear device containing a delay line and a coincidence circuit and reacting only to a certain sequence of impulses. If only a few sources operate at the same time, each receiver receives only the signals addressed to it. Crosstalk interference, which is called occurrence of "false addresses" occurs when many sources transmit because of the random combination of impulses which are transmitted by the different sources.

Such a system permits organizing radio communication with the same convenience for correspondents as is usually provided for in automatic dialing in long-distance telephonic communication.

5. (See Section 9.2) In a multiplexed communication channel the total number of messages for which a channel is designed is far from always used. This is especially characteristic of channels which are multiplexed by continuous telephonic messages when a channel is usually underused by 60-70% or more. However, this also occurs in certain measure when discrete messages are multiplexed. Therefore, it seems enticing to increase the multiplexing factor so as

to transfer information at an average rate reasonably close to the channel carrying capacity. In this case it is possible to find the number of messages transmitted simultaneously exceeding a certain magnitude k_0, and the total productivity of the sources becomes greater than the carrying capacity. In these cases a sharp drop in fidelity is inevitable but if the probability that the number of simultaneously transmitted messages exceeds k_0 is very small, this can be handled.

In the usual multiplexed systems channel carrying capacity is limited not by the average but by the maximum multiplexing factor. Thus, if during frequency multiplexing a certain part of the channel passband is set aside for the i-th message, it becomes possible to transmit an additional message because at a particular moment a certain message is not transmitted. This situation occurs in the case of time or combination multiplexing. However, it is possible to construct statistical multiplexing systems in which an interval in the transmission of one message permits transmitting another.

An example of a statistical multiplexing system is provided by the asynchronous system described in the preceding note. Another example is a separable systems in which each realization of an individual signal occupies the entire frequency band of the channel F and the entire time segment T set aside for transmission of a symbol. When the multiplexing factor is large such signals may be noise-like. If they are made orthogonal their total number, as shown in Section 9.5 will not exceed 2FT and, consequently, they can be used in the best case for transmitting 2FT binary messages (if keying in each message is done by shift or by using a passive interval system). But it is possible to select these signals randomly so that they will be orthogonal and increase their number. Then it is possible to increase the multiplexing factor to a magnitude greater than 2FT, but in this case crosstalk interference acting as a certain addition to fluctuation interference is inevitable. With the power of the individual signals the same, the strength of the crosstalk interference is equal to $(n - 1)P_s$ where P_s is the strength of one individual signal and n is the number of messages transmitted at a given instant. The ratio between the power of the signal and the spectral density of the crosstalk interference will be equal to

$$h^2 = \frac{P_s T}{(n-1)P_s/F} = \frac{FT}{n-1}$$

With a rather large FT product this crosstalk interference may be small and decrease the probability almost not at all as long as n does not exceed a certain magnitude as was shown in Chapter XII. Therefore, the maximum number of transmitted messages in such a system can be such that the probability that n will exceed a permissible magnitude will be sufficiently small.

This same idea of statistical multiplexing is used in asynchronous address systems which were mentioned in Note 2. The nonlinear method of selection (in a coincidence circuit) provides in them, for all practical purposes, for an absence of crosstalk interference when the number of simultaneously operating transmitters remains below a certain permissible level. With an increase in this number the crosstalk interference rapidly increases and the system becomes inoperable.

This method of statistical [illegible] considers [illegible] continuous [illegible]

[illegible] has been remarked [illegible] depends on [illegible] between signal and interference. Therefore, [illegible] Hence, [illegible] for an MRPI system, [illegible] a given passband, the rated speed of transmission [illegible] be [illegible] used.

Thus, with the usual [illegible] probability of error on the order of [illegible] or less is [illegible], the passband in front of the decision circuit must be about 2.1F [illegible] 2 about 3F [illegible], and when [illegible] 3.5 [illegible], etc.

In this respect mixed systems of the "[illegible]" type in which the sum of m signals is transmitted has the advantage: with a sufficiently large m the signal base is great and the required channel passband not much greater than [illegible], even if the MRPI multiplexing factor reaches [illegible] 3.

BIBLIOGRAPHY

1. Fano, R. M., "Trends of Development," in the book Lectures in the Theory of Communication Systems, edited by Ye. Zh. Bagdadi, "Mir" Press, 1964.

2. Ageyev, D. V., "Fundamentals in the Theory of Linear Selection," Nauchno-tekhnicheskiy sbornik LEIS, No. 10, 1935.

3. Kharkevich, A. A., "Essays in General Communication Theory," Gostekhizdat, 1955.

4. Mosier, R. R. and R. G. Clabaugh, "Kineplex, a Band with Efficient Binary Transmission System," Commun. and Electron, No. 34, 1958.

5. Fink, L. M. and V. S. Kotov, "Two Methods of Receiving DFT Signals," Radiotekhnika, No. 2, 1964.

6. [illegible] V., "[illegible] Resistance to Interference [illegible]," [illegible]

7. [illegible] "[illegible] Interference and Efficiency [illegible] Systems," [illegible]

8. [illegible] "[illegible] Multiple Phase [illegible]," [illegible] 196-.

9. [illegible] [illegible] Effects [illegible] Press, [illegible]

10. [illegible] [Theory of Random Processes and Its Application], [illegible], Sovetskoye Radio Press, 19[illegible].

11. Khvorostenko, N. P., [illegible] [Statistical Theory of Demodulation of Discrete Signals], Svyaz' Press, 1968.

12. [illegible] "[illegible] Impulses [illegible] Signal with Random Phase," [illegible]

13. Khvorostenko, N. P., "Noise Resistance [illegible] Discrete Information Receivers," Elektrosvyaz', No. 7, 1965.

14. Khvorostenko, N. P., Author's Certificate No. 158114, Class 21 A1, 1963.

15. Andronov, I. S., "Resistance to Interference of [illegible] Binary Channel [illegible]," Elektrosvyaz', No. [illegible], 1967.

16. Khvorostenko, N. P., "Resistance to Interference of One Method Receiving Fading Signals with Multiple PSM," Elektrosvyaz', No. 8, 1967.

17. Lydus, [illegible], V. I. Markov and V. D. Venediktov, "Asynchronous Address Communication Systems," Problemy peredachi informatsii, No. 1, 1965.

18. Levin, [illegible] A., [illegible] K. Levin, V. I. Aizenbert and V. S. Rozanov, "Increasing the Effectiveness of Multichannel Systems with Free Separation of Channels," Elektrosvyaz', No. 5, 1960.

19. Khvorostenko, N. P., "Interference Resistance of Multiple Phase Telegraphy," Radiotekhnika, No. 12, 1964.

CHAPTER X

RECEPTION OF REDUNDANCY-CODED MESSAGES

10.1. Element-by-Element Reception and "Entirety" Reception

The basic problems involved in the coding of messages have been examined in Chapters I and II where it was shown that the task of discriminating incoming signals may be settled with one of two types of reception. In the first case the signal segment corresponding to the elementary message "letter" is analyzed as a whole and on the basis of this analysis a decision is made as to what letter was transmitted ("entirety" reception) [1, [illegible]]. In the second case the individual signal elements are first analyzed which correspond to the code symbols and a decision is made by means of the first decision system as to what symbols were transmitted, and then the thus restored code combination is decoded, i.e., converted into a "letter" of the message (element-by-element reception). If redundancy coding is used, the code symbol sequence which is restored element by element forms an impermissible code combination.

In this case the decision as to the transmitted message letter is made with the aid of the second decision (decoding) system which identifies the received symbol sequence with the nearest (in the Hamming sense) permissible code combination (error-correcting method of reception), or only the permissible combinations are decoded, while all impermissible are rejected as incorrectly received (error-detecting method of reception, usually employed in systems with feedback connection).

The preceding chapters have dealt chiefly with element-by-element reception. Chapters III through IX are devoted to problems involving the first decision system; the second decision system was examined mainly in Chapter II. Nearly all the existing receiving units in actual use are based on element-by-element reception, as they are considerably simpler to realize than those based on entirety reception. Entirety reception, as was mentioned in Chapter I, has no advantages over element-by-element reception in non-redundancy encoding.[1] These facts explain the attention which is ordinary devoted to element-by-element reception.

Element-by-element reception, however, cannot be the optimum reception method if redundancy is used in encoding, even if the first and second decision systems are optimal (e.g., in the sense of the ideal observer). This can easily be understood from the following discussion.

---

[1] This result will be proved in passing in Section 10.5.

When analyzing signal element z'(t) the first decision system, based on the ideal observer criterion, determines the a posteriori probabilities of each code symbol x' and selects the symbol x'_i which has the greatest a posteriori probability. After this symbol has been selected the parameters of signal element z'(t), just like the computed a posteriori probabilities of the x' symbols, do not participate in the further reception process and only a completed but not final decision as to symbol x' is inserted into the second decision system. The operation of regenerating the code symbol sequence which is performed by the first decision system is irreversible and may be accompanied by loss of information. If the regenerated code symbol sequence forms a permitted code combination, the identification with the message letter is accomplished at the second decision system which does not have at its disposal all the information in the received z'(t) signal, but only that contained in symbol sequence x'. Only in a non-redundancy coding where all code symbol sequences are permissible and a priori equiprobable does the sequence of the most probable x' symbols (in the a posteriori sense) form the most probable code combination, and hence no loss of information in element-by-element reception occurs.

In entirety reception all the information contained in the received signal z'(t) relative to the transmitted message may be used by a combination decision system. Therefore with redundancy coding we may expect greater reliability from entirety reception than from element-by-element reception.

As has already been mentioned, the optimum decision system for entirety reception is complicated. Only in isolated cases can a relatively simple decision system be constructed [3]. Therefore the problem of designing decision systems which, although not optimal for entirety reception (i.e., do not use all the information contained in incoming signal z'(t)) still permit fewer errors than do element-by-element reception systems is of great interest. These systems, like the element-by-element systems, are designed on the two-stage principle, i.e., they first transform the signal z'(t) into a code symbol sequence which is then decoded. But in contrast to the usual systems of element-by-element reception these entirety reception systems keep the information about the values of the a posteriori probabilities of the regenerated symbols and use it in the decoding process. Examples of reception methods based on such systems are Wagner's method [4] and the method of reception based on the most reliable symbols [5, 6].

This chapter will examine some entirety reception methods with various channel characteristics, as well as the reception methods mentioned which occupy a middle ground between element-by-element and entirety reception. In addition, certain conditions for the use of redundancy codes with given channel features will be deduced.

So as not to complicate matters more, this chapter will study only the case of binary codes.

10.2. Entirety Reception with Fully Known Signal and Fluctuation Noise

In the idealized binary channel with constant parameters and fluctuation noise where all permissible signals are precisely known and the optimal reception is essentially no different from that of element-by-element reception. In fact, if the extent of the alphabet is N and some code combination consisting of n binary symbols corresponds to each of its letters, we may regard the signal element sequence corresponding to each code combination as a larger "element" of a new code of base N. Therefore all the results bearing on coherent reception of signals with code base m = 2 which were derived in Chapter III may be completely related to entirety reception with replacement of code base m by N. In particular, the optimal decision principle of expression (3.24) remains true, as well as the decision system of Figure 3.2, in which the local signal generators are to be understood to mean sources reproducing the z'(t) signals corresponding to entire code combinations.

It is easy to see that with equal power in the signal elements corresponding to each binary symbol the signals matching the different combinations of a uniform code also have the same power. Therefore the characteristics of an active-interval system are also kept in entirety reception, and this permits the decision principle of expression (3.28) and the corresponding decision system in Figure 3.3 to be used.

The practical utilization of these decision systems is rendered difficult chiefly by the fact that they must contain hard-to-realize sources which accurately imitate (including the initial phase) the signals corresponding to all the permissible code combinations. The decision system may, however, be substantially simplified by keeping in it only the source of a single continuous signal (or the filter matched to this signal) and N discrete sources emitting the permitted code combinations, e.g., as direct current pulses. A discrete source may be easily made of shift registers or by other simple means.

In order to provide a basis for this system, let us scrutinize the decision principle of expression (3.28) which may be written in the following way.

The decision system must register letter x_r if when all $q \neq r$ (q = 1,..., N)

$$\int_0^{nT} z_r(t)\, z'(t)\, dt > \int_0^{nT} z_q(t)\, z'(t)\, dt, \tag{10.1}$$

where $z_q(t)$ (q = 1,..., N) is the signal corresponding to the code combination representing message letter x_q and $z'(t)$ is the incoming signal (plus noise).

Every signal $z_q(t)$ may be represented in the form

$$z_q(t) = \begin{cases} z_q^{(1)}(t) \text{ when } 0 \leq t \leq T, \\ z_q^{(2)}(t) \text{ when } T < t \leq 2T, \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ z_q^{(i)}(t) \text{ when } (i-1)T < t \leq iT, \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ z_q^{(n)}(t) \text{ when } (n-1)T < t \leq nT, \end{cases} \tag{10.2}$$

where $z_q^{(i)}(t)$ (q = 1,..., N; i = 1,...,n) may represent one of two functions, either $z_0(t)$ matching symbol y = 0 or $z_1(t)$ matching symbol y = 1.

The condition from receiving letter x_r may now be written as

$$\begin{aligned}\int_0^T z_r^{(1)}(t)z'(t)dt &+ \int_T^{2T} z_r^{(2)}(t)z'(t)dt + \dots \\ &+ \int_{(n-1)T}^{nT} z_r^{(n)}(t)z'(t)dt > \int_0^T z_q^{(1)}(t)z'(t)dt + \\ &+ \int_T^{2T} z_q^{(2)}(t)z'(t)dt + \dots + \int_{(n-1)T}^{nT} z_q^{(n)}(t)z'(t)dt\end{aligned} \tag{10.3}$$

for all values of $q \neq r$.

We will introduce the notation $\bar{z}_q^{(i)}(t)$, to mean a value inverse to $z_q^{(i)}(t)$, i.e.,

$$\left.\begin{aligned}\bar{z}_q^{(i)} &= z_0(t), \quad \text{if } z_q^{(i)}(t) = z_1(t), \\ \bar{z}_q^{(i)}(t) &= z_1(t), \quad \text{if } z_q^{(i)}(t) = z_0(t).\end{aligned}\right\} \tag{10.4}$$

We easily satisfy ourselves that inequality (10.3) is equivalent to inequality

$$\begin{aligned}\int_0^T \bar{z}_r^{(1)}(t)z'(t)dt &+ \int_T^{2T} \bar{z}_r^{(2)}(t)z'(t)dt + \dots \\ &+ \int_{(n-1)T}^{nT} \bar{z}_r^{(n)}(t)z'(t)dt < \int_0^T \bar{z}_q^{(1)}(t)z'(t)dt + \\ &+ \int_T^{2T} \bar{z}_q^{(2)}(t)z'(t)dt + \dots + \int_{(n-1)T}^{nT} \bar{z}_q^{(n)}(t)z'(t)dt\end{aligned} \tag{10.3a}$$

In fact, the terms of inequality (10.3) in which $z_r^{(i)}(t) = z_q^{(i)}$ are equal to each other and therefore may be cancelled. The remaining terms are not equal to each other, and since $z_q^{(i)}$ may assume only two values, for them $z_q^{(i)} = \bar{z}_r^{(i)}$ and $\bar{z}_r^{(i)} = z_q^{(i)}$. Hence, if on both sides of inequality (10.3a) the terms which are equal to each other are removed, the left side of inequality (10.3a) will agree with the right side of inequality (10.3), and conversely. Therefore, the signs of the inequality in expressions (10.3a) and (10.3) are opposite. The addition to both sides of expression (10.3a) of terms in which $z_r^{(i)}(t) = z_q^{(i)}(t)$ obviously does not change the inequality.

Subtracting expression (10.3a) from (10.3) we obtain an inequality which is equivalent to them

$$\begin{aligned}\sum_{i=1}^{n} \int_{(i-1)T}^{iT} z'(t)\left[z_r^{(i)}(t) - \bar{z}_r^{(i)}(t)\right]dt \\ > \sum_{i=1}^{n} \int_{(i-1)T}^{iT} z'(t)\left[z_q^{(i)}(t) - \bar{z}_q^{(i)}(t)\right]dt.\end{aligned} \tag{10.5}$$

The differences $z_q^{(i)}(t) - \overline{z_q^{(i)}(t)}$ may assume only two values: $z_1(t) - z_0(t)$ or $z_0(t) - z_1(t)$. Let us denote the first of these by $z_\Delta(t)$, then the second by $-z_\Delta(t)$.

Let us introduce the further notation:

$$\left.\begin{array}{lll} \beta_{iq} = 1 & \text{if} & z_q^{(i)}(t) = z_1(t), \\ \beta_{iq} = -1, & \text{if} & z_q^{(i)}(t) = z_0(t) \end{array}\right| \qquad (10.6)$$

Then $z_q^{(i)}(t) - \overline{z_q^{(i)}(t)} = \beta_{iq} z_\Delta(t)$ and inequality (10.5) may be rewritten as

$$\sum_{i=1}^{n} \beta_{ir} \int_{(i-1)T}^{iT} z'(t) z_\Delta(t)\,dt > \sum_{i=1}^{n} \beta_{iq} \int_{(i-1)T}^{iT} z'(t) z_\Delta(t)\,dt \qquad (10.7)$$

This representation of the decision principle is convenient in that both sides of the inequality contain the same integrals which for further generalizations may be conveniently denoted by

$$c_i = \int_{(i-1)T}^{iT} z'(t) z_\Delta(t)\,dt, \qquad (10.8)$$

and different among themselves only by the β coefficient.

In this notation inequality (10.7) assumes the following simple form:

$$\sum_{i=1}^{n} \beta_{ir} c_i > \sum_{i=1}^{n} \beta_{iq} c_i \qquad (10.8a)$$

Therefore the decision system (Figure 10.1) corresponding to the principle of expression (10.7), contains only the source of the periodically repeating signal $z_\Delta(t)$ of period T which is multiplied with the incoming signal $z'(t)$. Their product is integrated over intervals of T, as occurred in element-by-element reception. At moments divisible by T the values of c_i come from the output of the integrator and are fed in parallel to M multipliers[1], to each of which proceeds the sequence of discrete values β_{iq} $(q = 1,\ldots,M)$ stored in the memory unit. It is easily seen that every such sequence is nothing else by the q-th code combination, in which the "0" symbols are replaced by "-1". The products taken from these multipliers are integrated (summated) over time nT and go to the comparator, which chooses the largest of them and determines from it the received message letter.

The decision principles derived may also be applied to the case of the channel with variable parameters if the parameters change slowly as compared to the length of the code combination and may be predicted with sufficient accuracy.

---

[1] These multipliers function essentially as polarity switches.

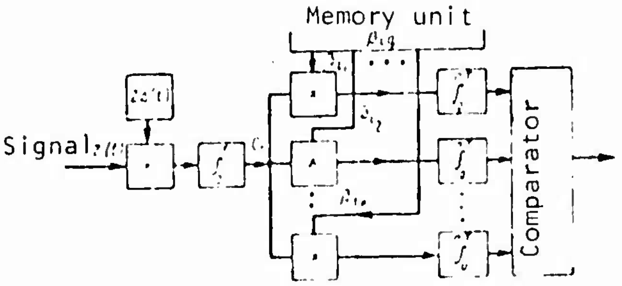


Figure 10.1. Decision System in Coherent Entirety Reception.

Calculation of the probability that inequality (10.7) will not be fulfilled, i.e., that a sign will be incorrectly received, encounters the same obstacles in entirety reception as were mentioned in Chapter III for the case of $m = 2$.

In some particular cases this probability may be expressed as integrals which can be numerically calculated. Further on we will give an evaluation of this probability which is not a very accurate one but allows for a comparison between entirety and element-by-element reception.

10.3. Incoherent Entirety Reception

If the initial phase of a transmitted signal is unknown, incoherent reception must be employed. Here two cases must be examined separately.

a) The initial phase of the signal corresponding to the code combination is random and unknown, but it is maintained during reception of the whole code combination. This case is no different from element-by-element incoherent reception, if "element" is understood to mean the whole signal corresponding to the code combination. The decision principle of (4.28) is obviously optimum for this case. For active-interval signals this principle is simplified and reduces to expression (4.30). The meaning of V_q must, of course, be

$$V_q = \frac{2a}{nT}\sqrt{\left[\int_0^{nT} x(t) z_q(t)\,dt\right]^2 + \left[\int_0^{nT} x(t)\tilde{z}_q(t)\,dt\right]^2}, \tag{10.9}$$

where $z_q(t)$ $(q = 1, \ldots, l)$ is the signal corresponding to the whole code combination of letters x_q and $z_q(t)$ is a function conjugate with $z_q(t)$.

Since the values of V_q are formed by adding the corresponding values for the elements of the incoming code combination with allowance made for the constancy of the initial phase, the reception method based on comparison of the values of (10.9) may be called the method of coherent cumulation. The decision system for the method of coherent cumulation is very complex, inasmuch as it must contain 21 generators of signals $z_q(t)$ and $z_q(t)$.

b) the initial phase of every element is random. This occurs, for example, in a channel with fading if the signal elements are diversified in time to effectuate decorrelation. In this case coherent cumulation is impossible. The optimum method of entirety reception under these conditions can be deduced by computing the a posteriori probabilities of each code combination.

This decision principle for an active-interval system with white noise proves to be the following: the sign of x_r must be registered if for all $q \neq r$ ($q = 1, \ldots, m$)

$$\sum_{i=1}^{n} \ln I_0 \left[\frac{2}{N_0} \sqrt{\left(\int_{(i-1)T}^{iT} z_r^{(i)}(t) x(t)\, dt \right)^2 + \left(\int_{(i-1)T}^{iT} \tilde{z}_r^{(i)}(t) x(t)\, dt \right)^2} \right] >$$

$$> \sum_{i=1}^{n} \ln I_0 \left[\frac{2}{N_0} \sqrt{\left(\int_{(i-1)T}^{iT} z_q^{(i)}(t) x(t)\, dt \right)^2 + \left(\int_{(i-1)T}^{iT} \tilde{z}_q^{(i)}(t) x(t)\, dt \right)^2} \right], \quad (10.10)$$

where $z_q^{(i)}(t)$ ($i = 1, \ldots, n$; $q = 1, \ldots, m$) is, as before, the signal element corresponding to the i-th symbol of the q-th code combination (with accuracy to an arbitrary initial phase), $\tilde{z}_q^{(i)}(t)$ is a function conjugate with $z_q^{(i)}(t)$, and N_0 is the spectral white noise density.

The decision principle derived may be called the principle of incoherent cumulation, inasmuch as the values obtained from [illegible] the individual elements are added without regard to the phase relationships between them.

The functions $z_q(t)$ [illegible] may represent signals [illegible] use the relation

[illegible] (10.11)

Further, let [illegible] when [illegible] and [illegible] when $z_q^{(i)}(t) = z_0(t)$. Then inequality (10.10) can be written as

$$\sum_{i=1}^{n} \ln I_0 \left[\frac{2\lambda}{N} (x_i a_i + y_i b_i) \right] \\ \sum_{i=1}^{n} \ln I_0 \left[\frac{2\lambda}{N} (x_i a_i + y_i b_i) \right] \qquad (10.12)$$

Figure 10.2 shows the decision principle designed from this principle. The values of a_i and b_i are obtained as the envelopes of the voltages at the output of filters matched to $s_1(t)$ and $s_0(t)$ (as in the system of Figure 3. [illegible]). Following detectors with characteristics of ln I_0 these values go to switching units controlled by the memory unit in which are fixed the discrete sequences x_{iq} which form the permissible code combination. The summators form the sums which appear in expression (10.12) and they are compared with each other at moment nT, with the largest value determining the received sign.

Figure 10.2. [illegible]
Key: 1, Matched filter; 2, Detector; 3, Memory unit [illegible];
4, Summator; 5, Comparator.

[illegible]

[illegible]

After the indicated replacement and the obvious simplifications the principle for registering letter x_k reduces to the following:

$$\sum_{i=1}^{n} (z_{1i}a_{ik} + z_{2i}b_{ik})^2 > \sum_{i=1}^{n} (z_{1i}a_{iq} + z_{2i}b_{iq})^2 \qquad (10.13)$$

Multiplying out and taking into account that by definition $a_{iq}^2 = a_{iq}$, $z_{iq}^2 = \bar{a}_{iq}$ and $a_{iq}\bar{z}_{iq} = 0$, we derive the inequality

$$\sum_{i=1}^{n} (z_{1i}^2 a_{ik} + z_{2i}^2 b_{ik}) > \sum_{i=1}^{n} (z_{1i}^2 a_{iq} + z_{2i}^2 b_{iq}) \qquad (10.14)$$

It is easily seen that the equivalent of this inequality is the inequality

$$\sum_{i=1}^{n} (z_{1i}^2 a_{iq} + z_{2i}^2 b_{iq}) < \sum_{i=1}^{n} (z_{1i}^2 a_{ik} + z_{2i}^2 b_{ik}) \qquad (10.14a)$$

Actually, only the terms for which $a_{ik} \neq a_{iq}$ are essential for every value of i in inequalities (10.14) and (10.14a). But in this case $a_{ik} = \bar{a}_{iq}$ and $b_{ik} = \bar{b}_{iq}$, therefore the conversion from inequality (10.14) to (10.14a) reduces to a transfer of all not identically equal terms from the left side to the right, and conversely. The result is that the inequality sign is reversed.

Let us subtract inequality (10.14a) from (10.14), meanwhile returning to the notation [illegible]

$$\sum_{i=1}^{n} [illegible] > \sum_{i=1}^{n} [illegible]$$

It is easily ascertained [illegible] Figure 10.3 shows the [illegible] expression [illegible]

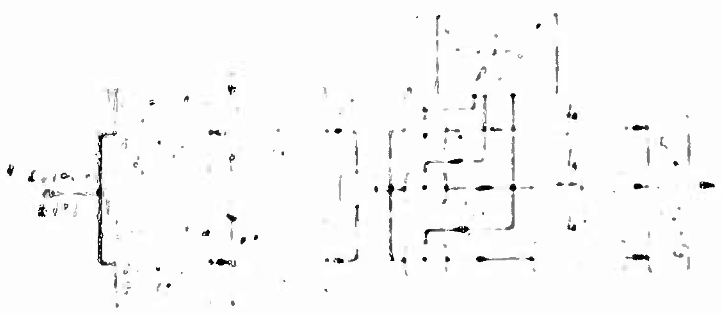


Figure 10.3. Decision System in Quadratic Incoherent Storage. Key: 1, Matched filter. 2, Quadratic detector. 3, Memory unit. 4, Summator. 5, Comparator.

If, finally we designate $a_i^2 + b_i^2 = c_i$ the registration principle of letter x_k reduces to the inequality

$$\sum_{i=1}^{n} \beta_{ki} c_i > \sum_{i=1}^{n} \beta_{si} c_i \qquad (10.15a)$$

which completely agrees in form with expression (10.8a). It should therefore be taken into consideration that c_i in expression (10.8a) and (10.15a) represent different quantities. In expression (10.8a) the value of c_i is determined by expression (10.8) and may be called the result of coherent differential detection of the i-th signal element. In expression (10.15a) the quantity c_i represents the result of incoherent (quadratic) differential detection of the i-th element. Nevertheless, the identical form of the decision principles allows comparison of entirety reception with element by element reception without taking any distinction between the coherent and the incoherent case and even without taking the channel characteristics into consideration. This comparison is made easier by the fact that the values of expressions (10.8a) and (10.15a) determine, respectively, the result of element by element coherent or incoherent reception of binary signals. It is easily ascertained that each time that [illegible] [illegible]:

$$\beta_{ki} [illegible] \qquad (10.1[illegible])$$

occurs during transmission of message x_k, element by element reception leads to [illegible] registration of the [illegible] which cannot be detected or corrected with a redundant code.

10.4. Evaluating Noise-Resistance in Entirety Reception

Let us deduce certain general relationships in order to compare the noise resistance of element by element reception and entirety reception. Let the result of demodulation of the i-th element of the code combination be some value [illegible]. In the first decision system in element by element reception each of the values [illegible] is replaced by the symbol [illegible], with the result that a certain code combination is produced [illegible] permissible combinations [illegible] is directly converted into the corresponding message letter. In the second case, depending on the design of the second decision system, there occurs either error detection with subsequent automatic interrogation or simple notation of the fact of the error, or "correction" of the error, i.e., identification of the received code combination with the nearest (in the Hamming sense) permissible combination. We will thus divide the methods of element by element reception into reception with detection and reception with correction of errors. Both methods are possible in any redundancy code.

In [illegible] reception the values of [illegible] are calculated by the [illegible] [illegible] 1, . . . , M; q = 1, . . . , [illegible] and the received signal is identified with the r-th letter of the message alphabet if

$$\sum [illegible] \quad \sum [illegible]$$

for all q ≠ r.

The product [illegible] is a random variable [illegible] mathematical expectation [illegible] alphabet is transmitted [illegible] relatively small [illegible] [illegible] will be [illegible] with large [illegible] of this value [illegible] the probability density [illegible]

[illegible]

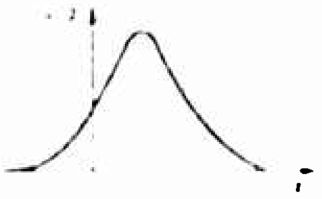


Figure 12-4. [illegible]

[illegible]

$$\sum [illegible] \quad \sum [illegible]$$

We will [illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

c_{ir} and c_{iq} do not agree) are negative. It is apparent that with these values of i the values of $c_{iq}c_i$ will be positive. Let us designate this event by A_2, and its probability by p_2.

In element-by-element reception with error-detection (without correction thereof) an undetected error will occur if each of the d products of $c_{ir}c_i$ corresponding to those i for which $c_{ir} \neq c_{iq}$ is negative. Let this event have probability p_1.

It is obvious that if event A_3 occurs, then events A_2 and A_1 always occur, [illegible]. If, however, event A_2 occurs, then event A_1 always occurs. Hence

$$p_3 \le p_2 \le p_1 \qquad (10.19)$$

Equality [illegible] occur only when [illegible], since in that, and only in that, case the events A_1, A_2 and A_3 coincide.

[illegible]

$$\sum [illegible] \sum [illegible]$$

[illegible]

[illegible]

$$\sum [illegible] \sum [illegible]$$

$$\sum [illegible]$$

[illegible]

[illegible] A_2 [illegible]

$$p \ [illegible] \ p$$

where equality takes place only when [illegible] A_2 [illegible]

[illegible] events A and A [illegible] occur simultaneously.

but one of these events may also occur without the other. Figure 10.5 gives a schematic representation of the relations between events A_1, A_2, A_3, and A_4.

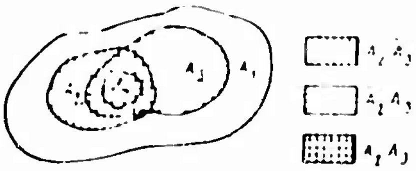


Figure 10.5. Relations Between Events A_1, A_2, A_3, and A_4.

Let us denote by $\bar{A}$ an event which is opposed to event A. The probabilities p_2 and p_3 may be presented as

$$p_2 = P(A_2A_3) + P(A_2\bar{A}_3),$$

$$p_3 = P(A_2A_3) + P(\bar{A}_2A_3) \qquad (10.23)$$

In order to prove that $p_2 \geq p_3$ it suffices to prove that

$$P(A_2\bar{A}_3) \geq P(\bar{A}_2A_3) \qquad (10.24)$$

[illegible] realization of the values [illegible] in which A_2 occurs and A_3 does not occur. This means that of the examined [illegible] half [illegible] are positive and at the same time

$$\sum [illegible] \qquad [illegible]$$

[illegible]

[illegible] are negative [illegible]

[illegible] means that A_2 does not occur.

[illegible] in which A_3 is fulfilled and A_2 is not fulfilled there corresponds a [illegible] more probable realization wherein A_2 is fulfilled and A_3 is not fulfilled. From this follows (10.24) and consequently

$$p_2 \geq p_3 \qquad (10.25)$$

The equal sign is valid when [illegible] where

$$P(A_2, \bar{A}_3) = P(\bar{A}_2, A_3) = 0 \quad \text{and} \quad p_2 \; p_3 \; [illegible]$$

Combining expressions (10.19), (10.22), and (10.25) we derive the inequality

$$P_1 \le P_2 \le P_3 \le P_4,$$

where inequality occurs when distance d [illegible] 1 between the i-th and q-th combinations.

Since (10.26) is true for any pair of code combinations, then by averaging it with respect to all possible pairs we obtain (10.18):

$$P_1 \; [illegible] \; P_2 \; [illegible] \; P \; [illegible] \; P_4$$

where inequality [illegible] only in the case when [illegible] combinations, i.e., when the code has no redundancy, which was to be proved.

Knowing the [illegible] of the channel [illegible] it is always possible to calculate the probabilities p_1 and p_2 and obtain [illegible] for p_4. However, the interval between p_1 and p_2 [illegible] calculations are [illegible] we will present [illegible]

[illegible]

Let [illegible]

[illegible]

$$[illegible]$$

[illegible]

[illegible]

For small values of n this evaluation characterizes the probability of error in entirety reception rather well. Computation of p_d can usually be reduced to computation of the probability of error in the case of diversity reception.

10.5. Examples

We present two simple examples to compare entirety reception with element-by-element reception.

Example 1. Let us find the relationships of p_1, p_2, p_3, and p_4 to error probability p in element-by-element reception for code (3,1). This code is the simplest code to permit error correction in element-by-element reception. It contains combinations 000 and 111. We will assume that the probability of incorrect reception p of an element is known and that errors in element-by-element reception occur independently from each other, i.e., that the channel is uniform or that the errors are decorrelated by spacing the elements of the combination in time.

Under these conditions

$$\left.\begin{aligned} P_1 &= 1-(1-p)^3 = 3p-3p^2+p^3,\\ P_2 &= p^3+3p^2(1-p) = 3p^2-2p^3,\\ P_4 &= p^3.\end{aligned}\right\} \qquad (10.28)$$

Figure 10.6 shows the relations derived.

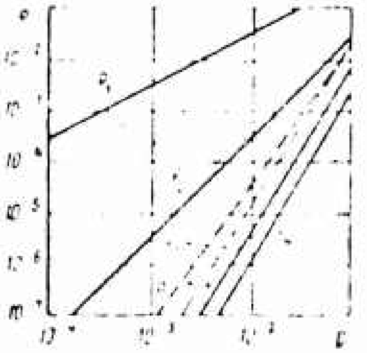


Figure 10.6. Dependence of Probabilities of Incorrect Code Combination Reception on Probability of Erroneous Element Reception for (3,1) Code.

The values of p_3 (the error probability in entirety reception) will be found separately for the case of coherent reception and two cases of incoherent reception. In coherent reception p_3 agrees with the probability of incorrect reception of an element of triple length. Therefore, in conformity with expression (3.61)

$$\left.\begin{aligned} p &= \frac{1}{2}[1-\Phi(\gamma h)], \\ \mathrm{P}_3 &= \frac{1}{2}[1-\Phi(\sqrt{3}\,\gamma h)]. \end{aligned}\right\} \qquad (10.29)$$

By regarding γh in these equalities as a parameter we may plot the dependence of P on p (curve in Figure 10.6).

For incoherent reception with coherent cumulation we assume the signals corresponding to 0 and 1 to be orthogonal in the intensified sense and obtain

$$p = \frac{1}{2}\exp\left(-\frac{h^2}{2}\right). \qquad (10.30)$$

Under this condition the full signals corresponding to the combinations 000 and 111 are also mutually orthogonal, but have triple length. Therefore

$$\mathrm{P}_3 = \frac{1}{2}\exp\left(-\frac{3}{2}h^2\right), \qquad (10.31)$$

whence

$$\mathrm{P}_3 = 4p^3 \qquad (10.32)$$

(curve b, Figure 10.6).

We will study the case of incoherent cumulation using the example of a channel with Rayleigh fading and assume that the code combination elements are adequately spaced for complete decorrelation. In this case P may be defined as the probability of error in triplex time-variant reception. From (6.58) we find

$$\mathrm{P}_3 = p^3[C_2^0 + C_3^1(1-p) + C_4^2(1-p)^2] = 10p^3 - 15p^4 + 6p^5 \qquad (10.33)$$

(curve c, Figure 10.6).

Inequality (10.18) is corroborated in all three cases, as the figure shows.

Example 2. Let us figure P_1 and P_4 and find an estimate for P_3 in Hamming code (6.5) which can detect any odd number of errors in a six-symbol combination. For this code $d_{min} = 2$ and $m = C_6^2 = 15$. In element-by-element reception, as is easily ascertained (assuming $p \ll 1$),

$$\begin{aligned} \mathrm{P}_1 &= 1-(1-p)^6 \approx 6p, \\ \mathrm{P}_4 &= 15p^2(1-p)^4 + 15p^4(1-p)^2 + p^6 \approx 15p^2, \end{aligned} \qquad (10.34)$$

where p is the error probability which in incoherent reception and with no fading is

$$p = \frac{1}{2}\exp\left(-\frac{h^2}{2}\right). \qquad (10.35)$$

Probability p_d in a coherent cumulation system may be defined as the probability of erroneous reception of an element of double length:

$$p_d \approx \frac{1}{2}\exp(-h^2) \qquad (10.35a)$$

Let us use (10.18) to bound P_3 from below according to which $P_3 \geq P_4$, and (10.27) to bound it from above. Then

$$P_3 \leq mp_d \approx 7.5\exp(-h^2). \qquad (10.36)$$

Figure 10.7 shows the dependence of P_1 and P_4 on h^2, figured by substituting expression (10.35) into expression (10.34), as well as the area of possible values of P_3 derived by means of the indicated estimate.

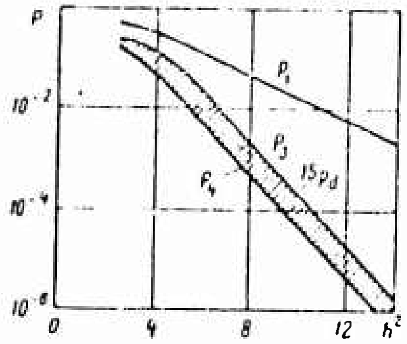


Figure 10.7. Probability of Incorrect Code Combination Reception with (6, 5) Code in Absence of Fading.

In slow Rayleigh fading P_1 and P_4 are figured by averaging expression (10.34) with respect to h. Here it is assumed that h hardly manages to change throughout the code combination reception. In the case where h_0^2 (the mathematical expectation of the value of h^2) is large enough the following approximations are defined:

$$P_1 \approx \frac{3.52}{h_0^2}; \quad P_4 \approx \frac{1.06}{h_0^2}. \qquad (10.37)$$

In the given case P_1 and P_4 are so close to each other that they give an adequately exact estimate of P_3 in confirmity with expression (10.18).

If measures are taken in a channel with Rayleigh fading to decorrelate errors, and the h values for the signal elements may be considered independent, then in accord with expression (5.17a)

$$p \approx \frac{1}{h_0^2+2}. \qquad (10.38)$$

In this case entirety reception may be accomplished by the incoherent cumulation method and the value of p_d is defined as the probability of error in duplex reception (see (6.37)):

$$P_4 = \frac{3h_0^2+4}{(h_0+2)^4} \quad (10.59)$$

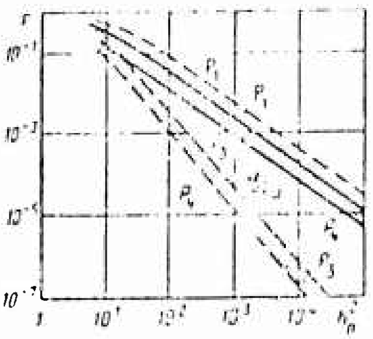


Figure 10.8. Probability of Incorrect Code Combination Reception with (6,5) Code in Rayleigh Fading: ————, Slow fading without decorrelation; -------, Fading with error decorrelation.

By substituting expression (10.58) in expression (10.51) we determine the relationships of P_1 and P_4 to h. To bound P_5 from above we must substitute expression (10.59) in expression (10.27). Figure 10.8 shows the results for a channel with Rayleigh fading. In analyzing them, it should be mentioned that error decorrelation is a necessary condition for effectively using the redundancy of a given code in both element-by-element and entirety reception.

Evaluations of the reliability of entirety reception for other codes may also be derived in similar fashion.

10.6. Reception Based on the Most Reliable Symbols and Wagner Method

The method of reception based on the most reliable symbols proposed by Borodin [6] occupies an intermediate position between the element-by-element and entirety methods of reception. Its basis is that in any code combination (d_{min}-1) symbols, and in some cases even more, may be erased and the transmitted letter recognized (decoded) from the remaining symbols. Inasmuch as any pair of code combinations has at least d_{min} locations where symbols do not agree, no less than d_{min} symbols must be erased to make this pair indistinguishable.

Let the first decision system, the same one as in element-by-element reception, determine the a posteriori probabilities of the symbols and make the preliminary decision that a symbol having the greatest a posteriori probability has been transmitted. The regenerated code combination derived in this way goes to the second decision system, but, in contrast to element-by-element reception, information about the a posteriori probability of every regenerated symbol is also fed to the second decision system (Figure 10.9). During decoding it is just the symbols which have the greatest a posteriori probabilities ("most reliable") that are taken into consideration in the number which is needed to distinguish one permissible combination from another. This number does not exceed $n-d_{min}+1$.

This method of reception must provide higher reliability than element-by-element reception because it uses information about the a posteriori probabilities of the regenerated symbols which is lost in element-by-element reception. Nevertheless, it must be in principle be inferior to entirety reception in fidelity because the information about the least reliability symbols is here completely lost. It may be said that reception with respect to the most reliable symbols is to entirety reception as diversity reception using a selection method is to diversity reception according to an optimum addition method.

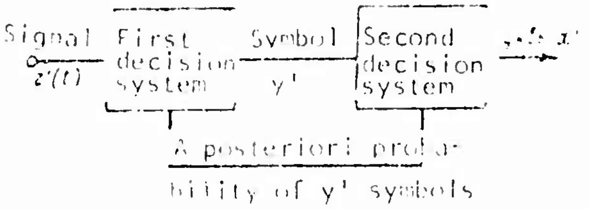


Figure 10.9. System of Reception with Respect to Most Reliable Symbols.

In the case of fluctuation noise it is easy to show that the a posteriori probability of a regenerated symbol is a monotonic function of c_i. Therefore, the most reliable symbols are those which have the higiest matching c_i values.

Let us pause on the particular case of codes of type (n, n-1) which differ in that n-1 symbols in the combination are informational and the n-th is a check symbol which is determined by checking for parity. The previous section examines a (6, 5) code which is an example of an (n, n-1) code. The least Hamming distance between any two combinations for such codes is $d_{min} = 2$.

Let us assume that a certain combination is being transmitted which corresponds to the letter x_r. An undetected error will occur in element-by-element reception if two elements are incorrectly received. Let them be the i-th and j-th element in order. In this case $\mu_{ir}c_i < 0$ and $\mu_{jr}c_j < 0$ and entirety reception will also result in an error since the code in question contains a combination corresponding to some letter x_q which differs from the transmitted combination only in the i-th and j-th symbols and for which $\mu_{iq}c_i > 0$ and $\mu_{jq}c_j > 0$. An error will also occur in reception with respect to the most reliable symbols since when $d_{min} = 2$ only one symbol may be erased so that at least one of the incorrectly received signals will be taken into consideration in decoding.

Now let $\mu_{jr}c_j < 0$ for only the j-th symbol while the others in element-by-element reception are received correctly. The result of element-by-element reception will be a detected error, but reception according to most reliable symbols will give a correct result only in case the erroneously received symbol is least reliable, i.e.,

$$|c_j| < |c_i| \quad (i = 1, \ldots, n,\ i \neq j). \tag{10.40}$$

It is easy to show that only when (10.40) is fulfilled does entirety reception also provide correct decoding. In fact, in that case

$$\beta_{jr}c_j + \beta_{ir}c_i > 0$$

for any i ≠ j, while $\beta_{ir}c_i > 0$ and $|\beta_{ir}c_i| > |\beta_{jr}c_j|$. If, however, expression (10.40) is not fulfilled, then there is some k-th symbol for which $|c_k| < |c_i|$ at all values of i = k. Then

$$\beta_{jr}c_j + \beta_{kr}c_k < 0,$$

since $\beta_{jr} < 0$ and $|\beta_{jr}| > |\beta_{kr}c_k|$. Hence, there is a code combination which corresponds to some letter x_q and differs from the transmitted combination only in the i-th and k-th symbols and for which

$$\beta_{jq}c_j + \beta_{kq}c_k > 0.$$

This reasoning shows that in the case of fluctuation noise and code (n, n-1) entirety reception and reception with respect to most reliable symbols are equivalent in fidelity. The same situation is also true for binary codes with a uniform weight. Work [3] gives proof of this. It would, however, be incorrect to generalize this conclusion to other codes or even to the class of codes whose $d_{min} = 2$.

Another receiving method which occupies middle ground between element-by-element and entirety reception is called the Wagner method [4]. This method is designed only for binary codes with even d_{min}. This type of code in element-by-element reception allows the correction of $[(d_{min}/2)-1]$ errors and, in addition, error detection if the number of erroneously received elements is $d_{min}/2$.

The Wagner system, like that in Figure 10.9, feeds the sequence of regenerated symbols to the second decision system, as well as the information about their a posteriori probabilities. This information, however is used only in case element-by-element decoding by the means of parity-checking indicates $d_{min}/2$ errors. In this case the least reliable symbol is changed to its opposite and, if it actually was incorrect, the number of errors is reduced to $(d_{min}/2)-1$. These other mistakes may be eliminated by parity-checks.

From the above it is evident that Wagner's method makes less use of the information about a posteriori probabilities than does Borodin's method (reception based on the most reliable symbols). Borodin's method allows the correction errors no more than $d_{min}-1$ in number on condition that the incorrectly received symbols have lower a posteriori probabilities than the correctly received symbols, whereas Wagner's method permits the correction of only $d_{min}/2$ errors if a symbol having the least a posteriori probability enters into the number of incorrectly received symbols.

In the particular case of (n, n-1) codes for which $d_{min} = 2$ the Wagner method is not essentially different from Borodin's method which, as we have seen, in this case provides the same reliability as does entirety reception. It should be noted that Borodin's method is applicable to codes of any base, whereas Wagner's method is designed only for binary codes.

10.7. Suboptimal Entirety Reception for Codes Permitting Majority Decoding

In element-by-element reception based on the criterion of maximal likelihood a symbol having the greatest likelihood function is determined from each value of c_i obtained as a result of demodulation. In optimal entirety reception likelihood functions are determined for all permissible code combinations for the entire aggregate of random values of c_i. This leads to a need to sort a large number of inequalities of the (10.8a) type and this leads to the complexity found in technological realization of entirety reception.

Let y_s ($s = 1,\ldots,k$) be the information symbol of a systematic (n, k) code. In element-by-element reception it is decided that $y_1 = 0$ if

$$w(c_1|u_1 = 0) > w(c_1|u_1 = 1) \tag{10.41}$$

In optimal entirety reception the decision to record letter x_r (i.e., the decision about the entire aggregate of information symbols in a combination), is made if

$$w(c_1,\ldots,c_n|x_r) > w(c_1,\ldots,c_n|x_q) \tag{10.42}$$

for all $q \neq r$.

Naturally thought arises about the possibility of constructing reception principles for which the likelihood function is determined for each information symbol separately but, in distinction from element-by-element reception, on the basis of an analysis of all values of c_i. In this process the decision that $y_1 = 0$ must be reached if

$$w(c_1,\ldots,c_n|u_1 = 0) > w(c_1,\ldots,c_n|u_1 = 1) \tag{10.43}$$

But the values of $c_1,\ldots,c_n$ depend not only on symbol y_i but on all remaining information symbols y_s of a transmitted code combination. Here only information symbols are considered inasmuch as the check symbols are determined by them unambiguously. If the a priori probabilities of information symbols are known,

[illegible] and [illegible]

[illegible] dependent [illegible] principle [illegible] the same [illegible] can be considered, [illegible] the equivalent probability of error

$$\sum$$

[illegible] probability of error [illegible]

[illegible] result for several codes has showed that with respect [illegible]

[illegible] this method [illegible] applicable [illegible] may [illegible] possible [illegible] described in Chapter II [illegible] cyclic codes, this method gives [illegible] information symbols are [illegible]

[illegible] follows that [illegible] distribution of the results [illegible]

[illegible] reception. The [illegible] method, and [illegible] experiment [illegible] according to [illegible] obtained in work [9]. The [illegible] over the length of a code [illegible]

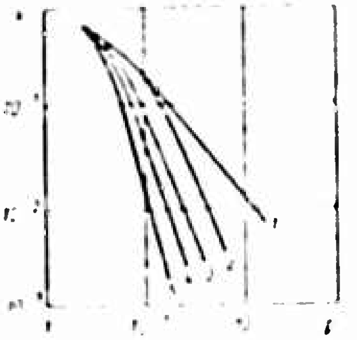


Figure 10.10. Comparison of Methods for Receiving Signals Using a (7, 3) Code. 1, Element-by-element reception with error correction, 2, Wagner method, 3, Borodin method, 4, Reception according to principle (1046), 5, Optimal entirety reception.

10.8. Conditions Under Which Use of Redundancy Codes is Advisable

The introduction of redundancy in message coding, if code base m is given, makes it necessary either to lengthen the time allocated to transmission of the given message or to shorten signal element length. In either case the message could be sent in the same time by means of a non-redundancy code and the length of the element correspondingly increased.

In most cases decreasing the element length leads to an increase in the probability of error. Thus, for example, in fluctuation noise this increase in error probability is caused by the decrease in signal power, and in lumped noise, also by expanding the frequency band occupied by the signal. Therefore, the probability of erroneous element reception is ordinarily greater with a redundancy code than with a code without redundancy. If this increase in error probability is not cancelled by the correcting capability of the code, the introduction of redundancy does not raise reception fidelity, but lowers it. The question here arises as to what conditions must be satisfied by a redundancy code for its use to enable reception fidelity to be increased as compared to non-redundancy encoding with the same message transmission period and the same signal power.

The answer to this question depends on the channel characteristics and the system of signals employed. The more sharply error probability rises as an element is shortened, the more difficult it is to cancel this rise in error probability by code redundancy. Thus, for example, it was shown in Chapter VII that in multipath propagation channels the reduction of element length to a value on the order of the beam path difference resulted in very abrupt increase in error probability. Under these conditions it is plainly inadvisable to employ a redundancy code and decrease element length. A more correct solution would be to use a code of large base m enabling element length to be increased with the same rate of information transmission.

In some cases the probability of error does not increase, or increases only very slightly, when the length of a signal element is shortened. In a channel with relatively rapid fading shortening the length of an element decreases its power but at the same time increases the correlation between values of the transmission factor for adjacent elements. Inasmuch as these two factors act in opposite directions, under these conditions shortening an element usually leads only to a small increase in the probability and sometimes (especially for RPT systems) even to a reduction. In the case of powerful but infrequent and brief impulse interference the probability of error depends practically not at all on the length of an element. When there is switching noise (brief interruptions in communication) which is characteristic of many cable channels, the probability of error increases with the rated transmission speed no faster than linearly (due to the increase in the number of elements falling in one interruption). In all these cases the use of correcting codes, as a rule, increases fidelity.

From now on in this section we shall limit ourselves to the case of binary codes in a channel with fluctuation noise.

Let us find the condition under which it is advisable to use a redundancy code in an active-interval system in coherent entirety reception. Let average signal power and time assigned to transmission of a message letter (code combination) T_c be given. We will compare reception conditions with a redundancy code, each code combination of which contains n elements, k of which are informational, and conditions with a non-redundancy code[1] in which the code combination contains k elements.

The noise-resistance of this system is adequately characterized by the minimum Kotelnikov distance between two code combinations

$$D = \int_0^T [z_r(t) - z_q(t)]^2\,dt \qquad (10.48)$$

where $z_r(t)$ and $z_q(t)$ are the signals corresponding to the two code combinations (see Chapter III).

In the case of the non-redundancy code the pair of nearest code combinations differs only in one element. Therefore throughout one element the integrand in expression (10.48) is different from zero and equals $[z_0(t) - z_1(t)]^2$. Since the element length in the non-redundancy code is T_c/k, then

---

[1] Let us remind the reader that entirety and element-by-element reception are equivalent to a non-redundancy code.

$$D_{min} = \int_{0}^{T_s} [illegible] \quad (10.49)$$

where P_s is the average power identical for all signal elements and [illegible] is a coefficient depending on the relation between signals $z_0(t)$ and $z_1(t)$ [illegible].

In the case of a redundancy code the two nearest code combinations differ by d_{min} elements, over the extent of which the integrand in expression (10.49) differs from zero. Since the element length in this code is $T_s k/n$, then

$$D_{min\,red} = \int_{0}^{T_s} [illegible] \frac{k}{n} [illegible] \quad (10.50)$$

consequently $D_{min\,red} > D_{min}$ in case

$$\frac{k}{n} d_{min} > 1$$

or

$$\frac{n}{k} < d_{min}. \quad (10.51)$$

If this condition is not fulfilled, then conversion to a redundancy code does not increase noise resistance, but diminishes it.

Condition (10.51) is satisfied by many systematic codes, but by no means all of them. Thus, for example, a code in which n = 7, k = 2, and d_{min} = 3 does not satisfy this condition although it belongs to Slepian's class of optimum codes [11]. The same is true of a code with n = 9, k = 3, and d_{min} = 3, as well as a number of others.

More rigid conditions must be imposed on the code parameters in the case of element-by-element reception with error correction. To deduce them it is convenient to use the concept of equivalent error probability p_e of the correcting code introduced in Chapter II. It is obvious that it is expedient to use a redundancy code only if

$$p_e < p'. \quad (10.52)$$

where p' designates the error probability which would prevail during the use of non-redundancy code when the rate of information transmission was maintained.

Let a code permit correcting all errors divisible by s or less and not correct N errors divisible by s + 1. With sufficiently small probabilities of

---

The signal $z_{0,1}(t)$ elements in expression (10.50) do not coincide with those of $z_{0,1}(t)$ in expression (10.49) because their length at given T_s differs, but P_s in both formulas is the same by stipulation.

error is, as can easily be seen, the [illegible] probability of error for [illegible] tematic [illegible] code is asymptotically equal to

$$P' \approx \text{[illegible]} \qquad (10.53)$$

where [illegible] k.

In the case of coherent reception

$$P = \frac{1}{2}[1 - \Phi(\text{[illegible]})],$$

$$P' = \frac{1}{2}\left[1 - \Phi\left(\text{[illegible]}\right)\right] \qquad (10.54)$$

[illegible] taken into account that [illegible] redundancy [illegible] non-redundancy code the signal element length, and hence also its power, increases by a factor of n/k.

By employing the known asymptotic expression for [illegible] we may obtain [1[illegible]] from (10.53) and (10.54):

$$\lim \text{[illegible]} \qquad (10.55)$$

$$\cdot \exp\left[\left(\text{[illegible]}\; m \;\; 1\right)\text{[illegible]}\right].$$

where [illegible] n is the code redundancy.

If [illegible] then the right side of (10.55) increases without limit and condition (10.52) is not met. If, on the other hand, [illegible] ≤ [illegible] + 1, then the limit of the right side is equal to zero and for sufficiently large k inequality (10.52) will be met. Thus, the following expression is the condition for determining the admissibility of [illegible] redundancy [illegible] in element-by-element reception:

$$m \;\text{[illegible]} \qquad (10.56)$$

In the case of incoherent reception for orthogonal signals with an active interval the following expression is obtained [12]

$$\lim \text{[illegible]} \left\{\left(\text{[illegible]}\right)\text{[illegible]}\right\}, \qquad (10.57)$$

whence the following expression [illegible] condition for determining the admissibility of [illegible] redundancy [illegible]

$$m \;\text{[illegible]} \qquad (10.58)$$

[illegible] can easily [illegible] that of all known [illegible] codes with n ≤ 12 only very few satisfy these conditions, [illegible] the value of n at which inequality (10.[illegible]) is met is very [illegible]. Apparently redundancy codes yield a significant gain in a channel with fluctuation interference only beginning with an n on the order of several tens.

Notes

1. (See Section 10.1) By "entirety reception" this chapter, as ordinarily, means the reception method in which the decision system analyzes in its entirety the signal segment which corresponds to the code combination. At times entirety reception is mentioned in the sense of analysis of a signal corresponding to an entire incoming message. It is easy to show that if any code combination sequence in this signal has the same a priori probability, then such reception of the entire signal has no advantage over entirety reception of individual combinations, just as in a non-redundancy code (when all symbol sequences are permissible and equiprobable) entirety reception has no advantage over element-by-element reception.

In actual fact, not all code combination sequences are equiprobable for many sources. This is a consequence of redundancy in the source alphabet. This redundancy is, however, ordinarily difficult to utilize to raise reception reliability. We will note that for a memory channel in which the values of c_i are correlated, if furthermore condition (10.17) is not met, entirety reception can have an advantage over element-by-element reception even in the case of primitive coding.

2. (See Sections 10.2 and 10.3) The main difficulty in realization of entirety reception and methods of reception approximating it is the need to remember continuous (continual) values of c_i which are obtained in processing the separate signal elements or their sum. For this purpose analog memory devices are needed, for example, the summators in Figures 10.2 and 10.3, which are more difficult to make than the discrete devices used in element-by-element reception. This problem is made simpler if entirety reception is used for signals with parallel coding (see Section 9.6), since in this case the results of demodulation of c_i are processed simultaneously and they need not be stored for an extended period.

We will note that the complex (wideband) signals mentioned in Chapters VII and VIII can also be considered as the result of sequential or parallel coding by the most redundant code (n, 1) considering one of the components an information element and all remaining the check elements. With such an approach [13] different methods of receiving such signals also amount to optimal coherent entirety reception, to incoherent entirety reception (with coherent or incoherent cumulation), and to element by element reception. Such a point of view is possible in investigating diversity reception [11]. In this case optimal coherent addition amounts to nothing other than coherent entirety reception and quadratic addition to incoherent entirety reception with incoherent cumulation. The method of selection in the case of diversity reception is a particular case of decoding based on the most reliable symbols, and the method of discrete addition is element-by-element reception with the correction of errors. Such a single approach to the different problems of receiving signals is very useful since it permits direct application of the results obtained in one area to the solution of many other problems. Furthermore, it leads to the idea that it might be possible to apply several methods (mainly suboptimal) developed for any other case (for example, for diversity reception) to the constraction of new systems in other areas (for example, in multiplexing and combining channels, etc.)

3. (See Section 10.4) The first inequality in expression (10.27) is obvious. Let us pause on the proof of the second inequality.

We will consider all examples of undetected errors. So that should an error occur (event A_5) in entirety reception it is necessary and sufficient that for the symbols corresponding to ones in one of these examples inequality (10.21) be met. We will use B_k $(k = 1, \dots, 2^{k} - 1)$ to indicate the event that for the k-th example of an undetected error (10.21) is met. Then A_5 is equivalent to occurrence of at least one event B_k.

Is a certain s-th example is the sum of the n th and v-th examples of an undetected error, event B_s can occur only when at least one event B_n or B_v occurs. It follows that for event A_5 it is necessary and sufficient that at least one of m events B_k pertaining to examples of error not represented in the form of a sum of other examples occur. Inasmuch as the probability of each event B_k is not greater than p_d, then

$$P_{н} = P(A_5) = P(B_1 \text{ or } B_2 \text{ or } B_m) \le P(B_1) + \dots + P(B_m) \le mp_d,$$

which was to be proved.

4. (See Section 10.7) We will present the derivation of principle (10.46) for a code permitting majority decoding using the system of separate checks in (10.45) and (10.45a). We will assume that the results of demodulation of c_{jm} which are mutually independent and proportional to the logarith of the likelihood ratio in element-by-element reception correspond to symbols y_{jm}:

$$c_{jm} = k \ln \frac{w(z_{jm} \mid y_{jm} = 0)}{w(z_{jm} \mid y_{jm} = 1)}, \qquad (10.59)$$

where k is a proportionality factor; and z_{jm} is a received signal element corresponding to symbol y_{jm}. We will note that for any optimal circuit in element-by-element reception of binary signals the results of demodulation, if not expressed by formula (10.59), in any case are monotonic reversible functions of the likelihood ratio. Therefore, they can be transformed into c_{jm} values which are presented by this formula.

We will find the likelihood ratio for symbol y_1 by assuming that c_1 and all c_{jm} corresponding to symbols on the right of system (10.45) are known. We will consider that symbols y_{jm} can have a value of 0 or 1, the only limitation being that equations (10.45) for a given y_1 must be met. This means that when $y_1 = 0$ the y_{jm} $(m = 1, \dots, r)$ entering into one equation there must be an even number of ones (or no ones at all) and when $y_i = 1$ there must be an odd number of ones.

The likelihood ratio for y_i under these conditions is

$$\frac{w(\ldots/0)}{w(\ldots/1)} = \frac{w(\ldots/0)}{w(\ldots/1)}\prod_{j=1}^{r}\frac{w(\ldots/0)}{w(\ldots/1)}. \tag{10.60}$$

The densities entering into the product on the right side of (10.60) depend on unknown parameters y_{jm}. By using a generalized criterion of maximal likelihood we will replace the numerator and denominator in (10.60) with their maximal values varying the values of symbols y_{jm} in light of the relationships imposed by equations (10.45), in other words, we will assume that $y_i = 0$ if

$$\frac{w(\ldots/0)}{w(\ldots/1)}\prod_{j=1}^{r}\frac{\max\, w(\ldots/0, y_{j1}, \ldots, y_{jr})}{\max\, w(\ldots/1, y_{j1}, \ldots, y_{jr})} > 1$$

Taking the logarithm of this inequality, we write the decision principle in the form

$$\ln\frac{w(\ldots/0)}{w(\ldots/1)} + \sum_{j=1}^{r}\max_{y_{jm}}\ln w(\ldots/0, y_{j1}, \ldots, y_{jr}) - \sum_{j=1}^{r}\max_{y_{jm}}\ln w(\ldots/1, y_{j1}, \ldots, y_{jr}) > 0. \tag{10.61}$$

We will consider one of the terms in the first sum

$$\max_{y_{jm}}\ln w(\ldots/0, y_{j1}, \ldots, y_{jr}) \tag{10.61a}$$

To find the maximum it is necessary to sort all possible sets of values y_{jm} [illegible] 0 satisfying equations (10.45), i.e., those containing an even number of ones. If this condition is met and the values of y_{jm} are fixed, then

$$w(\ldots/0, y_{j1}, \ldots, y_{jr}) = w(\ldots)\ldots w(\ldots)$$

Introducing instead of y_{jm} the values of ε_{jm} just as in (10.6[illegible]), or, more exactly, assuming $\varepsilon_{jm} = 1$ when $y_{jm} = 0$ and $\varepsilon_{jm} = -1$ when $y_{jm} = 1$, we may rewrite (10.61a) in the following form:

$$\max_{\varepsilon_{jm} = \pm 1}\sum_{m=1}^{r}\ln w(\ldots) = \max_{\varepsilon_{jm} = \pm 1}\sum\left[\frac{1+\varepsilon_{jm}}{2}\ln w_0(\ldots) + \frac{1-\varepsilon_{jm}}{2}\ln w_1(\ldots)\right]$$

$$-\frac{1}{2}\sum_{m=1}^{r}[\ln w_0(\varepsilon'_{jm}) + \ln w_1(\varepsilon'_{jm})] +$$

$$+\frac{1}{2}\max_{\xi_{jm}\in A}\sum_{m=1}^{r}\xi_{jm}\ln\frac{w_0(\varepsilon'_{jm})}{w_1(\varepsilon'_{jm})},$$

where

$w_0(\varepsilon'_{jm}) = w(\varepsilon'_{jm}|y_{jm} = 0)$ and

$w_1(\varepsilon'_{jm}) = w(\varepsilon'_{jm}|y_{jm} = 1)$.

and we will use A to denote a set of sequences of ξ_{jm} (m = 1,...,r) containing an even number of negative values.

The terms in the second sum in (10.61) can be represented similarly, the only exception being that it is necessary to maximize with respect to $\xi_{jm}\in B$ where B is the set of sequences of ξ_{jm} containing an odd number of negative values.

Substituting these expressions in (10.61) and also considering (10.59), we obtain after obvious transformation the following decision principle about $y_i = 0$:

$$c_i + \frac{1}{2}\sum_{j=1}^{l}\left[\max_{\xi_{jm}\in A}\sum_{m=1}^{r}\xi_{jm}c_{jm} - \max_{\xi_{jm}\in B}\sum_{m=1}^{r}\xi_{jm}c_{jm}\right] > 0 \qquad (10.62)$$

We will now seek values entering into this formula of maxima. We will assume that for a certain value j, i.e., for terms of a certain equation from system (10.45) there is among the c_{jm} an even number of negative ones. In order to maximize the first sum (when $\xi_{jm}\in A$) it is sufficient to set all ξ_{jm} corresponding to positive c_{jm} equal to +1 and the remaining ξ_{jm} equal to -1.

As a result the first maximum is equal to $\sum_{m=1}^{r}|c_{jm}|$. In maximizing the second sum (when $\xi_{jm}\in B$) it is not possible to make all $\xi_{jm}c_{jm}$ positive since the number of negative values of c_{jm} in the given example is even and the number of negative ξ_{jm} must be odd. Obviously, under these conditions the maximum of the second sum will occur if in it one term having the least absolute magnitude is negative. Thus, the maximum of the second sum will be $\sum_{m=1}^{r}|c_{jm}| - 2|c_{jm}|\min$ where $|c_{jm}|\min$ is the least value of the modulus of c_{jm} with a given j and when m = 1, ...,r.

Reasoning similarly for the case when among c_{jm} there is an odd number of negative ones, it can easily be seen that the maximum of the first sum will be

$$\sum_{m=1}^{r}|c_{jm}| - 2|c_{jm}|\min$$

and the maximum of the second sum will be $\sum_{m=1}^{r} |c_{jm}|$. Noting also that the function $\mathrm{sgn}\prod_{m=1}^{r}$ assumes the value of +1 if an even number of negative factors enters into the product and -1 otherwise, it is possible to represent the expression in the brackets of (10.62) in the form

$$\max_{\varepsilon_{jm}\in 1}\sum_{m=1}^{r}\varepsilon_{jm}c_{jm} - \max_{\varepsilon_{jm}\in H}\sum_{m=1}^{r}\varepsilon_{jm}c_{jm} = 2|c_{jm}|_{\min}\,\mathrm{sgn}\prod_{m=1}^{r}c_{jm} \qquad (10.63)$$

Finally, the decision principle for deciding that $y_i = 0$ takes the form

$$c_i + \sum_{j=1}^{s} |c_{jm}|_{\min}\,\mathrm{sgn}\prod_{m=1}^{r} c_{jm} > 0,$$

which coincides with (10.46).

We will note in conclusion that in ordinary element-by-element majority decoding the indicated principle can be written in the form

$$\mathrm{sgn}\, c_i + \sum_{j=1}^{s}\mathrm{sgn}\prod_{m=1}^{r} c_{jm} > 0.$$

Thus, the essence of analog decoding amounts to introducing weighting factors $c_{jm\ \min}$. In other words, the "weight" of each of the checks (10.45) is determined by the least modulus of the logarithm of the likelihood ratio of the symbols included in it.

BIBLIOGRAPHY

1. Meshkovskiy, K. A., "Problems of Resistance to Interference in Communication Systems Using Entirety Reception," Radiotekhnika, No. 6, 1958.

2. Meshkovskiy, K. A. and N. Ye. Kirillov, "[illegible]" [Coding in Communication Equipment], Svyaz' Press, 1966.

3. Pushnoy, B. M., "Ideal Entirety Signal Reception," Elektrosvyaz', No. 3, 1960.

4. Sil'vermen, R. A. and M. Bolser, "Coding in Transmitting Data at a Constant Rate," In the collection: Codes with the Detection and Correction of Errors, Foreign Literature Press, 1956.

5. Borodin, L. F., "[illegible]" [Introduction to the Theory of Interference-Resistance Coding], Sovetskoye Radio Press, 1968.

6. Borodin, L. F., "Receiver with a Floating (Random) Threshold," Elektorsvyaz', No. 2, 1960.

7. Korzhik, V. I., "Boundaries to Probabilities of Undetected Errors and Optimal Group Codes in Feedback Channels," Radiotekhnika, No. 1, 1965.

8. Kolesnik, V. D. and Ye. T. Mironchikov, "Several Cyclical Codes and Decoding Circuit Based on Majority of Checks," Problems of Information Transmission, Second Edition, 1965.

9. Kagan, B. D. and L. M. Fink, "Method of Sequential Entirety Reception for Codes Permitting Majority Decoding," Elektrosvyaz', No. 1, 1967.

10. Kagan, B. D. and L. M. Fink, "Entirety Reception for Codes Permitting Majority Decoding," Elektrosvyaz', No. 5, 1968.

11. Slepyan, D., "Class of Binary Signal Alphabets," In the collection: Theory of Message Transmission, Foreign Literature Press, 1957.

12. Fink, L. M., "Applicability of Binary Correcting Codes in Discrete Information Transmission Channels," Radiotekhnika, No. 10, 1961.

13. Zav[illegible]dnyy, V. M., Yu. B. Okunev and L. A. Yakovlev, "Theory of Wideband Communication Systems," Radiotekhnika, No. 5, 1967.

14. Khvorostenko, N. P., "[illegible]" (Statistical Theory of Demodulation of Discrete Signals), Svyaz' Press, 1968.

CHAPTER XI

FEEDBACK SYSTEMS

11.1. Method of Statistical Sequential Analysis for Signal Discrimination

In the preceding chapters we considered reception methods based on an analysis of an arriving signal realization as a result of which one of [illegible] hypotheses as to a transmitted code symbol or code combination was selected in accordance with a certain criterion (for example, the criterion of maximal likelihood). To achieve a high level of fidelity it is essential, along with providing for the best possible selection of transmitted signals and optimal (or close to it) design on a decision system, to provide for sufficient power in an arriving signal.

It is proved in statistical decision theory [1, 2*] that the effectiveness of this procedure can be heightened, if as a result of an analysis of an arriving signal to permit along with definite decisions also an indefinite outcome after the obtaining of which the analysis is continued, for which purpose the same signal must be transmitted again. This permits reducing the power in the signal, for example, by reducing the length of an element or by decreasing the redundancy introduced during [illegible]. Although [illegible] of a signal will be repeated many times [illegible], on the average it is possible to obtain a significant power gain.

[illegible]

[illegible]

---

* [illegible] the [illegible] in work [2].

is reached. In both these cases the sequential [illegible] can easily be followed inasmuch as the [illegible] the need to [illegible] is made by the experimenter himself.

The question is otherwise with sequential procedures when messages are transmitted. An arriving signal is analyzed [illegible] a decision [illegible] receiver and, if it is not possible to reach a final decision with sufficient confidence, the signal must be [illegible] repeated [illegible] a correspondent and this is possible only when there exists a channel [illegible] messages [illegible] the [illegible] message to the transmitter. Systems in which [illegible] channel [illegible] and is used to [illegible] the [illegible] are called feedback systems.

Sequential [illegible] procedure [illegible] effective for channels with changing parameters [illegible] the power and duration of signals [illegible]

In the second possibility the decision to repeat a signal is reached by the transmitter itself based on the information which is sent to it in the reverse channel. This information may consist of a sequence of received (decoded) symbols, or a received sum of signal and noise, or a certain stipulated signal formed from the signal received in accordance with a certain law, etc. Such a variation is called an information feedback system [26].

In addition to these two main variants of system sometimes mixed systems or complex feedback systems are considered in which a decision to repeat a signal is reached in some cases on the receiving end and in other cases on the transmitting end [12].

Regardless of where the decision is reached to repeat a signal, it may be received as a result of analysis of a continuous received signal [11] or of a discrete sequence of code symbols obtained following ordinary element-by-element demodulation of the received signal. In the first case we have feedback in a continuous channel or, using the terminology of F. Ye. Ginn [7], feedback before decision and in the second case feedback in a discrete channel (or after decision*). In principle, of course, mixed systems are also possible.

The rate of information contained in a rejected signal, i.e., in a signal after the analysis of which a decision to repeat a transmission is reached, is an important characteristic of a feedback system. Obviously, even a rejected signal contains certain information about a transmitted message which can to some degree be used to increase fidelity when reception is repeated. Such systems in which a rejected signal is retained and the information contained in it partially used are called memory systems. All known feedback systems in actual use are nonmemory systems, i.e., the information contained in a rejected signal is irretrievably lost. Nonmemory systems are much simpler than memory systems and under ordinary conditions are only slightly inferior to them in resistance to interference [11]. Only in those cases when the power of an arriving signal is very small in comparison with spectral noise density do memory systems have a pronounced advantage. In what follows we will consider only nonmemory systems.

In addition, feedback systems are subdivided into systems with a limited or unlimited number of repetitions. Systems with unlimited repetitions, strictly speaking, are only possible when messages originate from a source with a controllable rate. If a source has a fixed rate, to utilize feedback provision must be made for a buffer memory between source and modulator. Inasmuch as the size of such a memory if finite, the number of possible repetitions is limited since when the memory is overloaded repetitions must cease so that new information emitted by a source will not be completely lost. However, if the size of the buffer memory is sufficiently great, the probability of overloading is slight and in the first approximation the number of repetitions can be considered unlimited.

---

*Unfortunately, in the translation of work [7] the terminology used in it is distorted. The term feedback "before decision" (predecision feedback) and "after decision" (postdecision feedback) are translated as "information" and "decision" feedback, although these concepts do not quite coincide.

11.3. Interrogation Systems in a Discrete Channel

Principal Characteristics of the Simplest System

An overwhelming majority of existing feedback systems are interrogation systems using a discrete channel. Therefore, we will speak about them somewhat more in detail than all other types. Let a transmitted message be redundancy coded. In the simplest interrogation system only permissible code combinations are decoded and sent to the recipient (see Section 2.4). In this process a confirmation signal is sent over the reverse channel. If the code combination obtained at the output of the first decision circuit is forbidden, it is erased and an interrogation signal is sent over the reverse channel. In this way, a code combination which is received incorrectly may reach a recipient only when it is a permissible one. When an interrogation signal is received the approximate code combination is repeated and then information transmission continues.

Let's assume that a code is given (for simplicity we will limit ourselves to the case of a group binary (n,k)-code and we will also assume that the properties of the channel are known. Then it is possible to determine the probability of a detected error P_{de}, i.e., the probability that instead of the transmitted code combination some other of the forbidden code combinations is received and the probability of an undetected error is P_{ue}, i.e., the probability that a permitted combination is received which differs from the one transmitted. Obviously,

$$P_{de} + P_{ue} + Q = 1 \tag{11.1}$$

where Q is the probability of correct reception of a code combination.

Knowing these probabilities and also the length of a code combination n and the number of information symbols k in it, it is possible to determine the main characteristics of a given system to which the relative transmission rate S and the equivalent probability of error p_e pertain.

By relative transmission rate we mean the ratio between the mathematical expectation of a number of information symbols reaching a recipient and the total number of code symbols passed in the forward channel. When every code combination is received the probability is $1 - P_{de}$ that the recipient is issued k information symbols and the probability is P_{de} that not a single symbol is delivered to him. Therefore, for the simplest system

[illegible]

$$P_{res} = \frac{P_{ac}}{1 - P_{dc}}$$

[illegible]

$$P_{res} = P_{ac} + P_{ac}P_{dc} + P_{ac}P_{dc}^2 + \cdots$$

$$= \sum P_{ac}P_{dc}^{n} = \frac{P_{ac}}{1 - P_{dc}}$$

[illegible]

[illegible]

[illegible]

arrives at a decoder and there was no interrogation prior to this, it is erased and does not go to the recipient.

Another method [6] is to compare any received permitted code combination with the preceding one and, in the case of coincidence, to erase it. If in a transmitted message there are two identical combinations in a row, instead of repeating the combination a special combination is sent (we will denote it Y) which is assigned from the number of permissible combinations. Thus, if it is necessary to repeat combination X twice then XY is transmitted instead of XX. If, for example, XXXXX must be transmitted, then XYXYX is sent. In this system two identical combinations will never be sent in a row [illegible]. Mention should also be made of the method of distinguishing initial and repeated signals based on an additional modulation parameter [13]. All these methods, as can easily be seen, require some increase in redundancy.

Evaluations of Error Probability

Before turning to a description of other types of interrogation systems used in a discrete channel, we will pause on the computation of the probability of detected and undetected errors. We will assume that the parameters of the channel are constant or change slowly so that over the length of the code combination the probability of incorrect reception of a symbol can be considered constant or, finally, the parameters of a channel change very rapidly or methods of error decorrelation are used so that errors within the limits of a combination are used so that errors within the limits of a combination are practically independent and are determined by a certain average probability.

Even with such simplified assumptions, it is not always possible to compute the probability of an undetected error. Of course, if the code is small, then it is possible to sort all types of error vectors, to [illegible] and to compute their total probability. For group codes the task is somewhat simplified in that a certain type of error causes an undetected error if it coincides with one of the code combinations. This is apparent from the fact that a group code contains a combination of all zeros, and an error which transforms it into one of the other permitted combinations is undetected. It is necessary and sufficient that the errors occur in those cases in which the combination [illegible] from the properties of symmetry of a group code it follows that the same [illegible] causes an undetected error in the transmission of any code combination. Therefore, for computation of P_{ue} in a binary group code it is sufficient to know the list of code weights, i.e., the number of code combinations W_i [illegible], the weight (number of ones) [illegible]. Then, the probability of an undetected error is the probability of an error coinciding with one of the code combinations:

$$P_{ue} = \sum_{i} W_i p^i (1-p)^{n-i}$$

Unfortunately, a list of weights has been computed analytically only for Hamming codes where d = 3 and d = 4 [13] and also for Reed-Muller codes [illegible]. For other codes it can be determined only by sorting all combinations and this is possible in practice only when the size is small. Thus, when k [illegible] the number of code combinations exceeds 10^6 and such sorting is possible only by using an electronic computer. When k > 40 this task becomes impossible even for computers although for cyclical codes with k > 40 an encoder and decoder which detect errors are entirely realizable.

For codes with a large n and k it is necessary to use only estimates for P_{ue}. Several such estimates are presented in [14]. If along with n and k a minimal Hamming distance $d_{min} = 2t + 1$ is known, it is convenient to use the estimates shown in [15].

$$P_{ue} \; [illegible] \qquad (11.7)$$

$$P_{ue} \; \sum \; [illegible]$$

The second of these estimates is more exact but in the case of large n requires calculations on a computer.

The value P_{ue} [illegible], which is the probability of an undetected error in a binary channel when p [illegible], with a complete disruption in communication, is of interest. In this case any sequence of code symbols occurs with equal probability at the output of the decision system and the probability of an undetected error is equal to the ratio between the number of permitted combinations N and their total number [illegible]

$$P_{ue} \; [illegible] \qquad [illegible]$$

[illegible]

$$P_{ue} \; [illegible] \qquad [illegible]$$

[illegible] an undetected error with a [illegible] for example, P_{ue} [illegible]

[illegible] transmitted [illegible] because of the large [illegible] probability of error [illegible] However, [illegible] permits [illegible] false information [illegible] designed [illegible] system.

As far as the [illegible] undetected error P_{ue} is concerned, its approximate value [illegible] if it is remembered that [illegible] The latter inequality is met with any well designed [illegible] interrogation system

Thus,

$$P_{de} [illegible]$$

Figure 11.2 shows the dependences [illegible] from formulas [illegible] these dependences for [illegible] when [illegible] so that the values of [illegible] of these results.

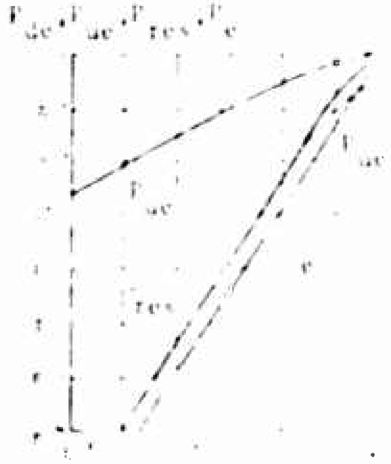


Figure 11.1. [illegible] Parameters [illegible] an Interrogation System Using [illegible] Hamming [illegible] Code

Figure 11.2. [illegible]

Blocking System

The simplest [illegible]

[illegible]

where [illegible]

In [illegible]

combination. All combinations received without detected errors are recorded in appropriate memory cells, and cells corresponding to combinations in which errors are detected remained unfilled. After completion of reception of the codogram numbers (addresses) of combinations not received are sent over the reverse channel and these combinations are then repeated over the forward channels. The process continues until the entire codogram, which is then sent to the recipient, is received (without detected errors).

Under ordinary conditions an address interrogation system does not have any noteworthy advantages over a blocking system and only when the values of M are very large does it give a certain gain in relative transmission rate. On the other hand, the address system has definite shortcomings. The principal one is the possibility of occurrence of specific errors in the case of incorrect reception of an address transmitted over a reverse channel. Therefore, it is necessary to employ complex encoding and decoding with error correction in the reverse channel.

Peculiarities of Duplex Interrogation Systems

Interrogation systems are usually duplex, i.e., in them information is exchanged in both directions. In this process each of the two one-way channels is used partially as a forward and partially as a reverse channel. By using any method of multiplexing, it is possible to transmit over one channel a main message as well as auxiliary signals of confirmation and interrogation. It is most convenient to use time multiplexing, alternating code combinations carrying the main message with service combinations. It is easy to see that the relative transmission rate for such a duplex blocking system is determined by formula (11.11) when $n + n_s$ replaces n, where n_s is the number of symbols in the service combination which, generally speaking, may differ from n:

$$S = \frac{k}{n + n_s}(1 - P_{de})^{M+1}. \tag{11.12}$$

We will call such a duplex system a system with separated service signals. However, more frequently a duplex interrogation system is implemented differently, i.e., with unseparated service signals, for the purpose of making better use of the channels. For this purpose confirmation signals are not transmitted but interrogation signals are transmitted whenever necessary, i.e., when an error is detected in a combination transmitted over the other channel, interrupting for this period of time the transmission of the main message. In this process, understandably, a code combination differing from those used for the main message is assigned for the interrogation signal. The transmission of a combination of main message is viewed simultaneously as confirmation of reception of the last combination.

At first glance such a system seems very simple and permits greatly increasing the use of channels, especially in channels of satisfactory quality when interrogation signals are transmitted infrequently. But in fact the system becomes more complex due to efforts to avoid serious distortions in a message caused by incorrect reception of interrogation instead of a transmitted message combination, and vice versa. It must be kept in mind that in such a duplex system an interrogation combination is protected from error no better than any other code combination. If an interrogation signal is received even with a detected error, it will not be recognized and instead of repetition of an incorrectly received combination an interrogation signal will be sent and at best this leads to added delay. If an error is not detected, a code combination in one of the messages will be lost and an extra one will appear in the the other.

To avoid this, use is made of the complicated operating algorithm of a duplex interrogation system which, in general terms, amounts to the following. Any detected error is considered simultaneously as an interrogation signal. Therefore, in the detection of an error, just as in obtaining an interrogation signal, an interrogation signal is also sent in the reverse direction, after which M + 1 combinations from a calculator-repeater are repeated. At the same time the receiver is blocked for M combinations. Thus, with the occurrence of a detected error in one direction, both receivers are blocked and the last M + 1 combinations are repeated in both directions. A more detailed description of this cross blocking algorithm can be found in work [8, 9].

The equivalent probability of error in such a duplex system in the first approximation (without consideration of undetected errors in the interrogation signal) is expressed by the same formula (11.4) as in the simplest system described earlier. The relative transmission rate can be determined by considering that the recipient is issued k information symbols on condition that the preceding M combinations in both directions are received without detected errors. Therefore,

$$S = \frac{k}{n}(1 - P_{de})^{M+1}(1 - P'_{de})^{M}, \tag{11.13}$$

where P'_{de} is the probability of detected error in the opposite direction.

Comparing (11.13) and (11.12), we can see that the system with the unseparated service signals permits increasing the relative transmission rate on condition that

$$(1 - P'_{de})^{M} > \frac{n}{n + n_s} \tag{11.14}$$

This condition is met only in rather good channels when $P'_{de} \ll 1$ and when the blocking of M is not long. In low quality channels, especially in transmission at a high rate and over long distances (which leads to a large value of M), the system with separated service signals is most advantageous.

Selection of a Code for an Interrogation System

At the present time no methods are known which permit finding for an interrogation system an optimal code providing for a maximum relative transmission rate with a given level of fidelity or maximum fidelity at a given rate.

However, several ideas about the trends in searching for a code can be expressed. We will begin with a consideration of a constant discrete channel.

First we will note that according to (11.4) the equivalent probability of error is proportional to the probability of an undetected error P_{ue}. Therefore, it is first necessary to provide for a sufficiently small value of P_{ue}. From (11.6) and (11.7) we may conclude that for this purpose a code must have a sufficiently large minimal Hamming distance d_{min}. On the other hand, the code must have small redundancy, i.e., the k/n ratio must not be very small since otherwise the relative transmission rate would be decreased. In order, with a small redundancy, to provide for a large d_{min}, we must use a code with a long n.

But increasing n leads to an increase in the probability of a detected error P_{de} which, in turn, lowers the relative transmission rate (11.2), (11.11), (11.15) and increases the equivalent probability of error (11.4). Obviously, there must be an optimal code length n depending on the probability of error in a channel and also on the rate of transmission and range of communication determining the blocking length of [illegible]. In selecting a value of n less than optimal the probability of an undetected error grows or, if redundancy is increased in order to retain fidelity, the relative transmission rate is decreased. In selecting an n larger than optimal, the probability of detecting an error increases and this reduces the fidelity and rate. It is not possible to find an optimal n analytically since we do not have an analytical expression for P_{ue}. Therefore, it is necessary to use iterative approximation.

Thus, to decrease P_{ue} it is necessary to increase n, and to lower P_{de} it must be decreased. This contradiction can be resolved by using an iterative code and an iterative interrogation procedure [11]. In this procedure use is made of several coding stages and several stages of checks. In the simplest case, for a two-stage code, the symbols of a message of information form a rectangular table (Figure 11.3). Check symbols are formed for each row and for each column. Each received row is checked for the presence of errors and when they are detected an interrogation signal is sent for the row which is repeated until it is received without errors. After the entire table has been received without a detected error with respect to rows, the check is performed with respect to columns. If in this process an error is detected if only in one column, the entire table is reprocessed.

With a rather short row the time taken to repeat rows is not great and reduces the relative rate of transmission little. It is true that there is a significant probability of an undetected error in a row but this is no great danger inasmuch as such errors will be detected when the check is performed for the column. On the other hand after correction of detected errors in the rows, the probability of error in the table is much reduced. Therefore, the need for table interrogation occurs much more rarely than when an ordinary code of the same length and same redundancy is used.

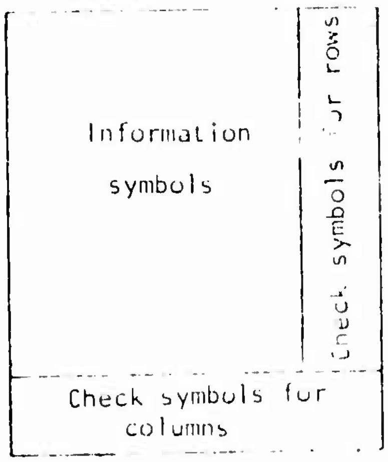


Figure 11.3.- Construction of a Two-Stage Iterative Code.

An iterative system of three stages or more is constructed similarly. Work [17] shows that a code with one check symbol and a simple check for parity is an optimal code for each stage. With an increase in the number of stages it is possible to provide for a small equivalent probability of error as desired with a transmission rate amounting to 27% of the channel carrying capacity.

The selection of a code for an interrogation system in a discrete channel having a memory is of great interest. In most cases encountered in practice when the probability of error p changes slowly, this code can be so selected as to provide for a given equivalent probability of error when using not less than 30-50% of channel carrying capacity. The selection of a code is based on the following ideas. In relatively good channel conditions when $p \ll 1$, it is easy to provide for a high level of fidelity with a sufficiently large rate of transmission. With an increase in p the equivalent probability of error inevitably increases but at the same time the relative transmission rate decreases. Let p_0 be that value of instantaneous probability of error in a channel at which the equivalent probability of error is equal to that permissible. By selecting a code with a sufficiently long n and sufficient redundancy, it is possible to have p_0 exceed the median probability of error in a given channel.

The average time required for transmission to the recipient of one code combination with a particular channel state is equal to

$$t_1 = \frac{n\tau}{s}, \tag{11.1}$$

where n, as above, τ is the length of a signal element.

We will use t_s to denote the average duration of time spent by a channel in states for which $p < p_0$. If the code is so selected that t_1 in these states satisfies the condition

$$t_1 \leq t_3. \qquad (11.16)$$

this means that in states when $p > p_0$ not a single code combination will reach the recipient, i.e., the transmitted combination will be repeated until the probability of error in the channel decreases. In "good" conditions when $p < p_0$ messages will be transmitted with a probability not less than that prescribed.

Condition (11.16) can always be met by selecting a sufficiently long combination (which increases P_{de} for large values of p) and a sufficient blocking length of M. At the same time, when $p < p_0$ the relative transmission rate S can be rather great. Inasmuch as p_0 is greater than the median value, "good" states of a channel when $p < p_0$ will occupy more than 50% of the time and the average transmission rate of information will be sufficiently great.

Without pausing over the details of selecting a code, we will present a somewhat simplified example. Let there be in a channel with equal probabilities three states described by the probabilities of error $p_1 = 10^{-3}$, p_2 on the order of 10^{-2}, and $p_3 \geq 10^{-1}$ or even $p_3 = 0.5$ (complete disruption in communication) and the average duration in each state be equal to 10 seconds. We will use a rated speed of transmission equal to 1,000 bauds and require that the equivalent probability of error not exceed $p_e = 10^{-10}$.

These conditions can be met by using a Bose-Choudkhuri [63,45] code in a blocking system where M = 3. Figure 11.4 shows for this instance the dependence of average time t_1 required for the transmission of the code combination on the probability of error p constructed in accordance with formulas (11.15), (11.11), and (11.9). The dependence of p_e on p from Figure 11.2 is also entered there. For an illustration of the role played by clocking the broken line shows the curve for t_1 in a system without blocking. As can be seen from the figure, in the first channel state ($p = 10^{-3}$) a message is transmitted at a great rate on the order of 16 combinations (about 720 bits) per second, and the equivalent probability of error is much less than 10^{-10}. In the second state ($p \approx 10^{-2}$) the rate of transmission is much reduced and in one second not more than the 1-2 combinations (45-90 bits) are transmitted and the equivalent probability of error reaches 10^{-10}. In the third state ($p \geq 0.1$) the code combination could be transmitted on the average over a period of two years or more if the state were maintained. In actual fact, transmission completely ceases in this state and the receiver remains blocked until the state changes. The probability of receiving a code combination in this state is so slight that it does not affect the average equivalent probability of error which, thusly, does not exceed 10^{-10}. The average transmission rate is about six combinations (270 bits) per second, i.e., higher than 27% of the channel carrying capacity in the best state.

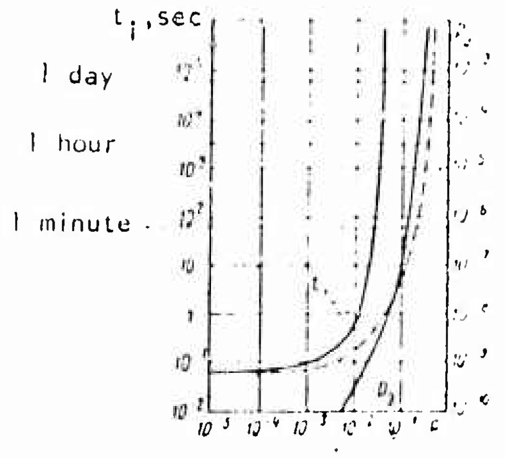


Figure 11.4. Equivalent Probability of Error p_e and Average Time of Transmission of the Code Combination t_1 for a (63, 45) Code When v = 1000 Bauds: -----, M = 3; - - - -, M = 0.

As can be seen from the figure, this code gives good results with a channel in which the median value of probability of error is close to 10[illegible]. With a smaller median value of p longer code combinations should be used and with a large median value shorter combinations should be used. Thus, an interrogation system permits successful use of "bad" channels in which the probability of error fluctuates around a rather large value and even brief disruptions in communication occur.

11.4. Interrogation Systems in a Continuous Channel

Interrogation systems in a continuous channel are different from those described above in that a decision to repeat a certain signal segment is reached not in the process of decoding but in the first decision system based on an analysis of a received continuous signal. Their advantage lies in the fact that for reaching a decision use is made of all information included in an arriving signal while with interrogation in a discrete channel some of the information is inevitably lost in the process of demodulation.

A transmitted message may be encoded by a primitive code. For detection of a possible error and reaching a decision about interrogation, the first decision system, as a rule, determines the likelihood functions for possible transmitted symbols and compares them. If one of them greatly exceeds the others, a final decision is reached and a received message after decoding is sent to the recipient. If for two symbols the values of the likelihood function are close to one another this signal is rejected and an interrogation signal is sent over the reverse channel.

For an example we will consider a binary system with signals orthogonal in the intensified sense with an active pause and in the case of incoherent reception. The a priori probabilities of both symbols will be considered to be the same. The a posteriori probability that signal z_0 was transmitted corresponding to symbol "0", according to the Bayes formula, is

$$p(0|z') = \frac{w(z'|0)}{w(z'|0) + w(z'|1)}, \qquad (11.17)$$

where $w(z'|0)$ and $w(z'|1)$ are likelihood functions for symbols "0" and "1" respectively.

If the decision is reached that symbol "0" was transmitted, the a posteriori probability $p(0|z')$ amounts to the probability of correct reception of an element of signal $z'(t)$ and $1 - p(0|z')$ is the probability of incorrect reception of this element. We will require that in a feedback system the probability of an undetected error for a symbol not exceed a certain given value of P_{ue}. For this purpose symbol "0" must be sent to the recipient only on condition that $1 - p(0|z') \leq P_{ue}$ and similarly symbol "1" if $1 - p(1|z') \leq P_{ue}$. In all remaining cases a signal element is rejected and an interrogation signal is sent over the reverse channel. In light of (11.17) this leads to the following algorithm:

Symbol "0" is recorded if

$$\frac{w(z'|0)}{w(z'|1)} \geq \frac{1 - P_{ue}}{P_{ue}},$$

symbol "1" is recorded if

$$\frac{w(z'|0)}{w(z'|1)} \leq \frac{P_{ue}}{1 - P_{ue}}, \qquad (11.18)$$

interrogation is sent if

$$\frac{P_{ue}}{1 - P_{ue}} < \frac{w(z'|0)}{w(z'|1)} < \frac{1 - P_{ue}}{P_{ue}},$$

in light of (4.27) for a channel without fading

$$\frac{w(z'|0)}{w(z'|1)} = \frac{I_0(V_0)}{I_0(V_1)}. \qquad (11.19)$$

In a channel with fading, having stipulated that the law governing the fading is unknown and a generalized criterion of maximal likelihood is used, according to (5.18a)

$$\frac{w(z'|0)}{w(z'|1)} = \exp\left[\frac{V_0 - V_1}{2P_s}\right], \qquad (11.20)$$

where V_0 and V_1 are determined by formula (4.29) and represent, for example, readouts of envelopes at the output of matched filters.

Inasmuch as $P_{ue} \ll 1$ should be selected, in (11.18) we may set $1 - P_{ue} \approx 1$. Then the algorithm for a channel with fading can be formulated thusly:

Symbol "0" is recorded if

$$V_0^2 - V_1^2 > 2\sigma^2 P_s \ln \frac{1}{P_{ue}} .$$

Symbol "1" is recorded if

$$V_0^2 - V_1^2 < -2\sigma^2 P_s \ln \frac{1}{P_{ue}} . \qquad (11.21)$$

Interrogation is sent if

$$-2\sigma^2 P_s \ln \frac{1}{P_{ue}} < V_0^2 - V_1^2 < 2\sigma^2 P_s \ln \frac{1}{P_{ue}} .$$

Thus, the decision system must, as in an ordinary receiver, determine the difference $V_0^2 - V_1^2$ and compare it with two symmetrical thresholds $\pm 2\sigma^2 P_s \ln(1/P_{ue})$. Such a receiver is usually called a receiver with a zero zone or with an erasure zone.

The probability of correct reception of an element q is easy to determine if the distribution of probabilities of the difference $V_0^2 - V_1^2$ is known. Thus, in the case of Rayleigh fading this difference has a bilateral exponential distribution and it can be shown by simple calculations that

$$q = [illegible] (1 - P_{ue}) [illegible] . \qquad (11.22)$$

When $h_0^2 \gg 1$ the magnitude of q is close to unity even with a very small P_{ue}. The probability of interrogation is

$$P_{inter} = 1 - q - P_{ue} . \qquad (11.23)$$

If there is no interference in the reverse channel it would be possible to implement a system with element-by-element check by repeating separately each symbol until it is received with a given level of fidelity. The interrogation and confirmation signals in this process would consist of one element. The equivalent probability of error in such a system can be found just as the residual probability of error in a discrete channel was determined in deducing formula (11.5):

$$P_e = \frac{P_{ue}}{1 - P_{de}} \qquad (11.24)$$

and the relative transmission rate is found as in deducing (11.11):

$$S = (1 - P_{de})^{M+1} , \qquad (11.25)$$

where M is the duration of blocking as determined by the number of signal elements transmitted during passage of the signal over the channel in both directions.

In actual fact there will be errors in the reverse channel and in order to reduce them to a permissible level it is necessary to transmit service signals of interrogation and confirmation in the form of rather long combinations. If the rated speed of transmission in the forward and reverse channels is the same, element-by-element check is no longer possible. It is necessary to check whole code combinations containing at least as many symbols as are in the service signals. An interrogation signal is sent in that case when during demodulation at least one of the elements of the code combination the difference $V_0^2 - V_1^2$ is in the erasure zone.

The maximal rate of transmission with a given level of fidelity or maximal fidelity with a given rate must be provided by selection of an optimal erasure zone [18] which plays the same role as selection of the code in a discrete channel.

In a continuous channel it is also possible to implement a duplex interrogation system in the same way as in a discrete channel, i.e., by using separated or unseparated service signals. In the latter case it is necessary to use cross blocking.

If a main message is encoded with redundancy, it is possible to effect entirety interrogation reception. In this case a single decision system evaluates the likelihood functions for all permissible combinations and if one of them greatly exceeds all others, it is issued to the recipient. If the difference between the two greatest values of the likelihood functions is not great, i.e., the level of fidelity of the decision reached is low, an interrogation is sent.

It is also possible to combine interrogation systems in discrete and continuous channels by effecting a check of each symbol of a combination falling in the erasure zone and then checking the entire combination for the presence of detected errors. According to some data such a system could be very effective if the code and erasure zone are selected properly.

11.5. Systems With Information Feedback

System With Reverse Check and Repeat

A system with reverse check and repeat is the simplest of the systems with information feedback in a discrete channel [26]. A message transmitted over a forward channel is encoded with the minimal redundancy required to discriminate one service combination of "negation." The last M transmitted code combinations, where M is determined from expression (11.10) are stored in the calculator-repeater of the transmitter. The received code symbols are recorded in a unit of the buffer memory of the receiver and sent over the reverse channel. The code symbols arriving over the reverse channel are compared with those stored in the repeater and if they do not coincide, a negation signal is sent over the forward channel and then all M combinations from the repeater are repeated.[1]

---

[1] In principle it should be possible to limit oneself to repetition of one combination of even one symbol, but this leads to complexity in the control system, an increase in the size of the buffer memory, and retention of the probability of overloading just as in an interrogation system without blocking.

Based on the received negation signal, M combinations are erased in the buffer memory of the receiver. Each received combination is issued to the recipient only when M combinations not contained an erasure signal have been received after it.

The possibility that there will be an incorrect symbol in a message issued to a recipient occurs only when the first symbol is received incorrectly in the reverse channel and a repeated incorrect symbol was transformed into a correct symbol in the reverse channel. Such a pair of errors is called an image error. In a binary system the probability of this is

$$p_{ue} = p_1 p_2 \tag{11.26}$$

where p_1 and p_2 are the probabilities of error in the forward and reverse channels respectively.

We will note that incorrect reception of a negation signal does not increase the probability of an undetected error. After it has been checked two negation signals will be sent over the reverse channel and two M combinations will be erased in the receiver buffer memory. It is only necessary to provide for a sufficient reserve. If the information combination is received as a negation signal, the erased symbols are simply repeated.

It is apparent from (11.26) that such a system can well be used when the probability of error in the reverse channel is much less than in the forward, for example, in the transmission of messages from a spacecraft when it is possible to use for the reverse channel a ground transmitter of much greater power than the onboard transmitter.

Reasoning as in the preceding section, we can show that the equivalent probability of error is

$$p_c = \frac{p_{ue}}{1 - p_{de}} \tag{11.27}$$

where p_{de} is the probability that an error which was detected occurred in the forward or reverse channel

$$p_{de} = (1 - p_1 p_2)^n - (1 - p_1)^n (1 - p_2)^n, \tag{11.28}$$

where n is the number of symbols in a combination.

The relative rate of transmission can be determined approximately by taking into account the fact that a code combination is emitted to a recipient if it is not a negation signal and if it and the following M combinations are received correctly in the forward and reverse channels or if errors were not detected. The probability that a transmitted combination is not a negation signal is equal to the probability that one combination passed without detected errors in the forward and reverse channels. Thus (if the probability of an undetected image error is ignored),

$$[illegible] \qquad (11.29)$$

It is apparent from this that if the probability of error in the forward channel is great, a good reverse channel permits attaining a rather high level of fidelity, but the rate of transmission [illegible] markedly small.

A system with a reverse check and repeat can be used in a duplex mode, alternating on the basis of time multiplexing the combination of the forward and reverse channels. The equivalent probability of error in this case will not change. The factor 1/2 appears only in formula (11.29) for a relative transmission rate in one direction [illegible] at the same time [illegible] is cut in [illegible].

System with Transmission of Check Symbols over the Reverse Channel

In this system [illegible] a message is encoded with a redundancy code but only information symbols are transmitted over the forward channel and check symbols are stored in a special memory unit. The received information symbols are also subjected to encoding but only the check symbols are sent over the reverse channel. On the transmitting end the check symbols received over the reverse channel are compared with those stored in the memory unit. If they do not coincide, a negation signal is sent over the forward channel and the given code combinations are repeated.

For simplification of analysis we will assume that the probabilities of error in both channels are the same: $p_1 = p_2 = p$. An error in the reception of a code combination is undetected if errors occur in the reverse channel, as a result of which the received check symbols approach the transmitted information symbols. It can easily be seen that [illegible] the transmitted combination into another. [illegible] P_{ne} is determined by the [illegible] and [illegible] the inappropriate [illegible] the probability [illegible] probability of a detected error [illegible] formula (11.2[illegible]) [illegible] just the same way [illegible] follows that [illegible] probability of [illegible]

We will find the [illegible] is transmitted [illegible] and the following [illegible] and their check [illegible] will ignore the [illegible] than the probability [illegible]

$$[illegible] \qquad (11.30)$$

or, when $p_1 = p_2 = p$

$$S = (1 - p)^{k(M+1)}(1 - P_{de})^{M+1}. \qquad (11.50)$$

The difference between this formula and (11.11) is caused by the fact that check signals are not transmitted over the forward channel. Thus, the system being discussed, at a given level of fidelity, exceeds an interrogation system in speed by n/k times due to the large load on the reverse channel.

The formulas obtained remain valid for a duplex system arrangement. In this case units consisting of n symbols are sent over each channel just as in a duplex interrogation system in a discrete channel, the only difference being that the check symbols in these units form a code combination, not with the information entering into this unit but with those which are contained in a unit obtained over another channel. Thus, until detection of errors occurs, a load on the channels in both systems is the same if one and the same code is used.

The difference between duplex systems with interrogation and transmission of check symbols over a reverse channel becomes noticeable in cases of error detection are taken into consideration. It amounts to the fact that a system with the transmission of check symbols does not need the cross blocking which is essential for a system with interrogation. Therefore, only coefficient k/n which allows for the use of a channel as a reverse channel must be introduced in formula (11.50) for the relative rate of transmission. Comparing this result with (11.12) and (11.13), we see that, all other things remaining equal, a duplex system with the transmission of check symbols is somewhat more effective than an interrogation system. In a technical respect they are approximately equivalent although a system with the transmission of check symbols needs a large memory and the functioning of its algorithm is somewhat more complex.

All the ideas about code selection and transmission of information in "bad" channels with memory as presented at the end of Section 11.5, with slight modifications, remain valid for the system under discussion. In systems with information feedback it is possible also to use an address repeat as in interrogation systems.

We will note that a system with reverse check and repeat can be viewed as a particular case of a system with transmission of check symbols occurring when n = 2k, i.e. code is used in which the check symbols are formed by repetition of the information symbols. Such a code is far from optimal since in it $d_{n,n}$ = 2 and therefore the probability of an undetected error is great despite the great redundancy. This leads to shortcomings in a system with a reverse check.

Information Feedback in a Continuous Channel

The possibilities of information feedback in a continuous channel have been little studied and have been considered mainly at a theoretical level (for example, [4, 19, 25]). Several methods which are possible in principle are considered in work [7]. Their general idea is that a received signal is sent over a reverse channel and information is extracted from it about the state of the forward channel. The information is used in the transmission of the following signals.

Duplex systems of radio communication involving reflection from meteor traces can be relegated to systems of information feedback in a continuous channel [3]. In them information is transmitted only over short segments of time while there is heightened ionization of the lower layers of the ionosphere caused by a passing meteor. During all remaining time sounding pulses are sent to both channels. Information about the possibility of transmitting information is extracted from the pulses arriving over the reverse channel.

The discontinuous communication based on such principles is also possible in shortwave channels when there are any other channels with slow fading. In this case by using the information obtained over the reverse channel messages are transmitted only when the transmission factor of the channel exceeds a certain threshold value $_0$. When $_0$ communication is interrupted and only sounding pulses essential to evaluate are transmitted. This permits, with a given level of fidelity, increasing the rated speed of transmission inasmuch as it is conducted only with a good channel state. The average rate of information transmission when the threshold $_0$ is selected optimally is much greater than in the case of ordinary continuous communication with the same level of fidelity [20-22].

11.6. Adaptive Methods of Encoding and Decoding

The existence of feedback permits adapting the methods of transmitting and receiving signals to channel state. Communication systems in which a code remains unchanged but the method of decoding and use of feedback change in accordance with a channel state are called systems with adaptive decoding. The same systems in which with a change in channel state the method of coding also changes (in the narrow or broad sense) are called systems with adaptive coding.

We will present an example to explain the possibilities of adaptive decoding. Let a message be encoded by a code with a minimal Hamming distance $d_{min} = 3$, specifically (7, 4) group code. In the receiver, along with an ordinary element-by-element decision system, there is a demodulator with a zero zone formed by two symmetrical thresholds selected in accordance with (11.21) and also a counter which counts the number of times the result of demodulation falls in the zero zone over a specific time interval. In the channel in a good state this is a little probable and with worsening in the state this probability increases. Thus, the readings of the counter permit judging the state of the channel.

If the number of times a demodulated signal falls in the zero zone is not recorded, a received code combination is decoded in the usual way and a single error can be corrected. When a demodulated signal falls in the zero zone once or twice over the length of a code combination, the corresponding symbols are rejected and decoding is done based on the remaining "most reliable" symbols (see Section 10.6). If the number of times reaches three or four, the code is used only to detect errors. In other words, the code combination is decoded only on condition that it is a permitted one. Otherwise an interrogation is sent over the reverse channel. Finally, if more than four values of the demodulated signal fall in the zero zone, decoding even of a permitted combination is not performed (inasmuch as in a channel in a bad state the probability of an undetected error is great) and an interrogation is sent over the reverse channel.

With a proper selection of thresholds such a system can provide for an exceedingly high level of fidelity. At the same time the average rate of information transmission turns out to be higher than in an ordinary interrogation system or an ordinary system with information system or an ordinary system with information feedback inasmuch as errors are corrected without interrogation and repeat in a channel with satisfactory states. Furthermore, with an ordinary interrogation system, and especially with a system providing for correction of errors, it would be necessary to use a much more complex code to obtain such a level of fidelity. A more detailed discussion of these methods and also a description of other methods of adaptive decoding can be found in work [23].

Methods of adaptive coding present additional possibilities for channels with slowly changing parameters. A simple code with low redundancy is used in a channel in a good state and with worsening in the state a switch is made toward a more complex code with greater redundancy, slowing the rate of information transmission but maintaining a given level of fidelity. The state of a channel can be judged from special signals which are sent over the reverse channel or, more simply, by counting the frequency of arrival of interrogation signals.

In multiplexed channels adaptive coding can be done by changing the multiplexing factor and this is easy to do in the case of sources with a controllable rate.

The theory of adaptive coding has not actually been worked out and therefore we must limit ourselves to the ideas expressed above.

Notes

1. The existence of a feedback channel in principle can increase the carrying capacity of a forward channel with memory. This increase occurs only because information is obtained about the state of a channel and cannot exceed the rate at which it is transmitted [10]. For a constant channel the channel capacity cannot be increased by using feedback [19].

In channels used in practice the rate of change of state is usually slow and the conditions themselves are measured without very great accuracy. Therefore, information about the state of a forward channel is extracted from a reverse channel with a slow rate and it can be considered that for all practical purposes the existence of feedback has no effect on the carrying capacity.

2. (See Section 11.5) In calculating the relative rate of transmission in the simplest system (11.2) and in a system with separated service signals (11.12) no account was taken of the delay incurred by incorrect reception of service signals. This delay occurs if an asymmetrical principle of decoding

service signals is used for protection against inserts and dropouts and the extra repeat code combinations are rejected. In this process a combination is not issued to a recipient if it is a repetition of a previously transmitted combination occurring as a consequence of transformation of a confirmation signal into an interrogation signal in the reverse channel.

We will use P_{conf} to denote the probability of incorrect reception of a confirmation signal. Then the probability that a combination used in the forward channel is not an extra repetition is equal to $1 - p_{conf}$. In light of this the rate of relative rate of transmission in a simplest system is

$$S = \frac{k}{n}(1 - P_{de})(1 - P_{conf}), \qquad (11.2a)$$

in a blocking system

$$S = \frac{k}{n}(1 - P_{de})^{M+1}(1 - P_{conf}) \qquad (11.11a)$$

and in a duplex system with separable service signals

$$S = \frac{k}{n + n_s}(1 - P_{de})^{M+1}(1 - P_{conf}). \qquad (11.12a)$$

The correction introduced here can greatly reduce the rate of transmission if a confirmation signal is often transformed into an interrogation signal. To avoid this the asymmetry of the principle for decoding service pulse trains should not be pushed to the limit.

3. (See Section (11.3) The value of the probability of an undetected error in the case of disruption in communication (11.5) is a very important characteristic of systems intended for channels in which such disruptions can occur. The less $P_{ue}(0.5)$ is, the greater is the certainty that during the time of such a disruption false information does not reach the recipient.

As follows from (11.8a) it is easy to provide for as small a value of $P_{ue}(0.5)$ as desired by selection of the code. For this purpose it is sufficient to have a large number of check symbols $n - k$. Thus when $n - k = 17$ $P_{ue}(0.5) <$ $< 10^{-5}$; when $n - k = 30$ $P_{ue}(0.5) \approx 10^{-9}$; when $n - k = 50$ $P_{ue}(0.5) \approx 10^{-15}$; etc. For this purpose it is not at all necessary for the code to have great redundancy. Thus, the code examined above (63, 45) providing for $P_{ue}(0.5) \leq 3 \cdot 10$[illegible] has a redundancy of only about 0.29.

In many channels "incomplete" disruptions in communication may occur when the probability of error p is close to 0.5 but does not reach this value. A question occurs as to whether it is possible to guarantee that in all states of a channel the probability of an undetected error does not exceed $P_{ue}(0.5)$ as calculated from formulas (11.8). The answer will be affirmative if P_{ue} is [illegible] 11.1 and 11.2 for codes [illegible] there are codes [illegible] demonstrate this

[illegible] symbols are arranged [illegible] can be easily [illegible] Therefore, according to [illegible] where [illegible] 0.1 [illegible] 10.1 [illegible] 10^{-4}.

[illegible] codes for [illegible] Unfortunately, as yet [illegible]

[illegible] interrogation [illegible] channels where the [illegible] Therefore, many authors [illegible] realization [illegible]

One such possible [illegible] blocking without introducing in the process a large buffer memory with complex control over input and output of information [illegible] simultaneous transmission of several [illegible] of the [illegible]. After [illegible] of the [illegible] of the first message [illegible] will be analyzed in the [illegible] signal corresponding [illegible] over the reverse channel. [illegible] the sender [illegible] of the first message [illegible] the need [illegible] possibility of continuous [illegible] In the case of [illegible] and there is no longer [illegible] transmitted in this way.

[illegible] number of [illegible] channel with a [illegible] relative speed of transmission [illegible] blocking [illegible] state contributes [illegible] blocked [illegible] to a high [illegible] shown in the [illegible]

[illegible] transmission [illegible]

about ten combinations will be recorded when $p_e \approx 10^{-7}$, i.e., three orders higher than the value which was considered as permissible. This does not occur in blocking since when $p \approx 5 \cdot 10^{-2}$ an average of one code combination will be recorded during a period of a day or longer. At the same time the existence of blocking has almost no effect on the average transmission rate as is apparent from a comparison of the curves for $p = 10^{-3}$, i.e., in that state when most of the information is transmitted.

5. (See Section 11.4) In most works interrogations systems in a continuous channel are considered in which the decision to interrogate is reached not by comparison of readings of the demodulator with the thresholds of the zero zone but by analysis of the shape of the envelope of the received sum of signal and interference which are not subjected to optimal (or suboptimal) processing in matched filters or devices equivalent to them. By way of criteria for evaluation of the shape of a signal, use is made of boundary distortions, splitting, or other parameters obtained from a comparison of signal shape with a certain standard. These methods are based on the fact that there is a correlation between states of a channel and distortions in the shape of the envelope. Nevertheless, they may not provide for an optimal statistical evaluation of the state of a channel and therefore lead, compared with the zero zone method, to a reduction either in level of fidelity or rate of transmission. At the same time they are no simpler than optimal or suboptimal methods with a zero zone.

6. (See Section 11.5) The principle problem in constructing a system with information feedback is protection against transformation of a negation signal into a combination of the main message or vice versa. Although these phenomena do not directly cause errors in a message arriving for a recipient (if the little likely cases of image errors are neglected), they may lead to overloading or units in the buffer memory on the transmitting or receiving end and thereby disrupt transmission. Therefore, it is always necessary to introduce a certain amount of redundancy in order to protect a negation signal from such transformations.

In transmission from sources with a controllable rate the need for a buffer memory on the transmitting end disappears. Therefore, in such systems the use of information feedback is more advisable especially if it is possible to use a memory unit with a large capacity on the receiving end. The indicated problems can be solved relatively easily in those cases when brief messages are to be transmitted. Nevertheless, concern should always be shown for protecting a negation signal against transformation [8].

BIBLIOGRAPHY

1. Val'd, A., "Sequential Analysis," Fizmatgiz, 1960.

2. Basharinov, A. Ye. and B. S. Fleyshman, *"Metody Statisticheskogo Posledovatel'nogo Analiza i Ikh Radiotekhnicheskiye Prilozheniya"* [Methods of Statistical Sequential Analysis and Their Radio Applications], Sovetskoye Radio Press, 1962.

3. Articles by various authors about radio communications involving reflection from meteor traces published in Proceedings of the IRE, No. 12, 1957.

4. Shannon, C., "Channels with Additional Information at the Transmitter," In the collection: Works on Information Theory and Cybernetics, Foreign Literature Press, 1963.

5. Chang, E. H., "Theory of Information Feedback Systems," IRE Trans., IT-2, No. 3, 1956.

6. Chang, E. H., "Improvement of Two-Way Communication by Means of Feedback," IRE Int. Conv. Rec., Pt. 4, 1961.

7. Grin, P. Ye., "Feedback Systems," In the book: Lectures on the Theory of Communication Systems, edited by Ye. Dzh. Bagdadi, Mir Press, 1964.

8. Kotov, P. A., "[illegible]" [Increasing Reliability in the Transmission of Digital Information], Svyaz' Press, 1966.

9. Shlyapoberskiy, V. I., "[illegible]" [Elements of Discrete Communication Systems], Voyenizdat, 1965.

10. Dobrushin, R. L., "Information Transmission in a Feedback Channel," Theory of Probability and Its Application, No. 4, 1958.

11. Blokh, E. L., "[illegible]" [Resistance to Interference in Communication Interrogation Systems], Press of the Academy of Sciences of the USSR, 1963.

12. Cherepov, N. S., "Main Characteristics of a System for Transmitting Binary Information with Complex Information Feedback," Elektrosvyaz', No. 1, 1966.

13. Piterson, U., "Codes Which Correct Errors," Mir Press, 1964.

14. Korzhik, V. I., "Probability Boundaries of Undetected Errors and Optimal Group Codes in Feedback Channels," Radiotekhnika, No. 1, 1965.

15. Korzhik, V. I., "One Evaluation of Detected Capabilities of Binary Group Codes," Radiotekhnika i elektronika, No. 11, 1965.

16. Martynov, Yu. M., "System for Transmitting Information with Address Interrogation of Code Words Rejected in Preceding Transmissions," Elektrosvyaz', No. 7, 1964.

17. Jacobs, I., "Optimum Error Detection Codes for Noiseless Decision Feedback", IRE Trans., IT-8, No. 6, 1962.

18. Borodin, L. F. and I. I. Grashko, "Advisability of Introducing Erasure Interval," Radiotekhnika, No. 8, 1960.

19. Shannon, C., "Carrying Capacity of a Noisy Channel When There is a Zero Error," In the book: Works on Information Theory and Cybernetics, Foreign Literature Press, 1963.

20. Montgomery, G. F., "Intermittant Communication with a Fluctuating Signal," PIRE, No. 12, 1957.

21. Mel'nikov, V. S., "Evaluation of Average Telegraphy Speed in Interrupted Radio Communication with Frequency Keying," Radiotekhnika, No. 4, 1959.

22. Klovskiy, D. D., "Resistance to Interference of Intermittent Communication Systems," Works of Communication Educational Institutions, Issue No. 8, 1961.

23. Borodin, L. F. and I. I. Grushko, "Adaptive Decoding Devices," Radiotekhnika, No. 6, 1966.

24. Dorman, M. I., "Transmission of Information in Systems with Decision Feedback in Binary Signal Modulation," Elektrosvyaz', No. 9, 1967.

25. Pinsker, M. S., "Error Probability in Blocking Transmission over a Gaussian Channel Without Memory and with Feedback," Problems of Information Transmission, Issue No. 4, 1968.

26. Kanevskiy, Z. M., "[illegible]" [Transmission of Messages with Information Feedback], Svyaz' Press, 1969.

27. Martynov, Yu. M., "[illegible]" [Processing Information in Data Transmission Systems], Svyaz' Press, 1969.

CONCLUSION

The questions examined in this book and the results derived enable us to compare different systems of transmitting discrete messages and wisely select the best system for any given conditions. This selection reduces to determination of the coding method, the system and method of signal formation, method of reception, etc. The author will deem his task fulfilled to a substantial degree if he has succeeded in convincing the reader of the importance of careful consideration of all channel characteristics, without which a chosen system will never make a good showing.

It would, however, be a mistake to try to find universal solutions here which would make it possible to design an optimum system of communications from a few given parameters. The relations given and formulas and graphs worked out can serve as basic data for the design of systems for transmitting discrete messages, but they are by no means ready-made prescriptions. There can be no such prescriptions because the conditions under which communications systems operate are extremely varied. Under different conditions various engineering, economic, tactical, or other demands affecting the design of the whole system or of individual parts thereof may play the dominant role.

In the design of communications systems certain supplementary stipulations may be made depending on the specific use to which they are to be put. In some cases, for example, the economic factor is decisive and the system must be so designed as to afford the least total expenditure on construction of the equipment and its operation for a certain period of time. In other cases the correct criterion may be minimum weight or volume of the whole apparatus. At times the requirements as to weight and volume may differ for the sending and receiving equipment, for example, when one end of the communications line is stationary and the other on a moving object.

Nevertheless, in all the numerous cases of communications systems design, skill in evaluating the probability of error and in determining how it will change in certain variations of the system is a necessary condition for a considered approach to the problem faced.

The vast majority of the systems for transmitting discrete messages which are at present in operation are far from being optimum. A partial explanation of this is that developing and putting into operation new means of communication usually takes many years; and this leads to a considerable lag of practical accomplishments behind theoretical advances. Another reason for this lag is the inadequate acquaintance on the part of many engineers engaged in the development and operation of communications systems with the latest theoretical findings. A not unimportant role in this is played by the inaccessibility of many theoretical works because of the complexity of the mathematics used and because they do not lead to precise recommendations.

In this connection it may be remarked that during the many years of the existence of general communication theory (approximately until the middle of the 1950's) the chief theoretical results consisted in explaining and generalizing the methods of communication which up to that time had already been formed mainly on the intuition of their developers (e.g., pulse-code modulation, seven-element error-detecting code, etc.). In the last few years the situation has changed and theory has begun to exert an active influence on the development of new systems (e.g., Kineplex, Rake, etc.).

Here it is appropriate to point out that the tremendous achievements in the field of developing various communication systems over a period of many decades during which communication theory was coming into being were the result of "natural selection." Along with methods of transmission and reception greatly advanced for their time and which became firmly implanted in the arsenal of modern technology, many inventors suggested and developed every year various systems which did not withstand the test of time. Even now great sums are wasted on experimental research of communication methods which could be immediately rejected as a result of theoretical analysis. At the same time many achievements in theory have been clearly put to inadequate use in practice.

Thus, in electric wire communication almost no use is made of optimal or close-to-optimal methods of processing a signal. Specialists in this area still are under the impression that the principal problem in a communication system is to reproduce the shape of a transmitted signal as accurately as possible while, in actuality, only extraction of the information contained in it is important. It is often said in justification of nonoptimal systems in which much information is lost that in cable channels fluctuation noise is so insignificant that existing methods of reception provide for a high level of fidelity. But this situation, as already noted, has led to clearly inadequate use of channel carrying capacity. The application of very simple methods for optimizing the shape of a signal and processing it would have made it possible to greatly increase the rate of information transmission (e.g., greatly increase the multiplexing factor) and would have yielded great economic gain.

A consideration of the radio communication system with frequency keying in widest use shows that modernization of it based on the use of only the simplest practicable recommendations of theory (application of matched filters for orthogonal signals, effective methods of adding in diversity reception, suitable methods for suppressing impulse interference, use of feedback channels, etc.) could have provided a power gain on the order of 10-20 db. This means that while retaining the same level of fidelity and reception it would have been possible to reduce transmitter power by ten times or while retaining the same power to greatly increase the level of fidelity. By using more complex methods it is possible with the present level of equipment to create communication systems yielding greater power gains and also permitting a great increase in the rate of information transmission.

Among the trends noted in the past few years in the development of systems for the transmission of discrete information particular attention should be devoted to the use of a feedback channel in all cases where possible. In conjunction with a wisely selected method of coding, feedback systems provide for a high level of fidelity and reception when the characteristics of a channel are least favorable.

Another promising trend (at least for radio channels employing ionospheric or tropospheric wave propagation) is the use of wideband signals. As was indicated, such systems permit actively combatting multibeam wave propagation and also using this phenomenon to increase the fidelity of reception. They provide, furthermore, for the possibility of reliable suppression of impulse interference and also used some conditions simplify the problem of assigning a large number of channels to a limited frequency range.

It should be noted that for the implementation of optimal or close-to-optimal communication systems, high frequency precision which is not always achievable with the present-day level of equipment to stabilize frequencies is required. This forces resort in some case to automatic frequency tuning. The essence of automatic tuning amounts to transmitting over a communication channel, along with the main message, information about a reference frequency used in shaping the signal. This information is extracted by the receiving device and used in the decision circuit for reception of the main message.

The transmission of information about frequency entails many interesting problems. Included in them are problems concerning the required additional channel carrying capacity, the best methods of extraction and use of this information, possible methods of coding, etc. Unfortunately, the extent of this work does not permit devoting attention to these problems.

The problem of synchronizing decision circuits is closely associated with this. Usually a distinction is made between beat (determination of instants of arrival of beginning of signal elements) and cyclical synchronization (determination of first symbol in a code combination). These problems have been rather well resolved in modern communication equipment, at least in incoherent reception circuits. However, synchronization theory has been little developed and it is difficult to say what possibilities will be found in it for improving and simplifying existing systems. Therefore, we are forced to limit ourselves to a consideration of the effects of inaccuracies in synchronization on interference resistance and to several general ideas.

A wide range of problems arises in studying methods for extracting information about the state of a channel from a signal (specifically, instantaneous values of the components of the transmission factor) and its use for optimizing signal processing. We were forced in most cases to limit ourselves to a consideration of two extreme situations when nothing was known about the value of the transmission factor and when it is known exactly. It is true that in many cases this information has little or no effect on resistance to interference but sometimes, for example, in diversity reception, selective fading, interrogation systems, etc., it is extremely important. The task amounts to predicting the values of the transmission factor based on observations of a signal over a certain segment of time and to synthesis of decision circuits.

Many other problems which have not yet been completely resolved were not considered in the book. A detailed listing of them would take too much space. Undoubtedly, with further development in the theory and technology of transmitting discrete messages, these problems will be resolved, but at the same time life will present us with new problems.

Only a thorough and comprehensive development of theory closely tied to practice will permit us in the future to rapidly and correctly find solutions to problems which arise in connection with transmitting discrete messages, the great variety and complexity of which can only be foreseen with great difficulty at the present time.

APPENDIX

[illegible]

PRINCIPAL SYMBOLS USED

$A(t)$	envelope of received signal
A_k, B_k	Fourier coefficients of a received signal
a_{rk}, b_{rk}	Fourier coefficients of a transmitted signal element corresponding to symbol n_r
B	base of signal (system) ($B = 2FT$)
C	channel passband
D	Kotelnikov distance between signals
d	Hamming distance between code combinations
$Ei(x)\ \int_{-\infty}^{x} \frac{e^t}{t}\,dt$	integral exponential function
F	stipulated signal frequency band (of a system)
$G(f)$	power spectral density
$g(t)$	impulse response of a linear system (of a filter)
$H(\ )$	entropy
$H'(x)$	productivity of message source, entropy per unit time
$h(x)$	differential entropy
h^2	ratio between power of signal element and interference spectral density
h_0^2	mathematical expectation of h^2 when there is fading
$I(x,y)$	amount of information contained in x relative to y
$I'(x,y)$	rate of transmission of information contained in x relative to y
$I_n(x) = j^{-n}J_n(jx)$	Bessel function of imaginary argument
$J_n(x)$	Bessel function of the n-th order
k	ratio between regular component of transmission coefficient and fluctuating component
k	divisibility of channel multiplexing
L	duration of channel reaction to impulse
L,M	parameters describing rate of fading
l	size of source alphabet
m	code base

n	number of symbols in code combination
(n,k)	systematic code, combinations of which contain k information symbols and n - k verification
$n(t)$	additive interference
P	power
P_s	signal power
P, p	probability
$P(\ )$, $p(\ )$	probability of error
$P_{[illegible]}$	probability of incorrect reception of a group signal in a multiplexed channel
p_e	equivalent probability of error
Q	number of branches in diversity reception
$Q(x,y) = \int_y^\infty u \exp\left(-\frac{x^2+u^2}{2}\right) I_0(ux)\,du$	Q-function
q	ratio between mean-square values of signal and interference at detector input
R, r, $R(r)$	correlation coefficient
S	state of source or channel
S	relative transmission rate
$S(j\omega)$	filter transfer function
$si(x) = -\int_0^\infty \frac{\sin t}{t}\,dt$	integral sine
T_e	duration of signal element
T	duration of part of signal element being analyzed
V_r, X_r, Y_r	magnitudes determined by formulas (3.10) and (4.11)
$w(\ )$	density of probability distribution of a random variable
x	transmitted message
x'	received message
y	transmitted code symbol or sequence of code symbols
y'	received code symbol or sequence of code symbols
z	transmitted signal
z'	received signal
$\tilde{z}(t)$	function conjugate with $z(t)$
α_k, β_k	Fourier coefficients of an element of interference

[illegible]

[illegible]	coefficient in (5.61) depending on signal [illegible]
[illegible] f	effective noise passband
[illegible]	indicator of transmitter linearity (see [illegible])
[illegible]	channel transmission factor
[illegible] c, [illegible] s	cophasal and quadrature components of transfer coefficient
[illegible] r	regular component of transfer coefficient
[illegible] f	fluctuating component of transfer coefficient
[illegible]	interference spectral density
[illegible]	parameter of nonorthogonality
[illegible]	dispersion
[illegible]	dispersion of Fourier coefficients of interference
[illegible] k	interval of correlation of fading
$\Phi(x) = \sqrt{\frac{2}{\pi}}\int_0^x e^{-\frac{x^2}{2}}dx$	Crampe function
[illegible]	initial phase
[illegible]	angular frequency

The mathematical expectation of a random variable is indicated by a horizontal line, for example: $\bar{x}$ is the mathematical expectation of the variable x.

SUBJECT INDEX

A posteriori probability 24
A priori probability 23, 27
Adaptive encoding and decoding 561-562
Address interrogation 548-549
Alphabet 8-9, 50
Amplitude modulation 142-145, 202-206, 284-286
Analog decoding (see also: Entirety reception) 523
Analytic signal 183
Asynchronous address systems 501

Base 117-118, 378, 396, 430-432
Bayes formula 23, 293, 355
Binary relative phase modulation (BRPM) 489-496
  --frequency modulation (BFM) 473, 476
  phase modulation (BPM) 478, 481
Binary unit 14
Blocking 547-548

Carrying capacity 158-167
  --of a fading channel 307-309
    --of undetermined phase 250-256
    --in the case of diversity reception 337
  --of a noiseless discrete channel 51
    --of an erasure channel 82
    --of a nonsymmetrical channel 84-85
    --of a symmetrical channel 69-70
Channel 4-6
  --discrete 48, 85, 459, 541
    --constant (uniform) 49
    --continuous 48, 459, 541
    --erasure 24, 82
    --Markov 87
    --with error grouping 87
    --with memory (nonuniform) 49
Code
  --binary 76, 79
  --blocking
  --Bose-Choudkhuri 81
  --cyclical 80
  --densely packed 73, 103
  --equidistant 99
  --Grey 182
  --group 79
  --interference resistant 65, 529-532
  --irreducible 57
  --keying 175, 479
  --optimal nonuniform 56, 59
  --packet 81
  --recurrent 81
  --systematic 77
  --uniform

Code combination 10, 54, 63
--permissible 63
Code symbol 9
Coding 9
--interference resistant 65, 496-500, 505, 529-532
--parallel 496
--primary (primitive) 55
--random 69
Combination multiplexing systems 460, 473-496
Combining of channels 452
Communication system 4
--dense 116
--equidistant 145-147
--simple 117, 447
--synchronous 11, 114
--with active delay 127, 137, 145
--using biorthogonal signals 149, 213
--using orthogonal signals 141, 146
--with a passive delay 141, 201-206, 284-285
--with interrogation 540, 542
--with opposed signals 137
Compensation of impulse interference 438, 441, 448
--of beams 594
Correlation matrix 208, 287, 315
--check 80
--generating 79
Correlation interval (see: Time correlation)
Correlation matrix 208, 287, 315
--check 80
--generating 79
Correlation time 231-234, 371-372
Decision system (circuit, principle) 19
Decoding 9, 10, 24
--analog 527
--majority 81, 527
--sequential 81
Decorrelation 20, 81
--of errors 88, 293
Delta-function 417, 441
Detecting, ideal 184
--linear 233-236
--quadratic 228-233
Detector, linear amplitude 184, 233
--frequency 223
--phase 139
--quadratic 140, 229
Discretization 6

Distribution normal 22, 121, 131
--"chi square" 336
-normal-logarithmic 276, 326
--Rayleigh 203, 270
--generalized (Rice distribution) 203, 370
Duplex systems with interrogation 529-530

Effective passband (noise) 174-176, 220-236
--of frequencies occupied by a signal (conditional) 116, 174-176, 429-434
Eigen functions and values 377
Element of a signal 114
of a message 7
Enlarged alphabet 12
Entirety reception 25, 505-538
--based on instantaneous frequency 225
--coherent 136, 236, 277
--coherent 506-510
--diversity 525, 562
--coherent 527
--frequency 525
--incoherent 527, 538
--of discrete addition 534
--of linear addition 533
--of quadratic addition 539-545
--selection diagram based on maximum power 548
--based on maximal likelihood 540, 546
spatial 525
--time 525
--element-by-element 26
--incoherent 181, 182, 240, 278
-incoherent 510-521
--with incoherent cumulation 510-511
--narrow-band based on envelope 219
--of general comparison 550
--quasi coherent 240, 248
-using the most reliable symbols (Borodin method) 523-526
using the Wagner method
--wideband with integration (filtration) following detector 225
Entropy 15, 42
--conditional 31, 41
--differential 41, 159-160
Envelope 183
Erasure zone 536

Factor of working order 326
--of correlation 274-326
-of transmission 111, 268-276

Fading 268
--fast 275, 372, 379-390, 407-410
--general 270
--quasi-Rayleigh 272
--Rayleigh 270
--selective 272, 390-393
--slow 275, 372
--zero rate 275
Feedback 516, 540-561
--decision (with interrogation) 541, 541-557
--information 541, 557-561
--with element-by-element check 557
Filter, ideal pi-shaped 220, 221, 224, 226, 232-235
--matched (optimal) 126, 128, 190, 243, 387, 402, 407, 422, 443
--of lower frequencies 235
--separating 220
--switched 130, 193, 247, 258
--whitening 151, 193, 425, 453
--with the characteristic of a single loop 220, 232-235
Fourier series 116
--conjugate 183
Frequency keying (FK) 141, 147, 193, 215, 225

Hamming distance 71, 519, 530
High-frequency filling 180, 184

Image errors 558
Information (amount of information) 11-15, 33-36
Instantaneous transfer function 365
--phase 183
--frequency 183
Integrator 124, 131, 242
Interference 4, 20
--additive 21
--cross 470
--Gaussian (fluctuation) 23
--impulse 417, 434-451, 457
--lumped 417-434, 449-457
--multiplicative 301
--mutual 86
--switching 86, 457
Interference impulse 417, 434-449
Interrupted communication 561
Isomorphic systems 144, 249, 493

Keying (modulation) 11
"Kiniplex" 246, 461
Kotel'nikov distance 136, 530
Kotel'nikov theorem 136

[illegible]
[illegible]
[illegible]

[illegible]
Message [illegible]
[illegible]
[illegible]
Message source [illegible]
[illegible]
[illegible]
[illegible]
[illegible]
Model of a channel [illegible]
Bennet [illegible]
--Gilbert 87
Multibeam propagation (multipath channels) 22, 504-426
Multiple relative phase modulation (MPM) [illegible]-494
--frequency keying [illegible] 473, [illegible]-496
--phase keying (MPK) 477-484
Multiplexing factor [illegible]
Multiplexing of channels [illegible] 501, [illegible], 457, (see also: Multiplexing system)
--statistical 502
Multiplexing system [illegible]-462
--address 501
--asynchronous 462, 500
--combination 462, 473-496
--frequency 459
--mixed 462
--quasi-separable 459
--separable 450, 458
--synchronous 462
--time 459

Nakagami probability [illegible] distribution 71
Noise [illegible]
--system 40[illegible]

Orthogonality 458
--in the interval [illegible]

Packet of errors 88, 294
[illegible] factor [illegible]
Phase difference modulation (see: Relative modulation)
Phase modulation (PM) 139, 140, 278
Potential resistance to interference 2, 133
Power gain [illegible]
Power of a signal 12[illegible]
--equivalent 134

Probability of error 69, 135-138, 463-465
  --detected 542-547
  --equivalent 101-104, 200-201, 531, 542
  --residual 544
  --uncorrected 99
  --undetected 542-546
Productivity of message source 21, 36
Protective interval 376

Quadratic form 286-289, 314-319, 336, 342, 388
Quadratic system 189
Quantizing 6, 10
Quasi-separable multiplexing systems 459, 470-473

"Rake" 402, 403-407, 450
Rate of information transmission 39
  --relative 542
Redundancy 18, 67, 305, 529-532, 543
Relative phase modulation (RPM) 94, 238-250, 278
Resistance impulse 417, 434-451
  --reaction impulse (transfer function) 128, 188, 210, 410-413

Shannon coding theorem 37, 52, 141
"ShOU" 444, 498
Signal 4
  --analytical 183
  --conjugate 183
    --wideband (noise-like) 138, 397, 403, 428, 431 434
  --group 458
Signal power 137, 138
  --equivalent 134
Simplex 119
Singular process 167, 173
Spectral power density 120, 138
Statistical criterion 26, 462, 464
  --Bayes 27
  --information 44
  --minimax 43
  --of maximal likelihood 29
    --generalized 32
  --of the ideal observer 28
Synchronization 133, 570-571

Telegraphic register 106-107
Transform of Gilbert 257
  --of Carson-Laplace 318

Unreliability 34, 36

White noise 23

Zero zone (see: Erasure zone)